Math 145 **Fall 2003**

Reveiw Problems - Solutions

1. Find $\frac{1}{2!}+\frac{2}{3!}+\frac{3}{4!}+\ldots+\frac{n-1}{n!}$.

Solution 1 (straightforward). Calculate the value of the expression for some small values of n, and notice the pattern.
The expression only makes sense for $n \geq 2$.
If $n=2$, then $\frac{1}{2!}=\frac{1}{2}$.

If $n=3$, then $\frac{1}{2!}+\frac{2}{3!}=\frac{5}{6}$.

If $n=4$, then $\frac{1}{2!}+\frac{2}{3!}+\frac{3}{4!}=\frac{23}{24}$.

It appears that $\frac{1}{2!}+\frac{2}{3!}+\frac{3}{4!}+\ldots+\frac{n-1}{n!}=\frac{n!-1}{n!}$.
Proof by induction.
The basis step (for $n=2$) is verified above.
Suppose the formula holds for $n=k$. We want to prove that it holds for $n=k+1$.

$$\frac{1}{2!}+\frac{2}{3!}+\frac{3}{4!}+\ldots+\frac{n-1}{n!}+\frac{n}{(n+1)!}=$$

$$\frac{n!-1}{n!}+\frac{n}{(n+1)!}=\frac{(n!-1)(n+1)}{(n+1)!}+\frac{n}{(n+1)!}=\frac{(n+1)!-n-1+n}{(n+1)!}=\frac{(n+1)!-1}{(n+1)!}.$$

Solution 2 (short by non-obvious). $\frac{1}{2!}+\frac{2}{3!}+\frac{3}{4!}+\ldots+\frac{n-1}{n!}=$

$$\frac{2-1}{2!}+\frac{3-1}{3!}+\frac{4-1}{4!}+\ldots+\frac{n-1}{n!}=\frac{2}{2!}-\frac{1}{2!}+\frac{3}{3!}-\frac{1}{3!}+\frac{4}{4!}\frac{1}{4!}+\ldots+\frac{n}{n!}-\frac{1}{n!}=$$

$$\frac{1}{1!}-\frac{1}{2!}+\frac{1}{2!}-\frac{1}{3!}+\frac{1}{3!}\frac{1}{4!}+\ldots+\frac{1}{(n-1)!}-\frac{1}{n!}=\frac{1}{1!}-\frac{1}{n!}=\frac{n!-1}{n!}$$

2. Prove that if $p > 3$ is prime, then $p^2 \equiv 1 \pmod{24}$.

   Solution 1 (straightforward) Consider all possible remainders mod 24, and eliminate the ones which are not possible for a prime number. Namely, eliminate numbers of the form $24n$, $24n+2$, $24n+3$, $24n+4$, $24n+6$, $24n+8$, $24n+9$, $24n+10$, $24n+12$, $24n+14$, $24n+15$, $24n+16$, $24n+18$, $24n+20$, $24n+21$, and $24n+22$ because all of these are either even or divisible by 3. We are left with the following possibilities: $24n+1$, $24n+5$, $24n+7$, $24n+11$, $24n+13$, $24n+17$, $24n+19$, $24n+23$. The remainder mod 24 of the square of such a number, $(24n+r)^2 = 24^2n^2 + 48nr + r^2$, is the same as the remainder of r^2 mod 24. But $1^1 = 1$, $5^2 = 25 = 24+1$, $7^2 = 49 = 24 \cdot 2 + 1$, $13^2 = 169 = 24 \cdot 7 + 1$, $17^2 = 289 = 24 \cdot 12 + 1$, $19^2 = 361 = 24 \cdot 15 + 1$, and $23^2 = 529 = 24 \cdot 22 + 1$ all have remainder 1 mod 24.

   Solution 2 (a bit shorter). It is actually sufficient to consider all possible remainders mod 12, since $(12n+r)^2 = 144n^2 + 24nr + r^2 \equiv r^2 \pmod{24}$. The proof is as above.

3. There are 8 people in a room. Every person counted how many people he knows. (Assume that if A knows B then B knows A.)

   (a) The numbers are 0, 1, 1, 2, 2, 3, 4, 4. Prove that somebody made a mistake.

