Practice Final - Solutions

1. (a) Prove that among 11 integer numbers, there are two numbers $a < b$ such that the difference $b - a$ ends with 0 (i.e. has the units digit 0).

   Since there are only 10 different digits, by Dirichlet's principle, among 11 integers there are at least two with the same last digit. Their difference ends with 0.

   (b) Is the above statement true for the tens digit?

   No. Example: 11, 110, 209, 308, 407, 506, 605, 704, 803, 902, 1001, 1100. These numbers are of the form $100k + (11 - k)$. *The difference of two such numbers is* $(100k + 11 - k) - (100n + 11 - n) = 100(k - n) - (k - n)$. *Since* $k - n < 12$, *the tens digit of the difference is* 8 *or* 9.

2. Let $F_1 = F_2 = 1$ and $F_{n+1} = F_n + F_{n-1}$. Prove that

   (a) (straightforward) $F_1F_2 + F_2F_3 + \ldots + F_{2n-1}F_{2n} = F_{2n}^2$

   Proof by induction.
   If $n = 1$, *then we have* $F_1F_2 = F_2^2$ *which means* $1 \cdot 1 = 1^2$ *which is true.*
   Assume the equality holds for $n = k$. *We want to prove that it holds for* $n = k + 1$.
   $F_1F_2 + F_2F_3 + \ldots + F_{2(k+1)-1}F_{2(k+1)} = F_1F_2 + F_2F_3 + \ldots + F_{2k-1}F_{2k} + F_{2k}F_{2k+1} + F_{2k+1}F_{2k+2} = F_{2k}^2 + F_{2k}F_{2k+1} + F_{2k+1}F_{2k+2} = F_{2k}(F_{2k} + F_{2k+1}) + F_{2k+1}F_{2k+2} = F_{2k}F_{2k+2} + F_{2k+1}F_{2k+2} = (F_{2k} + F_{2k+1})F_{2k+2} = F_{2k+2}^2 = F_{2(k+1)}^2$.

   (b) (a bit harder) $F_{n-1}^2 + F_n^2 = F_{2n-1}$

   Again, the proof is by induction.
   If $n = 1$, *then* F_0 *is undefined. So we start with* $n = 2$. *In this case we have* $F_1^2 + F_2^2 = F_3^2$ *which means* $1^2 + 1^2 = 2$ *which is true.*
   Now assume that it holds for all $n \le k$. *We want to prove that it holds for* $n = k+1$. *It is easier here to work from the right hand side.*
   $F_{2(k+1)-1} = F_{2k+1} = F_{2k} + F_{2k-1} = F_{2k-1} + F_{2k-2} + F_{2k-1} = 2F_{2k-1} + F_{2k-2} = 2F_{2k-1} + F_{2k-1} - F_{2k-3} = 3F_{2k-1} - F_{2k-3} = 3(F_{k-1}^2 + F_k^2) - (F_{(k-1)-1}^2 + F_{k-1}^2) = 3F_{k-1}^2 + 3F_k^2 - F_{k-2}^2 - F_{k-1}^2 = 2F_{k-1}^2 + 3F_k^2 - F_{k-2}^2 = 2F_{k-1}^2 + 3F_k^2 - (F_k - F_{k-1})^2 = 2F_{k-1}^2 + 3F_k^2 - F_k^2 + 2F_kF_{k-1} - F_{k-1}^2 = F_{k-1}^2 + 2F_k^2 + 2F_kF_{k-1} = F_{k-1}(F_{k-1} + F_k) + F_k(F_k + F_{k-1}) + F_k^2 = F_{k-1}F_{k+1} + F_kF_{k+1} + F_k^2 = (F_{k-1} + F_k)F_{k+1} + F_k^2 = F_k^2 + F_{k+1}^2 = F_{(k+1)-1}^2 + F_{k+1}^2$.
   Note: the idea of the above inductive step is the following: express F_{2k+1} *in terms of* F_i*'s with* i *odd and less than* $2k + 1$, *e.g. in terms of* F_{2k-1} *and* F_{2k-3}, *then use the inductive hypothesis to rewrite* F_{2k-1} *and* F_{2k-3} *as sums of squares (since we assume that the formula holds for smaller indices), and then rewrite the obtained expression in terms of* F_k *and* F_{k+1} *(because the formula we want to prove involves these terms).*

3. Prove that for any integer number n, $n^7 - n$ is divisible by 7.

   Idea 1 (straightforward). Consider all possible remainders of n mod 7. Calculate the remainder of $n^7 - n$. You'll get 0 in each case. Thus $n^7 - n$ is divisible by 7.

