MATH 114

Homework 7 - Solutions to selected problems

1.4, # 28. Determine the truth value of each of these statements if the universe of discourse of each variable consists of all real numbers.

(a) $\forall x \exists y (x^2 = y)$ True because for any x, we can choose $y = x^2$.

(b) $\forall x \exists y (x = y^2)$ False. Counterexample: if $x = -1$, there is no y such that $-1 = y^2$.

(c) $\exists x \forall y (xy = 0)$ True. Example: $x = 0$. Then for any y, $0 \cdot y = 0$.

(d) $\exists x \exists y (x + y \neq y + x)$ False because for all x and y, $x + y = y + x$ (commutativity law).

(e) $\forall x (x \neq 0 \to \exists y (xy = 1))$ True because any nonzero real number has a multiplicative inverse, namely, for any $x \neq 0$, we can choose $y = \frac{1}{x}$, then $xy = 1$.

(f) $\exists x \forall y (y \neq 0 \to xy = 1)$ False. Suppose there exists such an x. Then for $y = 2$ we have $x \cdot 2 = 1$, so $x = \frac{1}{2}$, and for $y = 3$ we have $x \cdot 3 = 1$, so $x = \frac{1}{3}$. But $\frac{1}{2} \neq \frac{1}{3}$. Contradiction.

(g) $\forall x \exists y (x + y = 1)$ True because for any x, we can choose $y = 1 - x$, and then $x + y = x + 1 - x = 1$.

(h) $\exists x \exists y (x + 2y = 2 \wedge 2x + 4y = 5)$ False because this system has no solutions: multiplying the first equation by 2 gives $2x + 4y = 4$, and subtracting this from the second equation gives $0 = 1$. Therefore the system is inconsistent.

(i) $\forall x \exists y (x + y = 2 \wedge 2x - y = 1)$ False. Counterexample: $x = 0$. Then the equations are $y = 2$ and $-y = 1$, or $y = 2$ and $y = -1$, but $2 \neq -1$. So for $x = 0$ there is no y that satisfies both equations.

(j) $\forall x \forall y \exists z (z = (x + y)/2)$ True because for any x and y, $z = (x + y)/2$ is a real number.

1.8, # 14. Determine whether $f : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ is onto if

(c) $f(m, n) = m + n + 1$**.** Yes because any integer y is in the image: $f(y - 1, 0) = y - 1 + 0 + 1 = y$.

(d) $f(m, n) = |m| - |n|$**.** Yes because any integer y is in the image: if $y \geq 0$ then $f(y, 0) = |y| - |0| = y - 0 = y$, and if $y < 0$ then $f(0, y) = |0| - |y| = 0 - (-y) = y$.

(e) $f(m, n) = m^2 - 4$**.** No because, for example, -5 is not in the image: there are no m and n such that $m^2 - 4 = -5$ (or $m^2 = -1$).

1.8, #36. Let f be a function from the set A to the set B. Let S be a subset of B. We define the inverse image of S to tbe the subset of A containing all pre-images of all elements of S. We denote the inverse image of S by $f^{-1}(S)$, so that $f^{-1}(S) = \{a \in A | f(a) \in S\}$.
Let S and T be subsets of B. Show that

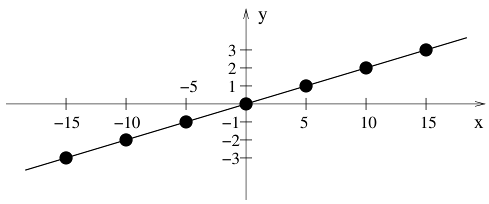
(a) $f^{-1}(S \cup T) = f^{-1}(S) \cup f^{-1}(T)$**.**
$f^{-1}(S \cup T) = \{x \in A \mid f(a) \in S \cup T\} = \{x \in A \mid f(a) \in S \vee f(a) \in T\}$
$= \{x \in A \mid f(a) \in S\} \cup \{x \in A \mid f(a) \in T\} = f^{-1}(S) \cup f^{-1}(T)$

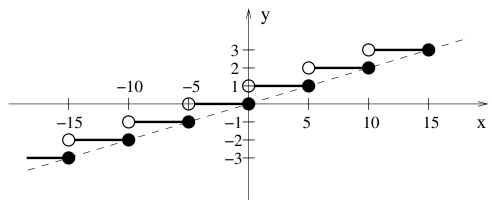
(a) $f^{-1}(S \cap T) = f^{-1}(S) \cap f^{-1}(T)$**.**
$f^{-1}(S \cap T) = \{x \in A \mid f(a) \in S \cap T\} = \{x \in A \mid f(a) \in S \wedge f(a) \in T\}$
$= \{x \in A \mid f(a) \in S\} \cap \{x \in A \mid f(a) \in T\} = f^{-1}(S) \cap f^{-1}(T)$

1.8, #60. Draw graphs of each of these functions.

