PROBLEM SOLVING

(Rough Draft)

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Chapter 1

Introduction

Solving mathematical problems is an art. It is impossible to learn how to solve every single problem... there are infinitely many of them...

Below are some problems.

1. Eleven children contributed money to buy a present for their classmate. The total amount of money collected was $30.00. Show that at least one child gave at least $2.73.

2. (a) Prove that any two-digit number is divisible by 3 if and only if the sum of its digits is divisible by 3.

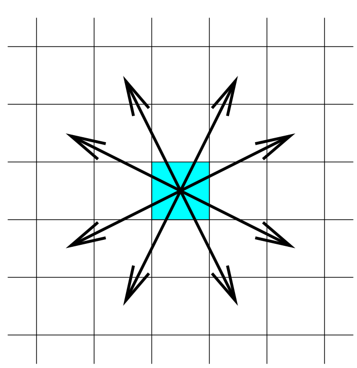
   (b) Prove that any natural number is divisible by 3 if and only if the sum of its digits is divisible by 3.

3. Is it true or false that for any natural number n, the number $n^2 + n + 41$ is prime?

4. In a 4×4 table six cells are marked with an * and all others are blank. Show that it is possible to cross out 2 columns and 2 rows so that the remaining cells are blank.

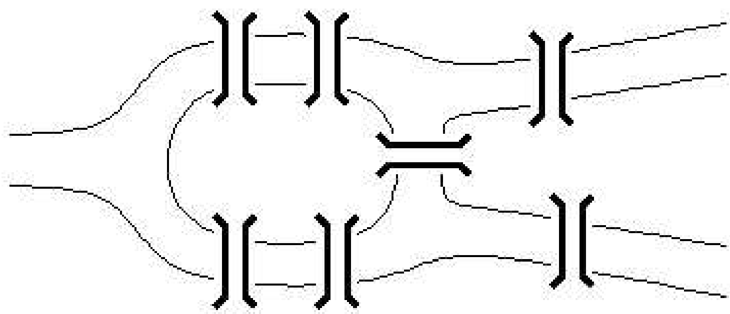
5. Is it true or false that for any natural number n, the number $n^3 + 2n$ is divisible by 3?

6. In chess, is it possible for a knight to start at the upper left corner and go through every square on the 8×8 chessboard exactly once? (A knight's move is 2 squares up, down, or to the right or left, and 1 square in a perpendicular direction. All allowed moves from a certain square are shown below:)



7. Sketch the graph of $f(x) = |x + 2| + |2x - 5|$.

8. Below is a plan of Konigsberg. Is it possible to design a tour of the town that crosses each of the bridges exactly once?



As said above, learning to solve problems is in part difficult because problems can be very different. However, there are a few basic principles that are good to know. There are a few approaches and methods that can be useful. In this book, we'll study some of them. After you study the material of this book you should be able to solve many problems pretty easily.

While using intuition and working out a few examples may help us find an idea, it is also important to write rigorous proofs. Since intuition is not always correct, we need to justify each step in a solution. We will therefore try to avoid words such as 'obviously'.

In each chapter, we provide basic definitions and facts to get you started. We do not prove the facts in this book, since our main goal is to learn how to solve problems, i.e. use these facts.

Chapter 2

Logic

Definition 2.1. A **proposition** is a statement that is either true of false.

For example, "3 plus 2 is 5" is a true proposition, "3 times 2 is 7" is a false proposition, while "x minus 4 is 8" is not a proposition because the value of x has not been defined.

Definition 2.2. Let p and q be propositions. Then:

- The **negation** of p, denoted by $\neg p$, is the proposition "not p".
- The **conjunction** of p and q, denoted by $p \wedge q$, is the proposition "p and q".
- The **disjunction** of p and q, denoted by $p \vee q$, is the proposition "p or q".
- The **exclusive or** of p and q, denoted by $p \oplus q$, is the proposition "either p or q but not both".
- The **implication** of p and q, denoted by $p \to q$, is the proposition that is false when p is true and q is false and true otherwise.
- The **biconditional** of p and q, denoted by $p \leftrightarrow q$, is the proposition that is true when p and q have the same truth values and is false otherwise.

Below is the so-called truth table that shows the truth values of the compound propositions defined above depending on the truth values of p and q.

p	q	$\neg p$	$p \wedge q$	$p \vee q$	$p \oplus q$	$p \to q$	$p \leftrightarrow q$
T	T	F	T	T	F	T	T
T	F	F	F	T	T	F	F
F	T	T	F	T	T	T	F
F	F	T	F	F	F	T	T

- A compound proposition that is always true, no matter what the truth values of the propositions that occur in it, is called a **tautology**.

  For example, $p \vee \neg p$ is a tautology.
- A compound proposition that is always false is called a **contradiction**.

  For example, $p \wedge \neg p$ is a contradiction.
- The propositions p and q are called **logically equivalent** if $p \leftrightarrow q$ is a tautology. The notation $p \Leftrightarrow q$ denotes that p and q are logically equivalent.

Example 2.3. Show that $\neg(p \vee q)$ and $(\neg p) \wedge (\neg q)$ are logically equivalent, i.e. "not (p or q) " is the same as "(not p) and (not q)".

Solution. *Construct the truth table:*

p	q	$p \vee q$	$\neg(p \vee q)$	$\neg p$	$\neg q$	$(\neg p) \wedge (\neg q)$
T	T	T	F	F	F	F
T	F	T	F	F	T	F
F	T	T	F	T	F	F
F	F	F	T	T	T	T

We see that the truth values of $\neg(p \vee q)$ and $(\neg p) \wedge (\neg q)$ are always the same, therefore the propositions are logically equivalent.

Definition 2.4. A statement $P(x)$ that depends on the value of a variable (x in this case) is called a **propositional function**. Once a value has been assigned to the variable x, the statement $P(x)$ becomes a proposition and has a truth value.

For example, if $P(x)$ is the statement "$x > 3$", then $P(4)$ is true and $P(2)$ is false.

- $\forall x P(x)$ means "for every x, $P(x)$ is true".
- $\exists x P(x)$ means "there exists x such that $P(x)$ is true".
- $\exists! x P(x)$ means "there exists a unique x such that $P(x)$ is true".

The symbols $\forall$ and $\exists$ are called quantifiers.

Propositional functions can be functions of two or more variables, and then we can use two or more quantifiers with them. It is important to realize that the order of quantifiers makes a difference. For example, below we will use the propositional function $F(x, y)$ which means that x and y are friends (the domain of this function can be a set of people). Then e.g. $\forall x \exists y F(x, y)$ means that everybody has at least friend, while $\exists y \forall x F(x, y)$ means that there is a person who is friends with everybody.

Propositions with negations can always be written so that negations appear only within predicates (that is, so that no negation is outside a quantifier or an expression involving logical connectives), for example:

- $\neg(p \wedge q) \Leftrightarrow \neg p \vee \neg q$
- $\neg(p \vee q) \Leftrightarrow \neg p \wedge \neg q$
- $\neg \forall x P(x) \Leftrightarrow \exists x \neg P(x)$
- $\neg \exists x P(x) \Leftrightarrow \forall x \neg P(x)$

Problems

1. Show that the following propositions are logically equivalent.

   (a) $p \to q$ and $\neg q \to \neg p$.

   (b) $p \to q$ and $\neg p \vee q$.

   (c) $p \vee (q \wedge r)$ and $(p \vee q) \wedge (p \vee r)$.

2. Let $P(x)$ denote the proposition "$(x < 3) \vee (x > 5)$". Determine the truth values of the following propositions (where the domain of x is the set of real numbers).
   (a) $P(2)$
   (b) $P(4)$
   (c) $P(2) \wedge P(4)$
   (d) $\forall x P(x)$
   (e) $\exists x P(x)$
   (f) $\exists! x P(x)$
   (g) $\forall x P(x) \vee P(-x)$

3. Translate the statement

$$\forall x(C(x) \vee \exists y(C(y) \wedge F(x,y)))$$

   into English, where $C(x)$ is "x has a computer", $F(x,y)$ is "x and y are friends", and the domain for both x and y is the set of all students at your university.

4. Let $F(x,y)$ be statement "x can fool y", where the domain for both variables is the set of all people in the world. Use quantifiers to express each of the following statements:
   (a) Everybody can fool Fred.
   (b) Mike can fool everybody.
   (c) Everybody can fool somebody.
   (d) There is no one who can fool everybody.
   (e) Everyone can be fooled by somebody.
   (f) No one can fool both Fred and Jerry.
   (g) Nancy can fool exactly two people.
   (h) There is exactly one person whom everybody can fool.
   (i) No one can fool himself or herself.
   (j) There is someone who can fool exactly one person besides himself or herself.

