MATH 111

Fundamental Logical Equivalences

The following equivalences can be verified by constructing the corresponding truth tables. T stands for True (or any compound statement that is always true), and F stands for False (or any compound statement that is always false).

Remark. The list below contains the most well-known and most often used equivalences. Some other operations share the same properties, some of which were discussed in class but omitted here. Don't memorize all those discussed in class, but the ones below are good to know.

1. Commutative laws:

$$P \vee Q \equiv Q \vee P$$

$$P \wedge Q \equiv Q \wedge P$$

$$P \Leftrightarrow Q \equiv Q \Leftrightarrow P$$

2. Associative laws:

$$(P \vee Q) \vee R \equiv P \vee (Q \vee R)$$

$$(P \wedge Q) \wedge R \equiv P \wedge (Q \wedge R)$$

3. Distributive laws:

$$P \vee (Q \wedge R) \equiv (P \vee Q) \wedge (P \vee R)$$

$$P \wedge (Q \vee R) \equiv (P \wedge Q) \vee (P \wedge R)$$

4. DeMorgan's laws:

$$\sim (P \vee Q) \equiv (\sim P) \wedge (\sim Q)$$

$$\sim (P \wedge Q) \equiv (\sim P) \vee (\sim Q)$$

5. Idempotent laws:

$$P \vee P \equiv P$$

$$P \wedge P \equiv P$$

6. Identity laws:

$$P \vee F \equiv P$$

$$P \wedge T \equiv P$$

7. Inverse laws:

$$P \vee (\sim P) \equiv T$$

$$P \wedge (\sim P) \equiv F$$

8. Domination laws:

$$P \vee T \equiv T$$

$$P \wedge F \equiv F$$

9. Absorption laws:

$$P \vee (P \wedge Q) \equiv P$$

$$P \wedge (P \vee Q) \equiv P$$

10. Double negation law:

$$\sim (\sim P) \equiv P$$

11. Implication identity:

$$P \Rightarrow Q \equiv (\sim P) \vee Q$$

12. Contrapositive identity:

$$P \Rightarrow Q \equiv (\sim Q) \Rightarrow (\sim P)$$

13. Biconditional identities:

$$P \Leftrightarrow Q \equiv (P \wedge Q) \vee ((\sim P) \wedge (\sim Q))$$

$$P \Leftrightarrow Q \equiv (P \Rightarrow Q) \wedge (Q \Rightarrow P)$$

We can use the above fundamental equivalences to prove other equivalences. Below are a couple of examples. More are studied in Math 110 (Symbolic Logic, elective class).

Example 1. Prove that $\sim (P \Rightarrow Q) \equiv P \wedge (\sim Q)$.

Proof:

$$\begin{aligned} \sim (P \Rightarrow Q) &\equiv \sim ((\sim P) \vee Q) \qquad \text{(Implication identity)} \\ &\equiv (\sim (\sim P)) \wedge (\sim Q) \qquad \text{(DeMorgan's law)} \\ &\equiv P \wedge (\sim Q) \qquad \text{(Double negation law)} \end{aligned}$$

Example 2. Prove that $\sim (P \Leftrightarrow Q) \equiv (P \wedge (\sim Q)) \vee ((\sim P) \wedge Q)$.

Proof:

$$\begin{aligned} \sim (P \Leftrightarrow Q) &\equiv \sim ((P \Rightarrow Q) \wedge (Q \Rightarrow P)) \qquad \text{(Biconditional identity)} \\ &\equiv (\sim (P \Rightarrow Q)) \vee (\sim (Q \Rightarrow P)) \qquad \text{(DeMorgan's law)} \\ &\equiv (P \wedge (\sim Q)) \vee (Q \wedge (\sim P)) \qquad \text{(by Example 1)} \\ &\equiv (P \wedge (\sim Q)) \vee ((\sim P) \wedge Q) \qquad \text{(Commutative law)} \end{aligned}$$