(b) $f(x) = \lceil 0.2x \rceil$

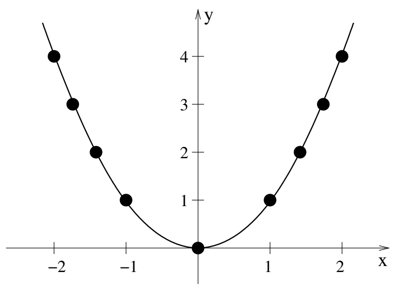
Draw the grapf of $y = 0.2x$. Mark all the points whose y-coordinate is an integer. These points belong to the graph $y = \lceil 0.2x \rceil$ too because for any integer number n, $\lceil n \rceil = n$. For all points whose y-coordinate is between two integers, say $n < y < n + 1$, $\lceil y \rceil = n + 1$. So you have to "lift" those points to the next integer value.

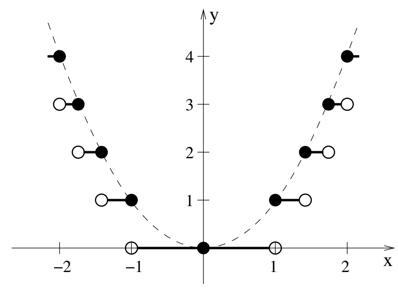




(d) $f(x) = \lfloor x^2 \rfloor$

Draw the grapf of $y = x^2$. Mark all the points whose y-coordinate is an integer. These points belong to the graph $y = \lfloor x^2 \rfloor$ too because for any integer number n, $\lfloor n \rfloor = n$. For all points whose y-coordinate is between two integers, say $n < y < n+1$, $\lfloor y \rfloor = n$. So you have to "lower" those points to the next integer value.





2.6, # 20. Find all solutions, if any, to the system of congruences.

$$\begin{cases} x \equiv 5 \pmod{6} \\ x \equiv 3 \pmod{10} \\ x \equiv 8 \pmod{15} \end{cases}$$

Since 6, 10, and 15 are not pairwise relatively prime, we can't use the Chinese Remainde Theorem (CRT). However, this does not mean that the system has no solution.

By CRT, $x \equiv 5 \pmod{6}$ is equivalent to the system $\begin{cases} x \equiv 5 \pmod{2} \\ x \equiv 5 \pmod{3} \end{cases}$

$x \equiv 3 \pmod{10}$ is equivalent to the system $\begin{cases} x \equiv 3 \pmod{2} \\ x \equiv 3 \pmod{5} \end{cases}$

$x \equiv 8 \pmod{15}$ is equivalent to the system $\begin{cases} x \equiv 8 \pmod{3} \\ x \equiv 8 \pmod{5} \end{cases}$

Therefore the original system is equivalent to $\begin{cases} x \equiv 5 \pmod{2} \\ x \equiv 5 \pmod{3} \\ x \equiv 3 \pmod{2} \\ x \equiv 3 \pmod{5} \\ x \equiv 8 \pmod{3} \\ x \equiv 8 \pmod{5} \end{cases}$

Here, the congruences $x \equiv 5 \pmod{2}$ and $x \equiv 3 \pmod{2}$ are equivalent since $3 \equiv 5 \pmod{2}$. The congruences $x \equiv 5 \pmod{3}$ and $x \equiv 8 \pmod{3}$ are equivalent. Also, the congruences $x \equiv 3 \pmod{5}$ and $x \equiv 8 \pmod{5}$ are equivalent.

Therefore the system is equivalent to $\begin{cases} x \equiv 5 \pmod{2} \\ x \equiv 5 \pmod{3} \\ x \equiv 3 \pmod{5} \end{cases}$

Now we can use CRT. Using the notations in the book, we have
$M = 30$, $M_1 = 15$, $M_2 = 10$, $M_3 = 6$. Then we need to solve: $15y_1 \equiv 1 \pmod{2}$, $10y_2 \equiv 1 \pmod{3}$, and $6y_3 \equiv 1 \pmod{5}$. By guessing (since these all are small numbers) or using the Euclidean algorithm, we find $y_1 = 1$, $y_2 = 1$, and $y_3 = 1$ satisfy these congruences.

Therefore $x \equiv 5 \cdot 15 \cdot 1 + 5 \cdot 10 \cdot 1 + 3 \cdot 6 \cdot 1 = 143 \equiv 23 \pmod{30}$.