   Proof 2. $n^7 - n = n(n^6 - 1) = n(n^3 - 1)(n^3 + 1) = n(n-1)(n^2+n+1)(n+1)(n^2-n+1) \equiv n(n-1)(n^2+n-6)(n+1)(n^2-n-6) = n(n-1)(n+3)(n-2)(n+1)(n-3)(n+2) = n(n-1)(n-2)(n-3)(n+3)(n+2)(n+1) \equiv n(n-1)(n-2)(n-3)(n-4)(n-5)(n-6) \pmod 7$. *Now, n has remainder 0 or 1 or ... or 6 mod 7. We see that in each case the product on the right is divisible by 7.*

   Proof 3. By induction.
   If $n = 1$, then $n^7 - n = 0$ is divisible by 7.
   Suppose $k^7 - k$ is divisible by 7. Then $(k+1)^7 - (k+1) = k^7 + 7k^6 + 21k^5 + 35k^4 + 35k^3 + 21k^2 + 7k + 1 - k - 1 = (k^7 - k) + 7k^6 + 21k^5 + 35k^4 + 35k^3 + 21k^2 + 7k$ is divisible by 7.

4. Explain the trick on the next page (see second page of the practice final).

   Let the numbers be $10a + b$ and $10a + (10 - b)$. Then $(10a + b)(10a + (10 - b)) = 100a^2 + 10ab + 10a(10 - b) + b(10 - b) = 100a^2 + 10ab + 100a - 10ab + b(10 - b) = 100a(a + 1) + b(10 - b)$.

5. What are the last two digits of 7^{50}?

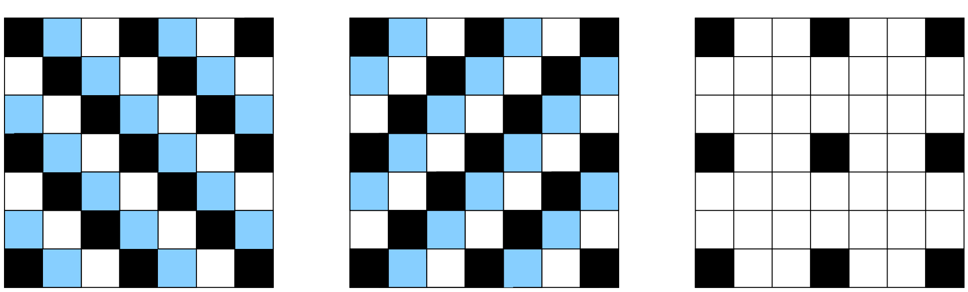
   $7^2 = 49$, $7^3 = 343$ ends with 43, 7^4 ends with 01. Therefore for any n, 7^{4n} ends with 01. Then $7^{50} = 7^{48} \cdot 49$ ends with 49.

6. The numbers from 0 to 9 are written along a circle in random order. Between every 2 neighboring numbers a and b we write $2b - a$. Then we erase the original numbers. This step is repeated. Show that it is not possible to reach ten 5's. (For example, the numbers could be written in the following order: 1, 5, 3, 9, 0, 2, 4, 6, 8, 7. Then the new number would be 9, 1, 15, −9, 4, 6, 8, 10, 6, −5.)

   The sum of the numbers does not change since a, b, c, d, ... are replaced by $2b - a$, $2c - b$, $2d - c$, ... The sum of the original numbers is 45. But the sum of ten 5's is 50. Therefore it is not possible to reach ten 5's.

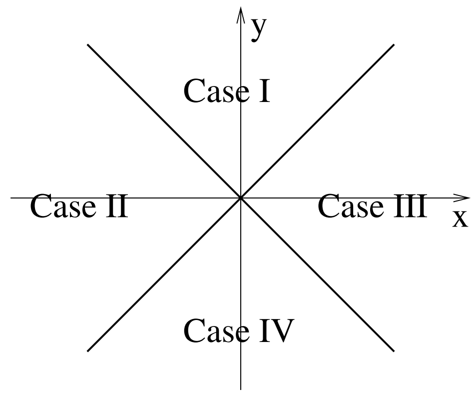
7. A 7×7 square is covered by sixteen 3×1 and one 1×1 tiles. What are the permissible positions of the 1×1 tile?