5. Let $P(x,y)$ denote the proposition "$x < y$" where x and y are real numbers. Determine the truth values of the following propositions (where the domain for both variables is the set of real numbers).
   (a) $\exists x \exists y P(x,y)$,
   (b) $\forall x \exists y P(x,y)$,
   (c) $\exists x \forall y P(x,y)$,
   (d) $\forall x \forall y P(x,y)$,
   (e) $\forall x P(-x,x)$.

6. Let $Q(x,y)$ denote "$x+y=0$". What are the truth values of the statements $\exists y \forall x Q(x,y)$ and $\forall x \exists y Q(x,y)$?

7. Let $Q(x,y)$ be the statement "$x+y = x-y$", and the domain for both variables is the set of integers. Find the truth values of the following statements. Explain.

(a) $Q(2,0)$

(b) $\forall y Q(1,y)$

(c) $\forall x \exists y Q(x,y)$

(d) $\forall y \exists x Q(x,y)$

(e) $\exists y \forall x Q(x,y)$

8. Express the definition of the limit $\lim_{x \to a} f(x) = L$ using quantifiers.

9. Express the definition of a convergent sequence $a_1, a_2, \ldots$ using quantifiers.

10. Rewrite each of the following statements so that negations appear only within predicates.

    (a) $\neg \forall x \forall y P(x,y)$

    (b) $\neg \forall y \exists x P(x,y)$

    (c) $\neg \forall y \forall x (P(x,y) \vee Q(x,y))$

    (d) $\neg(\exists x \exists y \neg P(x,y) \wedge \forall x \forall y Q(x,y))$

    (e) $\neg \forall x (\exists y \forall z P(x,y,z) \wedge \exists z \forall y P(x,y,z))$

    (f) $\neg \exists ! x P(x)$

Chapter 3

Types of proofs

Suppose we want to prove a proposition p.

- a **direct** proof just shows that p holds.
- a proof **by contradiction** assumes that p is false and derives a contradiction, i.e. both r and $\neg r$ for some proposition r.

If we want to prove an implication "if p then q", then

- a **direct** proof just shows how q follows from p.
- a proof **by contradiction** assumes that $p \to q$ is false, i.e. p and $\neg q$ are true, and derives a contradiction.
- a proof **by contrapositive** shows that $\neg q$ implies $\neg p$.

A proof of a statement of the form "$\exists x P(x)$" can be

- **constructive -** when you construct such an x explicitly, or
- **existential**, or **nonconstructive -** when you show the existence of such an x without actually constructing it.

To prove a statement of the form "$\forall x P(x)$" where the domain of x is a subset of integer numbers, it is often (but not always!) a good idea to use Mathematical Induction (see chapter 4).

To prove a statement of the form "$p \leftrightarrow q$", you can either

- prove $p \to q$ and $q \to p$ separately, or
- have each step of your proof of the form "if and only if".

To **disprove** a statement means to show that it is false. To disprove a statement of the form $\forall x P(x)$ it is sufficient to show that there exists at least one **counterexample** (that is, there exists at least one case when the statement does not hold).

Example 3.1. Prove that every odd integer is the difference of two perfect squares.

Direct proof: An odd integer has the form $2n+1$.
$2n+1 = (n+1)^2 - n^2$.

Example 3.2. Prove that $\sqrt{2}$ is irrational.

Proof by contradiction: Suppose $\sqrt{2}$ is rational. Then there exists an irreducible fraction $\frac{p}{q} = \sqrt{2}$. (Irreducible means that the greatest common divisor of p and q is 1.) Then
$\frac{p^2}{q^2} = 2$
$p^2 = 2q^2$
Then p^2 is even, so p is even. Let $p = 2m$, then $p = 4m^2$.
We have $4m^2 = 2q^2$
$2m^2 = q^2$
Now q is even. We get a contradiction because we have that on the one hand, p and q have the greatest common divisor 1, but on the other hand p and q are both even.

Example 3.3. Prove that if a and b are integers and ab is even, then either a or b is even (or both).

Proof by contrapositive: Suppose that neither a nor b is even, and we will prove that ab is not even. I.e. we suppose that both a and b are odd, and we will prove that ab is odd. Any odd numbers a and b can be written in the form $a = 2n + 1$ and $b = 2m + 1$ for some integers n and m. Then we have $ab = (2n + 1)(2m + 1) = 4nm + 2n + 2m + 1 = 2(2nm + n + m) + 1$ is an odd number.

Example 3.4. Prove that for every positive integer n there exist n consecutive composite numbers.

Constructive proof: We claim that $(n + 1)! + 2$, $(n + 1)! + 3$, ... , $(n + 1)! + (n + 1)$ are all composite. $(n+1)!$ is divisible by 2, by 3, ... , and by $n+1$. Therefore $(n+1)!+2$ is divisible by 2, $(n + 1)! + 3$ is divisible by 3, ... , $(n + 1)! + (n + 1)$ is divisible by $n + 1$.

Example 3.5. Prove that $x^3 + x - 1 = 0$ has a real root.

Nonconstructive proof: Let $f(x) = x^3+x-1$. Then $f(-1) = -3 < 0$ and $f(1) = 1 > 0$. Since $f(x)$ is a polynomial, it is continuous. By the Intermediate Value Theorem, there exists c between -1 and 1 such that $f(c) = 0$.

Example 3.6. Prove or disprove that every odd integer is prime.

Counterexample: 9 is odd but not prime. Thus the statement is false.

Problems

1. Prove that if n is an integer and $3n + 5$ is odd then n is even. Is your proof direct, by contradiction, or by contrapositive?
2. Prove that an integer a is even if and only if a^2 is even. Did you prove the two implications separately or simultaneously?
3. Prove or disprove that $2^n + 1$ is prime for all nonnegative integers n.
4. Prove that for any number n there is a prime number greater than n. Is your proof constructive?
5. Every odd number is either of the form $4n + 1$ (if it has remainder 1 when divided by 4) or of the form $4n + 3$ (if it has remainder 3). Prove that if an odd number is a perfect square, then it has the form $4n + 1$. Is your proof direct, by contradiction, or by contrapositive? State the converse. Prove or disprove the converse.

6. Prove or disprove that if a and b are rational numbers, then a^b is also rational.

7. Prove that the equation $x^{101} + x^{51} + x + 1 = 0$ has exactly one real solution. Split this into two statements:

   (a) the equation has at least one solution. Is your proof constructive or nonconstructive?

   (b) the equation can not have two distinct roots. Is your proof direct, by contradiction, or by contrapositive?

8. Prove that if the sum of two numbers is irrational then at least one of the numbers is irrational. Is your proof direct, by contradiction, or by contrapositive? State the converse. Prove or disprove the converse.

9. Prove that the equation $4\sin^2 x = 1$ has a real solution. Is your proof constructive?

10. Prove that the equation $x + \sin x = 1$ has a real solution. Is your proof constructive?

11. Prove that the equation $x^2 + x + 1 = 0$ has no rational solutions. Is your proof direct, by contradiction, or by contrapositive?

12. Prove that 0 is a root of the equation $a_n x^n + \dots a_1 x + a_0 = 0$ if and only if the free term $a_0 = 0$. Did you prove the two implications separately or simultaneously?

13. Prove that if a positive integer is divisible by 8 then it is the difference of two perfect squares. Is your proof direct, by contradiction, or by contrapositive? Is it constructive or nonconstructive?

14. Prove or disprove that if a and b are irrational numbers, then a^b is also irrational.

15. Prove that for any integers n and m, if $nm + 2n + 2m$ is odd then both n and m are odd. Is your proof direct, by contradiction, or by contrapositive?

Chapter 4

Principle of Mathematical Induction

Theorem 4.1. *(Principle of Mathematical Induction) Let S_n be a statement about a positive integer n. Suppose that*

1. *S_1 is true,*
2. *If $k \geq 1$ and S_k is true then S_{k+1} is true.*

Then S_n is true for all positive integers n.

Note. Conditions 1 and 2 in the above theorem are called the basis step and inductive step respectively.

This principle is easy to understand using the following example: suppose we know how to get to the first floor of a building (e.g. we know where an entrance is), and we also know how to get from any floor to the next one (e.g. we know where an elevator or a staircase is). Then we'll be able to get to any floor in this building. Namely, we'll get to the first floor, and then from the first to the second, and then from the second to the third, and so on. The same is true for any statement. If we can check that S_1 is true, then the second condition in theorem 4.1 ensures that S_2 follows from S_1, and S_3 follows from S_2, and so on. Thus S_n is true for any natural number n.

Mathematical Induction is used in all areas of mathematics. It can be used to prove summation formulas such as in the next example, various number theory, algebraic, and geometric statements.