   If these numbers were possible, we would be able to represent these people by a graph as follows. Let every person be represented by a vertex, and two vertices are connected if and only if the corresponding people know each other. Then the degree of each vertex is the number of people that person knows. The number of vertices of odd degrees must be even (or, equivalently, the sum of all degrees is even). But there are 3 odd numbers here (two 1's and one 3), so there is no such graph. Contradiction.

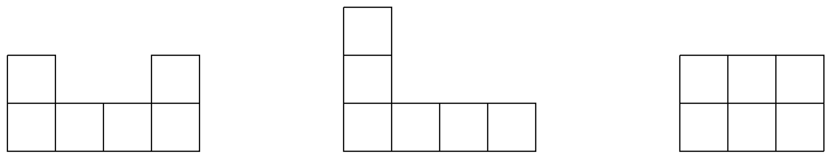
   (b) Can the numbers be 0, 1, 2, 3, 4, 5, 6, 7?

   No, because of 0 and 7: it is impossible that one person doesn't know anybody, and another person knows everybody.

4. We strike the first digit of the number 7^{2003}, and add it to the remaining number. This is repeated until a number with 10 digits remains. Prove that this number has 2 equal digit.

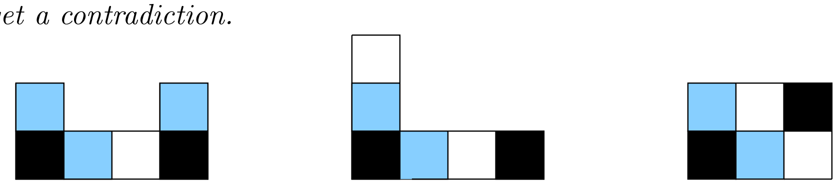
   This operation does not change the number mod 9. Namely, suppose that the first digit of the number is a. Then the number can be written as $10^n a + b$ (where b is the rest of the number). We replace it by $a+b$. The difference is $10^n a - a = (10^n - 1)a$ which is divisible by 9. The original number is a power of 7, hence not divisibly by 9. Thus, the resulting number with 10 digits is not divisibly by 9. Therefore the sum of its digits is not divisible by 9. But if all 10 digits are distinct, then their sum is 45 which is divisible by 9. Hence some digit repeats.

5. Show that it is not possible to cover any rectangle by one tile of type 1 shown below, one tile of type 2, and any number of tiles of type 3.



1 2 3

The first thing we try is the chessboard coloring. But each tile then covers 3 black and 3 white squares, so this will not give us a contradiction. We have to find a coloring so that for at least one color, different types of tiles cover different numbers of squares of that color. Try the diagonal coloring with 3 colors. Then a tile of type 1 covers 3 squares of one color, 2 squares of another color, and only 1 square of the third color. While tiles of types 2 and 3 cover 2 squares of each color. So suppose such tiles cover a rectangle. Each tile covers 6 squares, the area of the rectangle is divisible by 6, and hence divisible by 3. Then at least one its dimensions (say, length) is divisible by 3. Then if we color the board diagonally by 3 colors, each row will have the same number of squares of each color. So the number of squares of each color in the whole rectangle is the same. As we said, tiles of type 2 and 3 cover the same number of squares of each color. Since the only tile of type 1 covers 3 squares of one color and only 1 square of another color, we get a contradiction.



1 2 3

Note: We actually proved a stronger statement since we did not use here that there is only one tile of type 2. We could have any number of them. There may be another coloring for which you do have to use the fact that there is only one tile of type 2.

6. Let S be a set of 25 points such that, in any 3-subset of S, there are at least 2 points with distance less than 1. Prove that there exists a 13-subset of S which can be covered by a disk of radius 1.

Pick any point A. If any other point is within 1 unit of A, then all the points are covered by the disk with center at A and radius 1, and we are done. Now suppose there is a point B such that $|AB| > 1$. Then for any other point X in the set, either $|AX| < 1$ or $|BX| < 1$. Thus X is covered either by the disk with center at A and radius 1 or by the disk with center at B and radius 1. Thus every of the 25 points is covered by at least one of the 2 disks. By Dirichlet's principle, there is a disk that covers at least 13 points.