   Color the board diagonally with 3 colors. You'll get 16 squares of one color, 16 squares of another color, and 17 squares of the third color (these are shown in black below). Since each 3×1 tile covers one square of each color, the 1×1 tile must cover one of the 17 squares of the third color. Now, notice that we could color the board diagonally in the other direction. The 1×1 tile must cover one of the 17 squares for the second coloring (black again). Therefore it must be in the intersection of the two sets. The intersection consists of 9 squares (4 corners, 4 midpoints of edges, and the center). In each case, it is easy to find a covering by sixteen 3×1 and one 1×1 tiles. Thus the set of permissible positions of the 1×1 tile consists of those 9 squares.

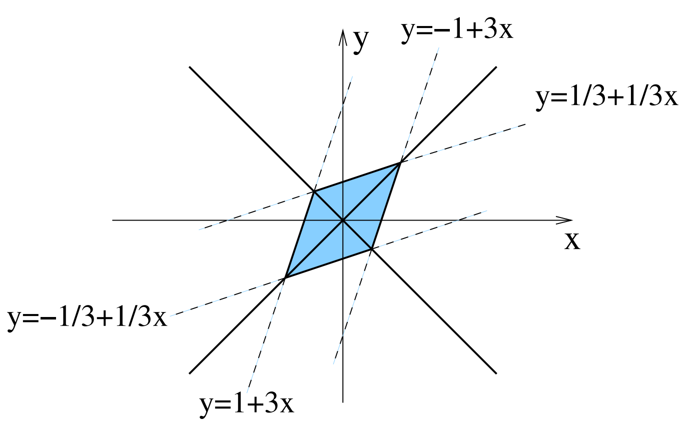


8. Sketch the region $\{(x, y) \mid 2|y-x| + |y+x| \leq 1\}$.

Consider 4 cases: $y - x$ positive or negative, and $x + y$ positive or negative. In each case, get rid of the absolute value, and solve for y.



Case I. $y - x \geq 0$, $x + y \geq 0$.
$2y - 2x + y + x \leq 1$
$3y \leq 1 + x$
$y \leq \frac{1}{3} + \frac{1}{3}x$
Case II. $y - x \geq 0$, $x + y < 0$.
$2y - 2x - y - x \leq 1$
$y \leq 1 + 3x$
Case III. $y - x < 0$, $x + y \geq 0$.
$-2y + 2x + y + x \leq 1$
$-y \leq 1 - 3x$
$y \geq 3x - 1$
Case IV. $y - x < 0$, $x + y < 0$.
$-2y + 2x - y - x \leq 1$
$-3y \leq 1 - x$
$y \geq -\frac{1}{3} + \frac{1}{3}x$
Now sketch the region in each case:



9. A connected bipartite graph G has 8 vertices. Recall that the vertices of a bipartite graph can be divided into 2 groups A and B so that every edge connects a vertex in group A and a vertex in group B. Both groups for G have 4 vertices. Three of the vertices in group A have degrees 4, 2, and 2. Three of the vertices in B have degrees 3, 1, and 1. What are the degrees of the remaining vertices?

   Let the degrees of the remaining vertices be a (in group A) and b (in group B). The sum of degrees of vertices in the first group must be equal to the sum of degrees of vertices in the second group. Thus $4 + 2 + 2 + a = 3 + 1 + 1 + b$*, or* $3 + a = b$*. Since the graph is connected, the degree of each vertex is at least 1. Thus* $a \geq 1$*. Now, it is easy to see that for every pair* a*,* b *satisfying* $3 + a = b$ *and* $a \geq 1$*, there exists a graph with vertices of such degrees. (Because we can have multiple edges between the vertices of degrees a and b.) Draw a few such graphs!*