Example 4.2. Prove that for any natural number n,

$$1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2}.$$

Proof. We will prove this identity using Mathematical Induction.
Basis step: if $n = 1$, the formula says that $1 = \dfrac{1 \cdot (1+1)}{2}$ which is true.
Inductive step: suppose the formula holds for $n = k$, i.e. that

$$1 + 2 + 3 + \ldots + k = \frac{k(k+1)}{2} \tag{4.1}$$

is true. We have to show that the formula holds for $n = k+1$, i.e. that

$$1 + 2 + 3 + \ldots + (k+1) = \frac{(k+1)((k+1)+1)}{2}$$

is true. Adding $k+1$ to both sides of (4.1) gives:

$$\begin{aligned}
1 + 2 + 3 + \ldots + k + (k+1) &= \frac{k(k+1)}{2} + (k+1) \\
&= \frac{k(k+1) + 2(k+1)}{2} \\
&= \frac{k^2 + k + 2k + 2}{2} \\
&= \frac{k^2 + 3k + 2}{2} \\
&= \frac{(k+2)(k+1)}{2} \\
&= \frac{((k+1)+1)(k+1)}{2}.
\end{aligned}$$

□

Note. For any specific value of n, it is easy to check that the identity holds. For example, for the first four natural numbers we have:

$$1 = \frac{1 \cdot (1+1)}{2}, \quad 1 + 2 = \frac{2 \cdot (2+1)}{2}, \quad 1 + 2 + 3 = \frac{3 \cdot (3+1)}{2}, \quad 1 + 2 + 3 + 4 = \frac{4 \cdot (4+1)}{2}.$$

However, remember that it is not sufficient to check *some* values of n. We had to prove the statement for *all* natural numbers n.

Remark. We might want to prove a statement S_n for all $n \geq 0$, or for all $n \geq 2$, etc., rather than for all $n \geq 1$. In this case, the basis step should check that the statement is valid for the smallest value of n, say, $n = 0$, or $n = 2$ in the above cases.

Sometimes to prove S_{k+1}, it is insufficient to assume S_k alone, but S_n for $n \leq k$ is needed. Then we use the so-called Strong Induction formulated below.

Theorem 4.3. *(Strong Mathematical Induction) Let S_n be a statement about a positive integer n. Suppose that*

1. *S_1 is true,*

2. *If $k \geq 1$ and S_n is true for all $1 \leq n \leq k$ then S_{k+1} is true.*

Then S_n is true for all positive integers n.

Remark. As above, we might want to start with 0 or 2 or something else rather than with 1.

Example 4.4. Prove that any integer $n \geq 2$ can be written in the form $n = 2a + 3b$ for some nonnegative integers a and b (we will say that n is a nonnegative linear combination of 2 and 3).

Proof. Basis step. If $n = 2$, we have $n = 2 \cdot 1 + 3 \cdot 0$.
Inductive step. Suppose that $k \geq 2$ and the statement holds for all $2 \leq n \leq k$. We want to prove it for $n = k + 1$.
Case I. $k = 2$, so $k + 1 = 3$. Then $k + 1 = 3 = 2 \cdot 0 + 3 \cdot 1$.
Case II. $k \geq 3$, then $2 \leq k - 1 \leq k$, thus the statement holds for $n = k - 1$. We have $k - 1 = 2a + 3b$ for some nonnegative integers a and b. Then $k + 1 = k - 1 + 2 = 2a + 3b + 2 = 2(a + 1) + 3b$, so $k + 1$ is a nonnegative linear combination of 2 and 3. $\square$

Remark. Notice that case I above just checks that the statement holds for $n = 3$. In literature, this calculation is often moved to the basis step.

Problems

1. Prove that the following formulas hold for any natural n.
   (a) $1^2 + 2^2 + 3^2 + \ldots + n^2 = \dfrac{n(n+1)(2n+1)}{6}$
   (b) $1^3 + 2^3 + 3^3 + \ldots + n^3 = \left(\dfrac{n(n+1)}{2}\right)^2$
   (c) $1 \cdot 1! + 2 \cdot 2! + \ldots + n \cdot n! = (n+1)! - 1$
   (d) $1 + 3 + 5 + \ldots + (2n - 1) = n^2$
   (e) $1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \ldots + n(n+1) = \dfrac{n(n+1)(n+2)}{3}$
2. Prove that for any positive integer n, $n < 2^n$.
3. Prove that if $m = 2^q$ where q is a positive integer, then $3^m - 1$ is divisible by 2^{q+2}.
4. Suppose that $2n$ points are given in space. Altogether $n^2 + 1$ line segments are drawn between these points. Prove that there is at least one set of three points which are joined pairwise by line segments.
5. Let $\{F_0,\ F_1,\ F_2,\ \ldots\}$ be the Fibonacci sequence defined by $F_0 = 0$, $F_1 = 1$, and $F_{n+1} = F_n + F_{n-1}$, $n \geq 1$. Prove the following identities.
   (a) $F_1F_2 + F_2F_3 + \ldots + F_{2n-1}F_{2n} = F_{2n}^2$
   (b) $\displaystyle\sum_{i=1}^{n} F_i^2 = F_nF_{n+1}$
   (c) $F_{n-1}F_{n+1} = F_n^2 + (-1)^n$
   (d) $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^n = \begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix}$
   (e) $F_{n-1}^2 + F_n^2 = F_{2n-1}$
6. There are n identical cars on a circular track. Among all of them, they have just enough gas for one car to complete a lap. Show that there is a car which can complete a lap by collecting gas from other cars on its way around.
7. Every road in Sikinia is one-way. Every pair of cities is connected by exactly one direct road. Show that there exists a city which can be reached from every other city either directly or via at most one other city.

8. Suppose that n lines are given in the plane. They divide the plane into parts. Show that it is possible color the plane with two colors, so that no parts with a common boundary line are colored the same way. Such a coloring is called a proper coloring.

9. Consider a few points in the plane and a few line segments connecting some of them so that (1) no two line segments intersect, and (2) each point is connected with at least two other points (so there are no isolated points and there are no "hanging" line segments). Such line segments divide the plane into several regions. Such a picture is called a map. Prove that a map can be properly colored with two colors if and only if each point is connected with an even number of other points. (See problem 8 for definition of a proper coloring)

10. Let α be any real number such that $\alpha + \dfrac{1}{\alpha} \in \mathbb{Z}$. Prove that $\alpha^n + \dfrac{1}{\alpha^n} \in \mathbb{Z}$ for any $n \in \mathbb{N}$.

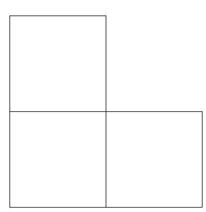
11. Prove that $1 < \dfrac{1}{n+1} + \dfrac{1}{n+2} + \ldots + \dfrac{1}{3n+1} < 2$.

12. Consider all nonempty subsets of the set $\{1, 2, \ldots, N\}$, which do not contain any neighboring elements. Then the sum of the squares of the products of all numbers in these subsets is $(N+1)! - 1$. (e.g. if $N = 3$, then such subsets of $\{1, 2, 3\}$ are $\{1\}$, $\{2\}$, $\{3\}$, and $\{1, 3\}$, and $1^2 + 2^2 + 3^2 + (1 \cdot 3)^2 = 23 = 4! - 1$.)

13. Find the determinant of the $n \times n$ matrix A_n with entries $a_{ij} = \begin{cases} 2 \text{ if } i = j \\ 1 \text{ if } |i - j| = 1 \\ 0 \text{ otherwise} \end{cases}$.
    Hint: calculate the determinants of A_1, A_2, A_3, and A_4. Notice the pattern. Prove your formula using Mathematical Induction.

14. Find the determinant of the $n \times n$ matrix M_n with entries $m_{ij} = \begin{cases} a \text{ if } i = j \\ b \text{ if } |i - j| = 1 \\ 0 \text{ otherwise} \end{cases}$ for arbitrary a and b. Suggestion: Find a recursive equation, prove it using Mathematical Induction, and then find an explicit formula for the determinant of such $n \times n$ matrix.

15. Prove that if one square of a $2^n \times 2^n$ chessboard is removed, then the remaining board can be covered by L-trominoes, i.e. the figures consisting of 3 squares as shown below.

    (You can choose which square you want to remove.)

16. Let f be a one-to-one function from $X = \{1, 2, \ldots, n\}$ onto X. Let $f^k = \underbrace{f \circ f \circ \ldots \circ f}_{k \text{ times}}$ denote the k-fold composition of f with itself. Show that for some positive integer m, $f^m(x) = x$ for all $x \in X$.

Chapter 5

Dirichlet's box principle

Theorem 5.1. *(Dirichlet's Box Principle) If $n+1$ or more objects are put into n boxes, then at least one box contains more than one object.*

Dirichlet's Box Principle is often called **the Pigeonhole Principle** and is formulated as follows.

Suppose there are n pigeonholes in the tree, and there are at least $n+1$ pigeons flying into these n holes. Then there is at least one hole containing more than one pigeon.

