MATH 114 Homework 12 - Selected solutions

4.3 #30 Seven women and nine men are on the faculty in the mathematics department at a school.

(a) How many ways are there to select a committee of five members of the department if at least one woman must be on the committee?

There are 16 faculty members in the department, therefore $C(16,5)$ different combinations of 5 people are possible. However, $C(9,5)$ of them consist entirely of men. Therefore there are $C(16,5) - C(9,5) = \frac{16!}{5!11!} - \frac{9!}{5!4!}$ ways to select a committee with at least one woman.

(b) How many ways are there to select a committee of five members of the department if at least one woman and at least one man must be on the committee?

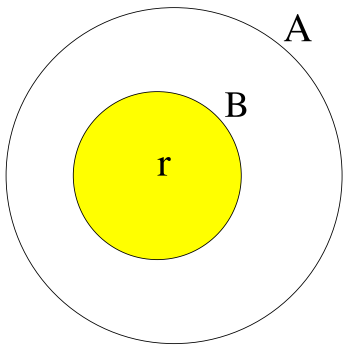
See part (a), but now we also have to subtract $C(7,5)$ (the number of combinations that consist entirely of women). So there are $C(16,5) - C(9,5) - C(7,5) = \frac{16!}{5!11!} - \frac{9!}{5!4!} - \frac{7!}{5!2!}$ ways to select a committee with at least one woman and at least one man.

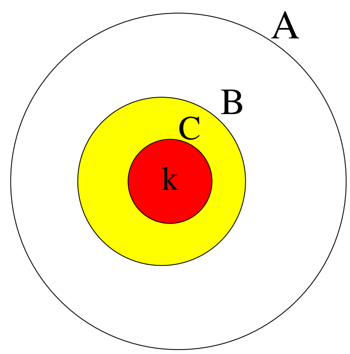
4.4 #24 Show that if p is a prime and k is an integer such that $1 \leq k \leq p-1$, then p divides $\binom{p}{k}$.

$\binom{p}{k} = \frac{p!}{k!(p-k)!} = \frac{1 \cdot 2 \cdot \ldots \cdot p}{1 \cdot 2 \cdot \ldots \cdot k \cdot 1 \cdot 2 \cdot \ldots (p-k)}$. The numerator and the denominator have prime factorizations. Notice that one of the primes in the numerator is p, and all the primes in the denominator are less than p because all the numbers in the product are less than p. We know that $\binom{p}{k}$ is an integer, so all the primes in the denominator must cancel with some primes in the numerator. Since there is no p in the denominator, p will stay in the numerator (there is no way to cancel it out). Therefore the obtained prime factorization of the integer $\binom{p}{k}$ will contain a p, so $\binom{p}{k}$ is divisible by p.

4.4 #22 (a) (optional) Prove the identity $\binom{n}{r}\binom{r}{k} = \binom{n}{k}\binom{n-k}{r-k}$, whenever n, r, and k are nonnegative integers with $r \le n$ and $k \le r$, using a combinatorial argument.

Let set A contain n elements. Then $\binom{n}{r}$ is the number of ways to choose a subset B of A containing r elements, and $\binom{r}{k}$ is the number of ways to choose a subset C of B containing k elements, so $\binom{n}{r}\binom{r}{k}$ is the number of ways to choose a pair (B, C) where B is a subset of A, C is a subset of B, $|B| = r$, and $|C| = k$.





Now we will show that the right hand side of the given identity counts the same thing. Namely, let's choose our subset C first, and then choose the rest of elements of B. There are $\binom{n}{k}$ ways to choose a subset C of A with k elements. The set B must contain C, so k elements of B (the ones that are in C) are already chosen. We have to choose the remaining $r-k$ elements of B out of the remaining $n-k$ elements of A. There are $\binom{n-k}{r-k}$ ways to do this.

Therefore, there are $\binom{n}{k}\binom{n-k}{r-k}$ ways to choose a pair (B, C) where B is a subset of A, C is a subset of B, $|B| = r$, and $|C| = k$.