10. Two players play the following game.

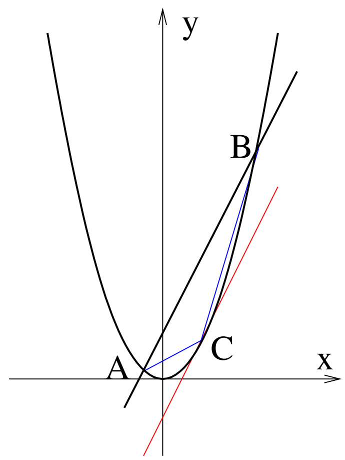
    - Turns alternate.
    - At each turn, a player removes 1, 2, 4, 8, 16, or 32 counters from a pile that had initially 50 counters.
    - The game ends when all counters have been removed.
    - The player who takes the last counter wins.

    Find a winning strategy for one of the players.

    We will work backwards. We want to take the last counter. How many counters should we leave on our next to last turn so that our opponent cannot take the last counter? We can't leave 1 because they will take it. We can't leave 2 either, since they can take 2. But we can leave 3. Then they can take either 1 or 2, that will leave 2 or 1 (respectively), and we will take them. So, we must leave 3 counters.
    How many should we leave on the turn before that so that our opponent cannot take all or leave 3? We can't leave 4 (they may take all of them). We can't leave 5 (they may take 2 and leave 3). How about 6? Let's see what choices our opponent has then. If they take 1 and leave 5, that's good - we'll take 2 then. If they take 2 and leave 4, also good for us - we'll take 1. They can also take 4 and leave 2. That's good too, we will then take the last 2. So, we must leave 6.
    The turn before that: we don't want to give our opponent an opportunity to leave 0 or 3 or 6. 7 is bad (they may take 1 and leave 6), 8 is bad (they may take 2 and leave 6). How about 9? Our opponent may take 1 (and leave 8) or take 2 (and leave 7) or take 4 (and leave 5) or take 8 (and leave 1). In each case, we'll be able to leave 6 or 3 or 0. So, we must leave 9.
    Now notice that the numbers 3, 6, 9 are multiples 3. Does this mean that leaving multiples of 3 is a winning strategy? Let's see... Suppose we leave a multiple of 3. Our opponent will take a power of 2. Since a power of 2 is not divisible by 3, they will leave a number not divisible by 3. Then we can take the remainder, and leave a multiple of 3 again. Thus we have to go first, take 2 counters, and leave 48 (or take 8 and leave 42, or take 32 and leave 18). Then, each time we'll be able to leave a multiple of 3. Thus sooner or later we'll leave 0, and we'll win.

11. The parabola $y = x^2$ and the line $y = mx + 1$ are given. They have two intersection points, A and B. Find the point C on the parabola that maximizes the area of $\triangle ABC$.

Since A and B are given, the length of AB is given. Now, to maximize the area of $\triangle ABC$, we have to maximize the height h_c. To do this, the point C must lie on the tangent line parallel to the given line. Thus the slope of the parabola at C must be equal to m. Then the x-coordinate of C is $\frac{m}{2}$ (since the slope is $2x$). The y-coordinate of C is then $\frac{m^2}{4}$.



12. Find a cubic polynomial $p(x) = ax^3 + bx^2 + cx + d$ that has a local maximum at $(0, 1)$ and a local minimum at $(1, 0)$.

We need the polynomial to pass through the given points, and have slope (which is $p'(x) = 3ax^2 + 2bx + c$) equal to 0 at both points.
The value at 0: $d = 1$.
The value at 1: $a + b + c + d = 0$.
The slope at 0: $c = 0$.
The slope at 1: $3a + 2b + c = 0$.
Since $d = 1$ and $c = 0$, the second and fourth equations become $a+b = -1$ and $3a+2b = 0$. Then $b = -\frac{3}{2}a$, and $a - \frac{3}{2}a = -1$. This gives $a = 2$. Then $b = -3$.
$p(x) = 2x^3 - 3x^2 + 1$.

13. Evaluate the integral $\int_0^{3\pi} \sin|x|dx$.

x is positive on the given interval, so $|x| = x$, and $\int_0^{3\pi} \sin|x|dx = \int_0^{3\pi} \sin x dx = \cos x \Big|_0^{3\pi} = -\cos(3\pi) + \cos 0 = -(-1) + 1 = 2$.