More formally and more generally, this principle can be formulated in the following way.

If the cardinality of a set S is bigger than the cardinality of a set T, and f is a function from S to T, then f is not one-to-one.

$$|S| > |T|$$

$$S \xrightarrow{f} T$$

e.g.

$$\{n+1 \text{ pearls}\} \longrightarrow \{n \text{ boxes}\}$$

$$\{n+1 \text{ pigeons}\} \longrightarrow \{n \text{ pigeonholes}\}$$

Theorem 5.2. *(Generalized Dirichlet's Box Principle) If $qn+1$ or more pearls are put into n boxes, then at least one box contains more than q pearls.*

Problems

1. Prove that among 13 persons, at least two were born in the same month.
2. Prove that among 50 persons, at least 5 were born in the same month.
3. Prove that among 120 integers, there are two whose difference ends with 00.

4. Nobody has more than 300,000 hairs on his head. The capital of Sikinia has 300,001 inhabitants. Can you assert with certainty that there are two persons with the same number of hairs on their heads?

5. 7 points are selected inside a regular hexagon whose sides have length 1. Prove that there are two points such that the distance between them is at most 1.

6. Suppose that 5 lattice points are chosen in the plane lattice. Prove that we can choose 2 of these points such that the segment joining these 2 points passes through another lattice point.

7. Prove that from any 12 distinct two-digit numbers, we can select two with a two-digit difference of the form aa.

8. (a) Prove that from any 52 positive integers, we can select two such that their sum or difference is divisible by 100.

   (b) Is the above assertion also valid for 51 positive integers?

9. Three hundred points are selected inside a cube with edge 7. Prove that we can place a small cube with edge 1 inside the big cube such that the interior of the small cube does not contain any of the selected points.

10. Prove that if there are n persons present in a room, and every person knows at least one other person, then among them there are 2 persons who have the same number of acquaintances.

11. Let a_1, a_2, and a_3 be integers. Show that the product $(a_1 - a_2)(a_1 - a_3)(a_2 - a_3)$ is even.

12. Let a_1, a_2, a_3, and a_4 be integers. Show that the product $\displaystyle\prod_{1 \le i < j \le 4} (a_i - a_j)$ is divisible by 12.

13. (a) Seven points are selected inside a 3×4 rectangle. Prove that there are two of them such that the distance between them is at most $\sqrt{5}$.

    (b) Six points are selected inside a 3×4 rectangle. Prove that there are two of them such that the distance between them is at most $\sqrt{5}$.

14. Suppose that fifty-one small insects are placed inside a square of side 1. Prove that at any moment there are at least three insects which can be covered by a single disk of radius 1/7.

15. Prove that in any convex $2n$-gon, there is a diagonal not parallel to any side.

16. Let every block of a 3×7 checkerboard be colored either black or white. Prove that in whichever way you color the checkerboard, it contains a rectangle consisting of more than one row and more than one column whose four corners have the same color.

17. Using 4 colors, we color a 5×41 block checkerboard. Prove that, whichever way we color the blocks, there exist at least one same-color-corner rectangle.

18. Let twenty pairwise distinct positive integers be all less than 70. Prove that among their pairwise differences there are four equal numbers.

19. Prove that among $n + 1$ positive integers all less than or equal to $2n$, there are two which are relatively prime.

20. Let $\{a_1, a_2, \ldots, a_{n+1}\}$ be numbers from the set $\{1, 2, \ldots, 2n\}$. Prove that one of the a_i's is divisible by another.

21. (a) Prove that among 11 distinct positive integer numbers, there are two numbers $a < b$ such that the difference $b - a$ ends with 0 (i.e. has the units digit 0).

    (b) Is the above statement true for the tens digit?

22. Prove that from 11 infinite decimals, we can select two numbers a and b so that the decimal representation of $a - b$ is finite or has infinitely many zeros.

23. 26 points are chosen inside a 20×15 rectangle. Prove that there are at least two points with distance less than or equal to 5.

24. Let n be a positive integer which is not divisible by 2 or 5. Prove that there is a multiple of n consisting entirely of ones.

25. Kevin is paid every other week on Friday. Show that every year, in some month he is paid three times.

Chapter 6

Number theory

In this chapter we recall basic properties of integers. (Some of them have been used in previous chapters.)

Note. **All numbers discussed in this chapter are integers.**

Definition 6.1. If $b = aq$, then we say that a **divides** b, and write $a|b$, or that b is **divisible** by a, and we write $b \vdots a$.

Fundamental properties

- $a|b$, $b|c \Rightarrow a|c$.
- $a|b$, $a|c \Rightarrow a|b \pm c$. More generally, $a|bx + cy$ for any x and y.

Definition 6.2. The largest number that divides both a and b is called **the greatest common divisor** of a and b, and is denoted by $\gcd(a, b)$ or just (a, b).

- $(a, a) = a$, $(a, 1) = 1$, $(a, 0) = a$, $(a, b) = (b, a)$.

Definition 6.3. An integer greater than 1 is called **prime** if it has exactly two divisors, 1 and itself. An integer greater than 1 that is not prime is called **composite**.

Theorem 6.4. *There are infinitely many primes.*

Theorem 6.5. *(Euclid's lemma) If p is prime, $p|ab$, then either $p|a$ or $p|b$.*

Theorem 6.6. *(Fundamental theorem of arithmetic) Every positive integer has a prime factorization, i.e. can be written as a product of primes (and such a product is unique up to order of the factors).*

Example 6.7. $12 = 2 \cdot 2 \cdot 3 = 2 \cdot 3 \cdot 2 = 3 \cdot 2 \cdot 2$ are the only prime factorizations of 12. The order of the factors is different, but the set of factors is the same.

- $p|a$, $q|a$, p and q are distinct primes $\Rightarrow (pq)|a$.

Definition 6.8. Integers a and b are called **relatively prime**, or coprime, if $(a, b) = 1$.

Remark. The numbers a and b may be relatively prime even if they are both composite. For example, $8 = 2 \cdot 2 \cdot 2$ and $15 = 3 \cdot 5$ are composite but relatively prime since they do not share any factors.

Theorem 6.9. *For any pair a, b, theit greatest common divisor (a, b) is a linear combination of a and b, i.e. there exist integers x and y such that $(a, b) = ax + by$. Moreover, (a, b) is the smallest positive integer that can be written in the form $ax + by$ for some x and y.*

Corollary 6.10. *Numbers a and b are relatively prime if and only if there exist x and y such that $ax + by = 1$.*

Definition 6.11. For every pair a, $b \neq 0$ there exist unique q and r such that

$$a = bq + r, \qquad 0 \leq r < b.$$

The numbers q and r are called the **quotient** and **remainder** upon division of a by b.

- Any integer can be written in the form $10q + r$ for some q and r where $0 \leq r \leq 9$.
- Any integer can be written in the form $10^n a_n + 10^{n-1} a_{n-1} + \ldots + 10^2 a_2 + 10 a_1 + a_0$ where a_n, ..., a_0 are the digits of the integer.
- If two numbers have the same remainder upon division by b, then they can be written as $bq_1 + r$ and $bq_2 + r$. Their difference is $b(q_1 - q_2)$, and thus it is divisible by b.
- Note that when we divide by b, there are b possible remainders. Thus given $b + 1$ numbers, by Dirichlet's Box Principle at least 2 of them have the same remainder. Their difference is divisible by b.

Definition 6.12. Integers a and b are said to be **congruent** mod m, and we write $a \equiv b (\text{mod } m)$ if $m | (a - b)$. Equivalently, $a - b = mq$ for some q, or $a = b + mq$, or a and b have the same remainder upon division by m.

For example, $12 \equiv 7 (\text{mod } 5)$ because they have the same remainder upon division by 5.

- Congruences can be added, subtracted, and multiplied: if $a \equiv b (\text{mod } m)$ and $c \equiv d (\text{mod } m)$, then $a \pm c \equiv b \pm d (\text{mod } m)$ and $ac \equiv bd (\text{mod } m)$.
- If $a \equiv b (\text{mod } m)$ then $a^c \equiv b^c (\text{mod} m)$.
- Cancellation rule: if $(c, m) = 1$, $ca = cb (\text{mod } m)$, then $a \equiv b (\text{mod } m)$.

Theorem 6.13. *(Fermat's theorem) If p is prime, then $a^p \equiv a (\text{mod } p)$.*

Corollary 6.14. *If p is prime and $p \not| a$, then $a^{p-1} \equiv 1 (\text{mod } p)$.*

Example 6.15. Let $p = 5$. Then $1^4 = 1 \equiv 1 (\text{mod } 5)$, $2^4 = 16 \equiv 1 (\text{mod } 5)$, $3^4 = 81 \equiv 1 (\text{mod } 5)$, $4^4 = 256 \equiv 1 (\text{mod } 5)$.

