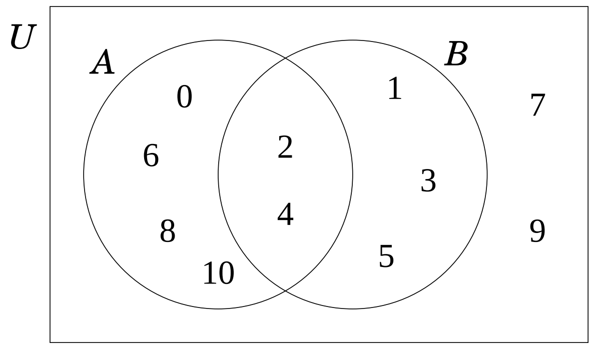
MATH 111

Practice Test 1 - Solutions

1. Read the textbook!
2. (a)



(b) $A \cap B = \{2, 4\}$, $\overline{A} = \{1, 3, 5, 7, 9\}$, $A \cup \overline{B} = \{0, 2, 4, 6, 7, 8, 9, 10\}$

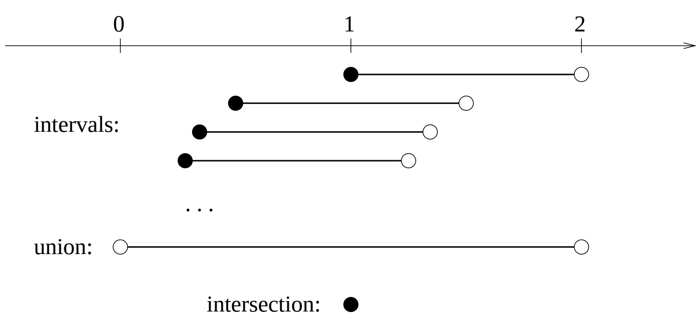
(c) Since A has six elements and B has five elements, $A \times B$ has $6 \cdot 5 = 30$ elements.

(d) $(0, 1)$, $(0, 2)$, $(10, 5)$.

3. (a) Statements $A \subset D$, $B \in D$, $\emptyset \subset D$ are true. The other statements are false.

(b) $|A| = |B| = |C| = 1$, $|D| = 3$.

4. First of all, let's rewrite the right endpoint: $A_n = \left[\frac{1}{n}, 1 + \frac{1}{n}\right)$. Then the first few intervals are: $A_1 = [1, 2)$, $A_2 = \left[\frac{1}{2}, 1 + \frac{1}{2}\right)$, $A_3 = \left[\frac{1}{3}, 1 + \frac{1}{3}\right)$, etc. We see that the left endpoint approaches 0 and the right endpoint approaches 1 as n gets larger:



Therefore the union of these intervals is $\cup_{n \in \mathbb{N}} A_n = (0, 2)$ and the intersection is $\cap_{n \in \mathbb{N}} A_n = \{1\}$.

5. Recall that $x,\ y,\ z \in \mathbb{R}$.

(a) $\exists x\ (x^2 = 8)$ is true, e.g. $x = 2\sqrt{2}$ satisfies this equation.

(b) $\forall x\ (x \neq -x)$ is false, e.g. for $x = 0$ we have $0 = -0$.

(c) $\exists x\ (x^2 - x + 1 = 0)$ is false as this quadratic equation has no real solutions (which can be verified using the quadratic formula).

(d) $\forall x\ (x^2 + 1 > 0)$ is true since $x^2 \geq 0$ for any real number x.

(e) $\forall x \forall y\ (xy = 0)$ is false, e.g. if $x = 1$ and $y = 1$, then $xy = 1$, so $xy \neq 0$.

(f) $\exists x \exists y\ (xy = 0)$ is true, e.g. for $x = 0$ and $y = 0$.

(g) $\forall x, y\ (x \neq y)$ is false, e.g. if $x = 1$ and $y = 1$ we have $x = y$.

(h) $\forall x \forall y \forall z\ (z = x + y)$ is false, e.g. for $x = 1$, $y = 1$, and $z = 1$, we have $z \neq x + y$.

(i) $\exists x, y, z\ (z = x + y \wedge x \neq y)$ is true, e.g. for $x = 1$, $y = 2$, and $z = 3$ both $z = x + y$ and $x \neq y$ are satisfied.

