Math 145 **Fall 2003**

Practice Final

On the actual exam, you will be given 8 problems, and you will have to choose any 6. You will have 2 hours to complete the exam.

1. (a) Prove that among 11 integer numbers, there are two numbers $a < b$ such that the difference $b - a$ ends with 0 (i.e. has the units digit 0).

   (b) Is the above statement true for the tens digit?

2. Let $F_1 = F_2 = 1$ and $F_{n+1} = F_n + F_{n-1}$. Prove that

   (a) (straightforward) $F_1F_2 + F_2F_3 + \ldots + F_{2n-1}F_{2n} = F_{2n}^2$

   (b) (a bit harder) $F_{n-1}^2 + F_n^2 = F_{2n-1}$

3. Prove that for any integer number n, $n^7 - n$ is divisible by 7.

4. Explain the trick on the next page.

5. What are the last two digits of 7^{50}?

6. The numbers from 0 to 9 are written along a circle in random order. Between every 2 neighboring numbers a and b we write $2b - a$. Then we erase the original numbers. This step is repeated. Show that it is not possible to reach ten 5's. (For example, the numbers could be written in the following order: 1, 5, 3, 9, 0, 2, 4, 6, 8, 7. Then the new number would be 9, 1, 15, -9, 4, 6, 8, 10, 6, -5.)

7. A 7×7 square is covered by sixteen 3×1 and one 1×1 tiles. What are the permissible positions of the 1×1 tile?

8. Sketch the region $\{(x, y) \mid 2|y - x| + |y + x| \leq 1\}$.

9. A connected bipartite graph G has 8 vertices. Recall that the vertices of a bipartite graph can be divided into 2 groups A and B so that every edge connects a vertex in group A and a vertex in group B. Both groups for G have 4 vertices. Three of the vertices in group A have degrees 4, 2, and 2. Three of the vertices in B have degrees 3, 1, and 1. What are the degrees of the remaining vertices?

10. Two players play the following game.

    - Turns alternate.
    - At each turn, a player removes 1, 2, 4, 8, 16, or 32 counters from a pile that had initially 50 counters.
    - The game ends when all counters have been removed.
    - The player who takes the last counter wins.

    Find a winning strategy for one of the players.

11. The parabola $y = x^2$ and the line $y = mx+1$ are given. They have two intersection points, A and B. Find the point C on the parabola that maximizes the area of $\triangle ABC$.

12. Find a cubic polynomial $p(x) = ax^3 + bx^2 + cx + d$ that has a local maximum at $(0, 1)$ and a local minimum at $(1, 0)$.

13. Evaluate the integral $\displaystyle\int_0^{3\pi} \sin|x|dx$.