Useful formulas

- $a^n - b^n = (a - b)(a^{n-1} + a^{n-2} b + \ldots + ab^{n-2} + b^{n-1})$
- if n is odd, $a^n + b^n = (a + b)(a^{n-1} - a^{n-2} b + \ldots + (-1)^{n-2} ab^{n-2} + (-1)^{n-1} b^{n-1})$

Problems

1. Show that $\sqrt[3]{25}$ is irrational.

2. Show that $\log_2 5$ is irrational.
3. (a) Prove that a natural number is divisible by 9 iff the sum of its digits is divisible by 9.
   (b) Prove that if the sum of the digits of a number is 66 then it is not a perfect square.
4. Show that 3 divides both a and b iff 3 divides $a^2 + b^2$.
5. (a) If c is a perfect square (the square of an integer), what are the possible values of its last (units) digit?
   (b) Conclude that a number ending with 3 cannot be a perfect square.
6. (a) If c is a perfect square, what are the possible values of its remainder upon division by 4?
   (b) Conclude that a number ending with 66 cannot be a perfect square.
7. Can a number ending with 65 be a perfect square?
8. The four-digit number $aabb$ is a perfect square. Find it.
9. Find 2^{100} mod 5 (that is, find the remainder upon division of 2^{100} by 5).
10. Show that $A = 3^{105} + 4^{105}$ is divisible by 7. Find A mod 11 and A mod 13.
11. Show that if the units digit of a natural number n is 3 then $5|(n^2 + 1)$.
12. Show that for for any natural n, $6|(n^3 + 5n)$.
13. Show that if n is not prime, then $2^n - 1$ is not prime.
14. Show than $2^n \nmid n!$ for any $n \in \mathbb{N}$.
15. Find all primes p and q such that $p^2 - 2q^2 = 1$.
16. Find all the integral solutions of $x + y = xy$.
17. Show that $x^2 - 3y^2 = 17$ has no integral solutions.
18. Find all integral solutions of $x + y = x^2 - xy + y^2$.
19. How many pairs of positive integers are solutions to the equation $2x + 3y = 100$?
20. How many pairs of positive integers are solutions to the equation $5x + 7y = 1234$?
21. Does there exist a number that starts with 123 and is divisible by 4567? If so, find it.
22. Does there exist a number that ends with 123 and is divisible by 4567? If so, find it.
23. Show that $2^{457} + 3^{457}$ is divisible by 5.

Chapter 7

Case study

We have seen in the previous chapter that some number theory problems can be solved by considering all possible remainders mod n for a variable. The problem about roads in Sikinia (problem 7 in chapter 4) also required considering some cases separately. As we will see in future chapters, the technique of considering all possible cases can be used in many different problems of very different types.

Here is a couple of typical examples where we need to consider two or more cases.

If $a^b = 1$ (and $a, b \in \mathbb{R}$), then there are 3 possibilities:

- $a = 1$
- $a \neq 0$, $b = 0$
- $a = -1$ and b is even

Example 7.1. Find all values of x for which $(x^2 - 5x + 5)^{x^2-9x+20} = 1$.

Solution. *Let* $a = x^2 - 5x + 5$ *and* $b = x^2 - 9x + 20$. *Then we have* $a^b = 1$.

Case I. $a = 1$:
$x^2 - 5x + 5 = 1$
$x = 1$ *or* $x = 4$

Case II. $a \neq 0$, $b = 0$.
First solve $b = 0$:
$x^2 - 9x + 20 = 0$
$x = 4$ *or* $x = 5$
For both roots $a \neq 0$.

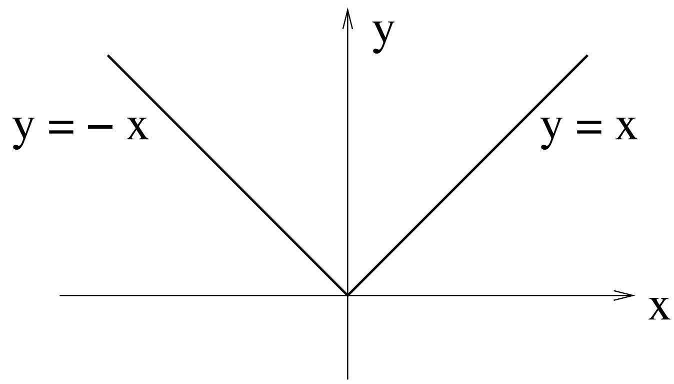
Case III. $a = -1$ *and* b *is even.*
First solve $a = -1$:
$x^2 - 5x + 5 = -1$
$x = 2$ *or* $x = 3$
For both roots b *is even.*

Therefore, the solutions are 1, 2, 3, 4, and 5.

Recall that the absolute value of a real number x is denoted $|x|$ and is given by

$$|x| = \begin{cases} x & \text{if} \quad x \geq 0 \\ -x & \text{if} \quad x < 0 \end{cases}$$

Here is the graph of $|x|$:



To solve problems involving the absolute value, consider 2 cases: when the expression inside the absolute value is positive or 0, and when it is negative. If there are several absolute values, consider 2 cases for each absolute value.

Example 7.2. Solve $3|x^2 - 9| - 11x + 7 = 0$.

Solution. *Case I.* *If $x^2 - 9 \geq 0$, then $|x^2 - 9| = x^2 - 9$, and the equation becomes*
$3(x^2 - 9) - 11x + 7 = 0$
$3x^2 - 11x - 20 = 0$
Using the quadratic formula, we find $x = 5$ or $x = -4/3$
$x = 5$ satisfies the condition $x^2 - 9 \geq 0$, but $x = -4/3$ does not, so we throw it away.

Case II. *If $x^2 - 9 < 0$, then $|x^2 - 9| = -(x^2 - 9)$, and the equation becomes*
$-3(x^2 - 9) - 11x + 7 = 0$
$-3x^2 - 11x + 34 = 0$
the roots are 2 and $-17/3$, but the second root does not satisfy the condition $x^2 - 9 < 0$, so we throw it away.

Thus the only roots are 5 and 2.

Problems

1. Prove that if n is an integer then
   (a) $n^3 + 5n$ is divisible by 3,
   (b) $n^2 + 2$ is not divisible by 5.
2. Solve for x:
   (a) $x^{x^2-7x+12} = 1$
   (b) $(x^{x+1})^{x^2} = 1$
   (c) $x^{(x^2)} = x^2$
   (d) $x^{((x+1)^2)} = x^{16}$
   (e) $x^{(x^x)} = (x^x)^x$
   (f) $\sqrt{x^{x+1}} = x^{\sqrt{x+1}}$
   (g) $(x-3)^{x^2-8x+15} = 1$
3. Find all pairs (x, y) that satisfy the system

(a) $\begin{cases} x^{2x} & = & y+1 \\ x^y & = & 1 \end{cases}$

(b) $\begin{cases} x^{x+y} & = & y^4 \\ y^{x+y} & = & x \end{cases}$

4. Solve for x:

   (a) $x^2 + |2x-2| = 1$
   (b) $x^2 - |5x-6| \leq 0$
   (c) $|2x+3| - |x| = 3$
   (d) $|2x-1| - |x+5| = 3$
   (e) $|x-5| + |2x-4| \leq 6$
   (f) $|x-1| - |x-3| \geq 5$
   (g) $|x+1| + 5 - x^2 \geq 0$

5. Sketch the graph of

   (a) $f(x) = |x^2 - 4| + 2$
   (b) $y = |x^2 - 1| - |x^2 - 4|$
   (c) $f(x) = |x + |x+2||$.
   (d) $y = |x^2 - 4|x| + 3|$
   (e) $|x| + |y| = 1 + |xy|$

6. Sketch the region:

   (a) $\{(x,y) \mid |x| + |y^3| < 8\}$
   (b) $\{(x,y) \mid |x-y| + |x| - |y| \leq 2\}$
   (c) $\{(x,y) \mid 2|y-x| + |y+x| \leq 1\}$

7. Find all integral solutions of

   (a) $a^b = 625$
   (b) $(a^b)^c = 64$

   (i.e. list all the solutions and show that there are no other solutions).

Chapter 8

Finding a pattern

Example 8.1. Find the n-th derivative of $f(x) = 5^x$.

Solution. *Find the first few derivatives (until you can see a pattern):*
$f'(x) = \ln 5 \cdot 5^x$
$f''(x) = \ln 5 \cdot \ln 5 \cdot 5^x = (\ln 5)^2 \cdot 5^x$
$f'''(x) = (\ln 5)^2 \cdot \ln 5 \cdot 5^x = (\ln 5)^3 \cdot 5^x$
We notice that $f^{(n)}(x) = (\ln 5)^n \cdot 5^x$.
It is easy to prove this formula using Mathematical Induction.
The basis step is $f'(x) = \ln 5 \cdot 5^x$.
Inductive step: suppose $f^{(k)}(x) = (\ln 5)^k \cdot 5^x$ *is true. Then*
$f^{(k+1)}(x) = (f^{(k)}(x))' = ((\ln 5)^k \cdot 5^x)' = (\ln 5)^k \cdot \ln 5 \cdot 5^x = (\ln 5)^{k+1} \cdot 5^x$.