(j) $\forall x, y\ (x \leq y \vee x \geq y)$ is true since for any real number x, any real number y is either less than, or equal to, or larger then x.

6. (a) We will use a truth table to show that $P \Leftrightarrow Q$ and $(P \wedge Q) \vee ((\neg P) \wedge (\neg Q))$ are logically equivalent.

P	Q	$P \Leftrightarrow Q$	$P \wedge Q$	$\neg P$	$\neg Q$	$(\neg P) \wedge (\neg Q)$	$(P \wedge Q) \vee ((\neg P) \wedge (\neg Q))$
T	T	T	T	F	F	F	T
T	F	F	F	F	T	F	F
F	T	F	F	T	F	F	F
F	F	T	F	T	T	T	T

Since the truth values of $P \Leftrightarrow Q$ and $(P \wedge Q) \vee ((\neg P) \wedge (\neg Q))$ are the same for all possible combinations of truth values of P and Q, these compound propositions are logically equivalent.

(b) The compound statement $(P \Leftrightarrow Q) \Leftrightarrow ((P \wedge Q) \vee ((\neg P) \wedge (\neg Q)))$ is a <u>tautology</u>.

(c) The compound statement $(P \Leftrightarrow Q) \Leftrightarrow \neg((P \wedge Q) \vee ((\neg P) \wedge (\neg Q)))$ is a <u>contradiction</u>.

7. (a) We will show that for any integer n, the number $3n^2 + 5n$ is even. To do this, we will consider two cases:
<u>Case I:</u> n is even. Then $n = 2k$ for some $k \in \mathbb{Z}$, and $3n^2 + 5n = 3(2k)^2 + 5(2k) = 12k^2 + 10k = 2(6k^2 + 5k)$. Since $6k^2 + 5k \in \mathbb{Z}$, the number $3n^2 + 5n$ is even.
<u>Case II:</u> n is odd. Then $n = 2k + 1$ for some $k \in \mathbb{Z}$, and $3n^2 + 5n = 3(2k + 1)^2 + 5(2k+1) = 12k^2 + 12k + 3 + 10k + 5 = 12k^2 + 22k + 8 = 2(6k^2 + 11k + 4)$. Since $6k^2 + 11k + 4 \in \mathbb{Z}$, the number $3n^2 + 5n$ is even.
Since $3n^2 + 5n$ is never odd, the implication follows. (This is a vacuous proof.)

(b) If n is even, then $n = 2k$ for some $k \in \mathbb{Z}$, and $3n^2 - 2n - 5 = 3(2k)^2 - 2(2k) - 5 = 12k^2 - 4k - 5 = 12k^2 - 4k - 6 + 1 = 2(6k^2 - 2k - 3) + 1$. Since $6k^2 - 2k - 3 \in \mathbb{Z}$, the number $3n^2 - 2n - 5$ is odd. (This is a direct proof.)

(c) We will prove the statement by contrapositive, namely, we will prove that if n and m are of the same parity, then $n - 5m$ is even, and thus not odd.
Let's consider two cases:
Case I: n and m are both even. Then $n = 2k$ and $m = 2l$ for some $k, l \in \mathbb{Z}$. Then $n - 5m = 2k - 5(2l) = 2k - 10l = 2(k - 5l)$. Since $k - 5l \in \mathbb{Z}$, the number $n - 5m$ is even.
Case II: n and m are both odd. Then $n = 2k + 1$ and $m = 2l + 1$ for some $k, l \in \mathbb{Z}$. Then $n - 5m = 2k + 1 - 5(2l + 1) = 2k + 1 - 10l - 5 = 2k - 10l - 4 = 2(k - 5l - 2)$. Since $k - 5l - 2 \in \mathbb{Z}$, the number $n - 5m$ is even.

8. (a) For any real number x, $x^2 \geq 0$, therefore $-x^2 \leq 0$, and $-5 - x^2 \leq -5 + 0 = -5 < 0$. (This is a trivial proof.)

(b) If $|x| = 5$, then either $x = 5$ or $x = -5$. Thus we can consider the following two cases:
Case I: $x = 5$. Then $x^2 + x + 1 = 5^2 + 5 + 1 = 31 > 20$.
Case II: $x = -5$. Then $x^2 + x + 1 = (-5)^2 + (-5) + 1 = 21 > 20$.
(This is a proof by cases.)