MATH 111

Practice Test 3 - Solutions

1. Read the textbook.
2. (a) This statement is true. For example, if $a = -1$, then for every real number b, we have $b^2 \geq 0 \geq -1$, so $b^2 \geq a$.

   (b) This statement is false. For any integer a, either $a \leq 4$ or $a \geq 5$. If $a \leq 4$, then $a^3 + 2a + 3 \leq 64 + 8 + 3 = 75 < 100$, so $a^3 + 2a + 3 \neq 100$. If $a \geq 5$, then $a^3 + 2a + 3 \geq 125 + 10 + 3 = 138 > 100$, so $a^3 + 2a + 3 \neq 100$.

   (c) This statement is false. For example, if $a = -1$, then there is no integer b such that $b^2 = -1$.

   (d) This statement is false. For example, $\sqrt{2} + (2 - \sqrt{2}) = 2$. We know that $\sqrt{2}$ is irrational (we proved such a theorem). The fact that $2 - \sqrt{2}$ is irrational can be proved by contradiction. Namely, assume that $2 - \sqrt{2}$ is rational, then $2 - \sqrt{2} = \frac{m}{n}$ for some $m, n \in \mathbb{Z}$, $n \neq 0$. Then $\sqrt{2} = 2 - \frac{m}{n} = \frac{2n-m}{n}$. Since $2n - m \in \mathbb{Z}$ and $n \neq 0$, $\sqrt{2}$ is rational. Contradiction. Finally, $2 = \frac{2}{1}$ is rational.

   (e) This statement is true. Let a be any irrational number. Then $a = 1 + (a - 1)$. Observe that 1 is rational, and $a - 1$ is irrational (the proof of this is similar to the proof given in previous problem, and is omitted here).

   (f) This statement is true. For any sets A and B, let $C = A \cup B$. Then $A \cup C = A \cup A \cup B = A \cup B$ and $B \cup C = B \cup A \cup B = A \cup B$, so $A \cup C = B \cup C$.

   (g) This statement is false. For example, if $A = \{1\}$, $B = \{2\}$, $C = \{1, 2\}$, $D = \{2, 3\}$, then $A \subset C$, $B \subset D$, and $A \cap B = \emptyset$, however, $C \cap D \neq \emptyset$.

   (h) This statement if true. Suppose that $A \subset C$, $B \subset D$, $C \cap D = \emptyset$, but $A \cap B \neq \emptyset$. Then there is an element $x \in A \cap B$, so $x \in A$ and $x \in B$. Since $A \subset C$ and $B \subset D$, it follows that $x \in C$ and $x \in D$. Then $x \in C \cap D$, thus $C \cap D \neq \emptyset$. We get a contradiction.
3. (a) $\{(a, 1), (b, 2), (c, 3)\}$ is a relation from B to A (since it is a subset of $B \times A$). Moreover, it is a function from B to A (since each element of B is the first coordinate of exactly one pair in the relation).

   (b) $\{(1, b), (1, c), (3, a), (4, b)\}$ is a relation from A to B (since it is a subset of $A \times B$), but it is not a function (e.g. since the image of 1 is not well-defined).
4. (a) The relation R is not reflexive: e.g. $(1, 1) \notin R$ since $1 + 1 \neq 0$;
   R is symmetric since if $(a, b) \in R$, then $a + b = 0$, then $b + a = 0$, so $(b, a) \in R$;
   R is not transitive: e.g. $(1, -1) \in R$ and $(-1, 1) \in R$, but $(1, 1) \notin R$;
   R is not an equivalence relation: e.g. R is not reflexive.

(b) The relation R is not reflexive: e.g. $(0,0) \notin R$ since $\dfrac{0}{0}$ is undefined, so it is not an element of $\mathbb{Q}$;
R is not symmetric: e.g. $(0,1) \in R$ since $\dfrac{0}{1} \in \mathbb{Q}$, but $(1,0) \notin R$ since $\dfrac{1}{0}$ is undefined;
R is transitive since if $(a,b) \in R$ and $(b,c) \in R$, then $\dfrac{a}{b} \in \mathbb{Q}$ and $\dfrac{b}{c} \in \mathbb{Q}$, and then $\dfrac{a}{c} \in \mathbb{Q}$;
R is not an equivalence relation: e.g. R is not reflexive.

(c) The relation R is not reflexive: $(0,0) \notin R$ since $0 \cdot 0 \not> 0$;
R is symmetric since if $(a,b) \in R$, then $ab > 0$, then $ba > 0$, so $(b,a) \in R$;
R is transitive since if $(a,b) \in R$ and $(b,c) \in R$, then $ab > 0$ and $bc > 0$, then either all of a, b, and c are positive or all of them are negative; in either case, $ac > 0$, so $(a,c) \in R$;
R is not an equivalence relation since R is not reflexive.

(d) The relation R is reflexive since for any $a \in \mathbb{Z}$, $a \equiv a \pmod 3$, so $(a,a) \in R$;
R is symmetric since if $(a,b) \in R$, then $a \equiv b \pmod 3$, then $b \equiv a \pmod 3$, and then $(b,a) \in R$;
R is transitive since if $(a,b) \in R$ and $(b,c) \in R$, then $a \equiv b \pmod 3$ and $b \equiv c \pmod 3$, then $a \equiv c \pmod 3$, so $(a,c) \in R$;
R is an equivalence relation since it is reflexive, symmetric, and transitive.
The equivalence classes are $[0] = \{a \in \mathbb{Z} \mid a \equiv 0 \pmod 3\}$, $[1] = \{a \in \mathbb{Z} \mid a \equiv 1 \pmod 3\}$, and $[2] = \{a \in \mathbb{Z} \mid a \equiv 2 \pmod 3\}$.

(e) The relation R is not reflexive: e.g. $(1,1) \notin R$ since $1 \not> 1$;
R is not symmetric: e.g. $(2,1) \in R$ and $(1,2) \notin R$;
R is transitive since if $(a,b) \in R$ and $(b,c) \in R$, then $a > b$ and $b > c$, then $a > c$, so $(a,c) \in R$;
R is not an equivalence relation since R is not symmetric.

5. (a) The function f is not one-to-one: e.g. $5 \cdot 1^2 + 2 = 5(-1)^2 = 2$, but $1 \neq -1$;
f is not onto: e.g. there is no integer n such that $5n + 2 = 3$ since the only real solution of this equation is $n = \frac{1}{5}$ which is not an integer;
f is not bijective: e.g. it is not one-to-one;

(b) The function f is one-to-one since if $\frac{1}{x} = \frac{1}{y}$, then $x = y$;
f is not onto: e.g. there is no natural number n such that $\frac{1}{n} = 2$ since the only real solution of this equation is $n = \frac{1}{2}$ which is not a natural number;
f is not bijective since it is not onto.

(c) The function f is one-to-one: let $f(x) = f(y)$ where $x, y \in \mathbb{R}$. If $f(x) \neq 0$, then $x \neq 0$ and $y \neq 0$, so $\frac{1}{x} = \frac{1}{y}$. Therefore $x = y$.
The function f is onto: let $y \in \mathbb{R}$. If $y \neq 0$, let $x = \frac{1}{y}$. Then $f(x) = \frac{1}{1/y} = y$. If $y = 0$, then $f(0) = y$.
This function is bijective since it is both one-to-one and onto.

(d) The function f is not one-to-one: e.g. $f(1) = f(0)$ but $1 \neq 0$;
f is onto since it is a continuous function with $\lim_{x \to -\infty} = -\infty$ and $\lim_{x \to \infty} = \infty$;
f is not bijective since it is not one-to-one.