Note. When you notice a pattern, it is often needed to guess a formula (which you can then prove by Mathematical Induction or some other method).

Example 8.2. Guess a formula for the n-th term of the sequence: 1, 3, 6, 10, 15, 21, ...

Solution 1. *Notice that the difference between the first and the second terms is 2, the difference between the second and the third terms is 3, and then the differences are 4, 5, 6, ... Thus*
$a_1 = 1,$
$a_2 = 1 + 2,$
$a_3 = 1 + 2 + 3,$
$a_4 = 1 + 2 + 3 + 4,$
$a_5 = 1 + 2 + 3 + 4 + 5,$
$a_6 = 1 + 2 + 3 + 4 + 5 + 6.$
So it appears that $a_n = 1 + 2 + \ldots + n = \dfrac{n(n+1)}{2}$.

Note. Of course for the problems like this one, since only a few terms of a sequence are given, there may be several different formulas valid for these few terms. We tried to find a simple one.

Problems

1. Guess a formula for the n-th term of the sequence $\{a_1, a_2, a_3, \ldots\}$ whose first few terms are given. Try to find the simplest possible formula, but any correct formula (that is, any formula that works for the given terms) will be accepted.

(a) -1, 0, 1, 2, 3, 4, 5, ...

(b) 5, 7, 9, 11, 13, 15, ...

(c) 1, 3, 4, 6, 7, 9, 10, 12, ...

(d) $\frac{1}{2}, \frac{1}{2}, \frac{3}{8}, \frac{1}{4}, \frac{5}{32}, \frac{3}{32}, \frac{7}{128}, \ldots$

2. Guess a formula for the n-th term of the sequence whose first few terms are given.

   (a) 1, 4, 9, 16, 25, 36, 49, ...

   (b) 8, 10, 12, 14, 16, 18, ...

   (c) $3, 1, -1, -3, -5, -7$, ...

   (d) 1, 2, 1, 4, 1, 6, 1, 8, ...

   (e) 0, 1, 3, 7, 15, 31, ...

3. Compute $A_n = 1+3+5+\ldots+(2n-1)$ for some small values of n. Notice the pattern. Write a formula for A_n and prove it using Mathematical Induction.

4. Find $\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\ldots+\frac{1}{(n-1)n}$. Prove your formula.

5. Let $f_1(x) = 2x+1$ and $f_n = f_1 \circ f_{n-1}$. Compute f_n for some small values of n. Notice the pattern. Write a formula for f_n and prove it using Mathematical Induction.

6. Let $f_1(x) = \frac{1}{2-x}$ and $f_{n+1} = f_1 \circ f_n$ for $n \geq 1$. Find an expression for $f_n(x)$ and use Mathematical Induction to prove it.

7. What is the last digit of 107^{107}? (Hint: find the last digit of 107^n for small values of n and notice the pattern.)

8. What is the last digit of 1234^{5678}?

9. What are the last two digits of 7^{50}?

10. Find the remainder of 5^{4321} upon division by 11.

11. Find the n-th derivative of

    (a) $g(x) = \sin(x)$,

    (b) $g(x) = \ln(x)$,

    (c) $h(x) = 2e^{5x}$.

12. n lines in general position are given in a plane. (General position means that no two lines are parallel, and no 3 lines have a common point.) Into how many regions do they divide the plane?

13. Suppose n circles are given in a plane, such that every pair of circles has 2 intersection points, but no 3 circles have a common point. Into how many regions do they divide the plane?

14. Amanda is training her rabbit to climb a flight of 10 steps. The rabbit can hop up 1 or 2 steps each time he hops. He never hops down, only up. How many different ways can he hop up the flight of 10 steps? (Don't even try to list all the ways. There are too many of them! Better replace 10 by small numbers, and guess the pattern. Prove your guess using Mathematical Induction.)

    Let $F_0 = 0$, $F_1 = 1$, $F_2 = 1$, …, F_{99} be the first 100 Fibonacci numbers (recall that $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$). How many of them are even?

15. Find a formula for

$$\prod_{i=1}^{2n-1} \left(1 - \frac{(-1)^i}{i}\right) = \left(1 - \frac{-1}{1}\right)\left(1 - \frac{1}{2}\right)\left(1 - \frac{-1}{3}\right)\cdots\left(1 - \frac{-1}{2n-1}\right)$$

    and prove it.

Chapter 9

Invariants

Definition 9.1. An **invariant** is something that doesn't change.

Example 9.2. The numbers $1, 2, \ldots, 10$ are written on the blackboard. We pick any two numbers, let's call them a and b, we erase them, and write $a+1$ and $b-1$ instead. Is it possible to get ten 5's by a sequence of such operations?

Solution. *Notice that when we increase a by 1 and decrease b by 1, the sum of the numbers does not change. Initially the sum is* $1+2+\ldots+10=55$, *and* $10 \cdot 5 = 50$, *so it is not possible to get ten 5's.*

Example 9.3. Each of the numbers $a_1, a_2, \ldots, a_n$ is 1 or -1, and

$$S = a_1a_2a_3a_4 + a_2a_3a_4a_5 + \ldots + a_{n-3}a_{n-2}a_{n-1}a_n + a_{n-2}a_{n-1}a_na_1 + a_{n-1}a_na_1a_2 + a_na_1a_2a_3 = 0.$$

Prove that $4|n$.

Solution. *If we replace* a_i *by* $-a_i$, *then* S *does not change mod* 4 *since four terms (containing* a_i*) change their sign. Indeed, if all four terms are of the same sign, then their sum changes from* 4 *to* -4 *or from* -4 *to* 4, *thus* S *changes by* ± 8. *If one or three have the same sign, then their sum changes from* 2 *to* -2 *or from* -2 *to* 2, *thus* S *changes by* ± 4. *Finally, if two are positive and two are negative, then the sum doesn't change. Initially, we have* $S = 0$ *which implies* $S \equiv 0 (mod\ 4)$. *Now, step-by-step, we can change each negative* -1 *into a positive* 1. *This does not change* S *mod* 4. *At the end, we still have* $S \equiv 0\ (mod\ 4)$, *but also* $S = n$, *so* $4|n$.

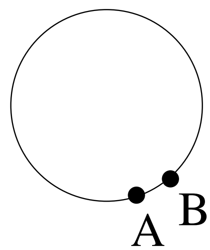
Here are a few things that are very often invariants in problems involving sets of numbers and allowed operations, so you may want to try look at them. Sometimes, of course, you have to be very creative!

- The sum or the product of all given numbers
- The number of positive or negative numbers
- The number of even or odd numbers
- The number of numbers congruent to a modulo b
- One of the above modulo a positive number (e.g. the sum modulo 2, i.e. the parity of the sum; the product modulo 3; the number of positive numbers modulo 4; etc.)

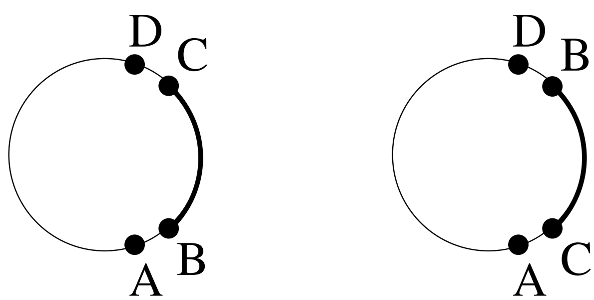
In the example below, we find a positive decreasing function rather than a constant function. The idea is that the value of that function must be non-negative. We apply a series of steps each of which decreases the value of the function. Since the value cannot become negative, sooner or later it will reach 0.

Example 9.4. $2n$ ambassadors are invited to a banquet. Every ambassador has at most $n-1$ enemies. Prove that the ambassadors can be seated around a round table, so that nobody sits next to an enemy.

Solution. *First, we seat the ambassadors randomly. Let H be the number of neighboring hostile couples. We must find an algorithm which reduces this number whenever $H > 0$. Let (A, B) be a hostile couple with B sitting to the right of A:*



We want to separate them so as to cause as little disturbance as possible (moreover, we do not want to gain any new neighboring hostile couples). This will be achieved if we reverse some arc BC as shown below. H will be reduced if (A, C) and (B, D) are friendly couples.



It remains to be shown that such a couple always exists. We start at A and go around the table counterclockwise. We will encounter at least n friends of A. To their right, there are at least n seats. They cannot all be occupied by enemies of B since B has at most $n-1$ enemies. Thus, there is a friend C of A with right neighbor D, a friend of B.

Problems

1. Start with the set $\{-3, -2, -1, 1, 2, 3\}$. In each step you may choose any two of these numbers and change their signs. Show that it is not possible to reach the set $\{3, 2, 1, 1, 2, 3\}$.

2. Start with the set $\{-3, -2, -1, 1, 2, 3\}$. In each step you may multiply or divide any of these numbers by any positive number. Show that it is not possible to reach the set $\{-2, -1, 1, 2, 3, 4\}$.

3. Start with the set $\{1, 1, 1, 1\}$. In each step, you may either multiply one of the numbers by 3, or subtract 2 from it. Show that it is not possible to reach the set $\{1, 2, 3, 4\}$.

4. Start with the set $\{1, 2, 3, 4, 5, 6\}$. In each step, you may add 2 to any 5 numbers or subtract 1 from any 5 numbers. Can you reach $\{1, 2, 4, 8, 16, 32\}$?

5. Start with the set $\{1, 2, 3, 4\}$. In each step you may add or subtract 2 times one of the numbers to/from another number. Say, you can replace 1 by $1 + 2 \cdot 2$, or by $1 - 2 \cdot 2$, or by $1 + 2 \cdot 3$, etc. Can you reach $\{10, 20, 30, 40\}$?

6. Start with the set $\{1, 3, 6\}$. In each step you may choose two of the numbers, let's call them a and b, and replace them by $0.6a - 0.8b$ and $0.8a + 0.6b$. Can you reach $\{2, 4, 5\}$?

7. Start with the table shown below. In one step, you may either add 1 to all the numbers in any row or column, or subtract 1 from all the numbers in any row or column. Prove that it is not possible to reach nine 1's.

0	0	0
0	1	0
0	0	0

8. Initially 1 is written in every cell of a 5×5 table. You may change the signs of the numbers in any two adjacent cells. Is it possible to make all of the numbers -1?

9. There are several $+$ and $-$ signs on a blackboard. You may erase two signs and write, instead, $+$ if they are equal and $-$ if they are unequal. Prove that the last sign on the board does not depend on the order of erasure.

10. Assume we have an 8×8 chessboard with the usual coloring. You may repaint all squares in any row or column. The goal is to attain just one black square. Can you reach the goal? What if you are allowed to repaint all squares in a 2×2 square?

11. Each of the numbers 1 to 10^6 is repeatedly replaced by its digital sum until we reach 10^6 one-digit numbers. For example, 987654 is replaced by $9 + 8 + 7 + 6 + 5 + 4 = 39$, then 39 is replaced by $3 + 9 = 12$, and finally, 12 is replaced by $1 + 2 = 3$. Among these 10^6 one-digit numbers, will we have more 1's or 2's?

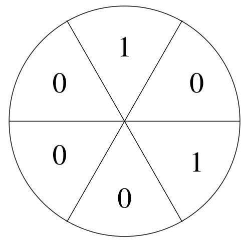
12. You can write all the digits from 1 to 9 in a row in any order you like, and then you write plus signs between some digits (as many plus signs as you like). For example, you could write $7 + 35 + 19 + 4 + 2 + 8 + 6$. Finally, you evaluate of the obtained expression. Prove that there is no way to get the value of 100. Or 101. Or 102. Or 103... What is the smallest possible three-digit number that can be obtained in this game?

13. Start with the positive integers $1, 2, \ldots, 4n - 1$. In each step you may replace any two integers by their difference. Prove that an even integer will be left after $4n - 2$ steps.

14. Let n be an odd positive integer. First we write the numbers $1, 2, 3, \ldots, 2n$ on the blackboard. Then we pick any two numbers, a and b, erase them, and write, instead, $|a - b|$. We do this until only one number remains. Prove that an odd number will remain at the end.

15. The numbers from 0 to 9 are written along a circle in random order. Between every 2 neighboring numbers a and b (in the clockwise order) we write $2b - a$. Then we erase the original numbers. This step is repeated. Show that it is not possible to reach ten 5's. (For example, the numbers could be written in the following order: 1, 5, 3, 9, 0, 2, 4, 6, 8, 7. Then the new numbers would be 9, 1, 15, -9, 4, 6, 8, 10, 6, -5.)

16. A circle is divided into six sectors. Then the numbers 1, 0, 1, 0, 0, 0 are written into the sectors as shown on the picture below. You may increase any two neighboring numbers by 1. Is it possible to equalize all numbers by a sequence of such steps?



17. The integers $1, 2, 3, 4, 5, 6$ are arranged in any order on 6 places numbered 1 through 6. Now we add its place number to each integer. Prove that there are two among the sums which have the same remainder mod 6.

18. There are a white, b black, and c red chips on a table. In one step, you may choose two chips of different colors and replace them by one chip of the third color. If just one chip will remain at the end, prove that its color does not depend on the evolution of the game, but it only depends on the numbers a, b, and c.

19. Nine 1×1 cells of a 10×10 square are infected. Two cells are called neighbors if they have a common side. In one time unit, the cells with at least two infected neighbors become infected. Can the infection spread to the whole square (in any amount of time)?

20. Twelve 1×1 cells of a 10×10 square are infected. Two cells are called neighbors if they share at least one vertex (thus an inner cell has 8 neighbors). In one time unit, the cells with at least 4 infected neighbors become infected. Can the infection spread to the whole square (in any amount of time)?

21. In the Parliament of Sikinia, each member has at most three enemies. Prove that the house can be separated into two houses, so that each member has at most one enemy in his own house.

22. In the table below, you may switch the signs of all numbers of a row, column, or parallel to one of the diagonals, In particular, you may switch the sign of each corner square. Prove that at least one -1 will remain in the table.

-1	1	-1	1
1	1	1	1
1	1	-1	-1
1	-1	1	1

23. There are seven 1's and eight -1's on a blackboard. In each step, you may erase any two numbers, say, a and b, and write $-ab$ instead. Show that no matter in what order we erase the numbers, 1 will remain in the end.

24. Start with the set $\{1, 4, 32, 128, 256\}$. In each step, you may divide one number by 2 and multiply another number by 2. Is it possible to reach the set $\{512, 32, 16, 16, 2\}$?

Chapter 10

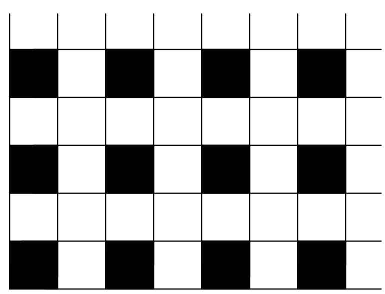
Coloring

Example 10.1. In 1961, the British theoretical physicist M. E. Fisher solved a famous and very tough problem. He showed that an 8×8 chessboard can be covered by 2×1 dominoes in $2^4 \times 901^2 = 12{,}988{,}816$ ways. Now let us cut out two diagonally opposite corners of the board. In how many ways can you cover the 62 squares of the mutilated chessboard with 31 dominoes?

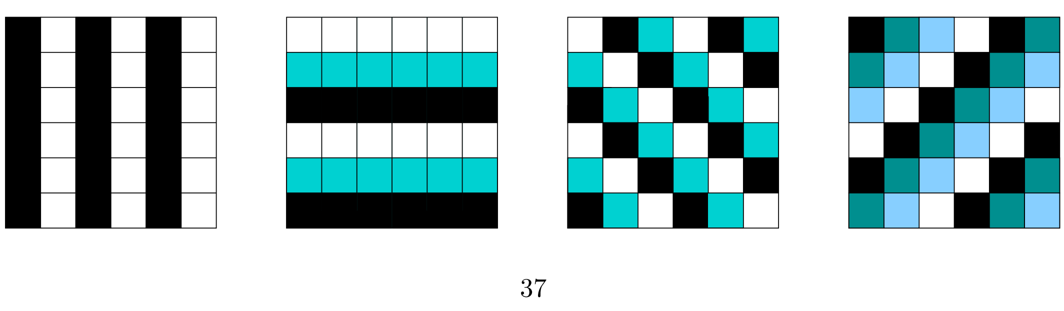
Solution. *Zero. There is no way to cover the mutilated chessboard. Each domino covers one black and one white square. If a covering of the board existed, it would cover* 31 *black and* 31 *white squares. But the mutilated chessboard has* 30 *squares of one color and* 32 *squares of the other color.*

Example 10.2. A rectangular floor is covered by 2×2 and 4×1 tiles. One tile got smashed. There is a tile of the other kind available. Show that the floor cannot be covered by rearranging the tiles.

Solution. *Color the floor as shown on the picture below. A* 4×1 *tile always covers either* 0 *or* 2 *black squares. A* 2×2 *tile always covers one black square. Therefore it is impossible to exchange one tile for a tile of the other kind.*



Besides the colorings used in the above examples, the “stripe colorings” and the ”diagonal colorings” shown below are often helpful.



Also, you can use the stripe or diagonal pattern with more colors; or the stripe pattern with one stripe of one color followed by several stripes of another color; and so on. Of course, sometimes you have to be creative and find your own coloring that would work for a particular problem!

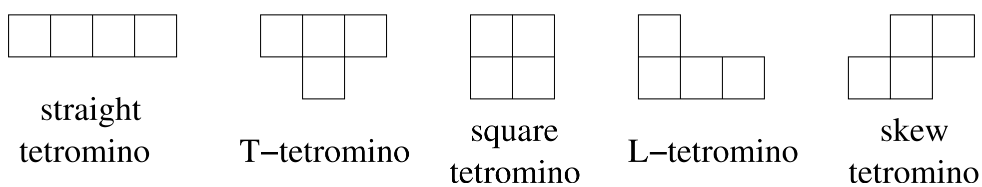
Problems

1. Prove that a 14×14 board cannot be covered by 49 T-tetrominoes (see pictures of tetrominoes below).

2. Prove that an 8×8 chessboard cannot be covered by 15 T-tetrominoes and one square tetromino.

3. Prove that a 10×10 board cannot be covered by 15 T-tetrominoes and 10 L-tetrominoes.

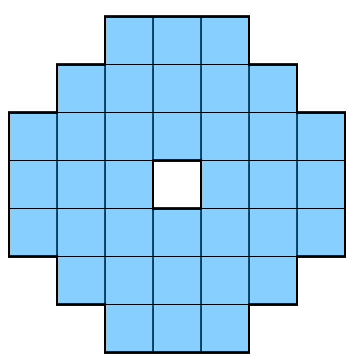
4. Is it possible to form a rectangle with the five tetrominoes shown below (using one tetromino of each kind)?



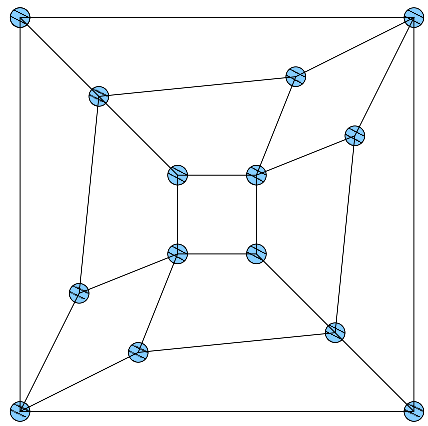
5. An 8×8 chessboard is covered by tetrominoes. Prove that the number of T-tetrominoes is even.

6. In chess, is it possible for a knight to start at the upper left corner, go throught every square on the chessboard exactly once and reach the lower right corner? (See allowed moves in chapter 1.)

7. Prove that the figure shown below (with center block removed) cannot be covered by dominoes.



8. The figure below shows a road map connecting 14 cities. Is there a path passing through each city exactly once?



9. Prove that a 6×6 board cannot be covered by 9 L-tetrominoes.

10. Prove that an 8×8 square cannot be covered by 11 straight tetrominoes and 5 L-tetrominoes.

11. Prove that an 8×8 board with one corner square removed (so, 63 squares remain) cannot be covered by 21 straight trominoes (i.e. 3×1 tiles).

12. Prove that a 15×8 board cannot be covered by 2 L-tetrominoes and 28 skew tetrominoes.

13. Prove that a 23×23 square cannot be covered by 2×2 and 3×3 tiles.

14. Prove that a 10×10 board cannot be covered by 25 straight tetrominoes.

15. Prove that an $a \times b$ rectangle can be covered by $1 \times n$ rectangles iff $n|a$ or $n|b$.

16. Prove that there is no way to pack fifty-four $1 \times 1 \times 4$ bricks into a $6 \times 6 \times 6$ box.

17. A 7×7 square is covered by sixteen 3×1 and one 1×1 tiles. What are the permissible positions of the 1×1 tile?

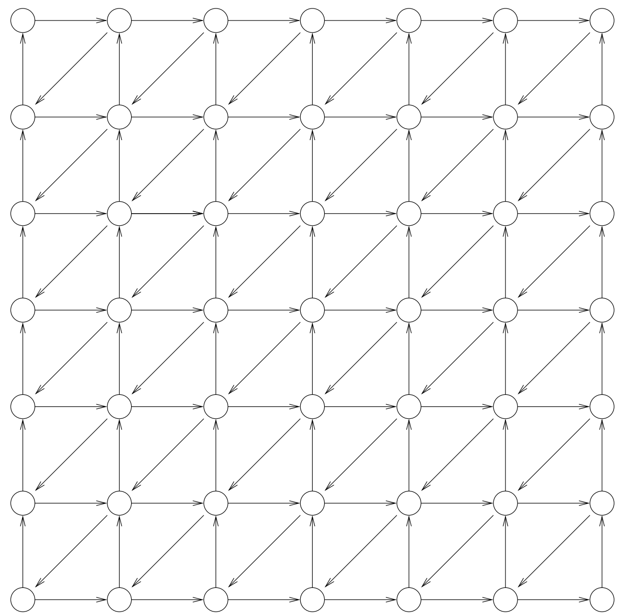
18. The map below shows the cities and one-way roads in Sikinia.

    (a) Prove that there is no closed path (a path is closed if it starts and quits in the same city) that visits every city exactly once.

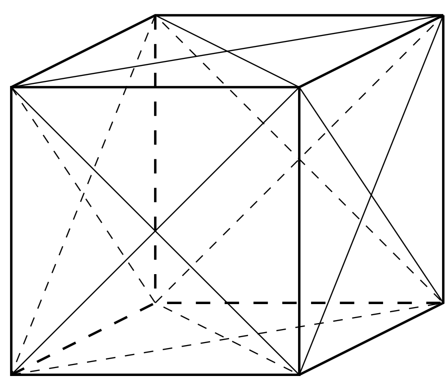
    (b) Is there a closed path that visits every city exactly twice?

    (c) How about a path, not necessary closed, that starts in the upper left corner and visits every city exactly once?

    (d) Finally, is there a path, not necessarily closed, that starts in the upper left corner and visits every city exactly twice?



19. (a) The vertices and midpoints of the faces are marked on a cube, and all face diagonals are drawn. Prove that there is no path along the face diagonals that visits each marked point exactly once.



(b) Show that if one walk along an edge is allowed, then there is a path visiting all the marked points. (Find such a path.)

20. Show that if $4 \times 1 \times 1$ bricks and $2 \times 2 \times 2$ cubes fill (without overlap) an $8 \times 8 \times 8$ cube, then the number of $2 \times 2 \times 2$ cubes is even.

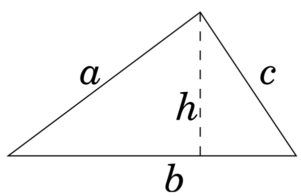
21. Prove that an 8×8 board cannot be covered by 7 T-tetrominoes and 9 L-tetrominoes.

Chapter 11

Areas and Volumes

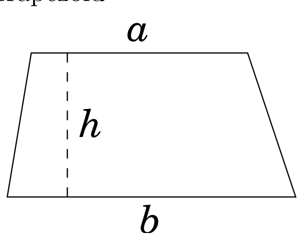
Recall the following area and volume formulas:

1. Triangle



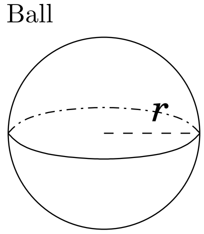
Area $A = \frac{1}{2}bh = \sqrt{\frac{p}{2}\left(\frac{p}{2}-a\right)\left(\frac{p}{2}-b\right)\left(\frac{p}{2}-c\right)}$
where $p = a + b + c$

2. Trapezoid



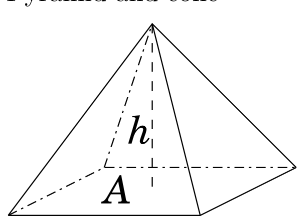
Area $A = \frac{1}{2}(a+b)h$

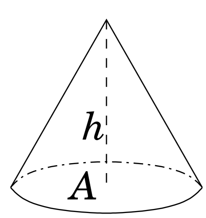
3. Ball



Volume $V = \frac{4}{3}\pi r^3$
where r is the radius

4. Pyramid and cone



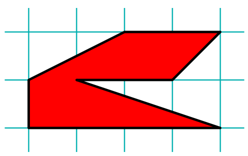


Volume $V = \frac{1}{3}Ah$
where A is the area of the base

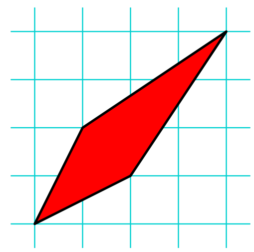
Problems

Find the areas of the following (shaded) regions. If a grid is given, assume each small square is 1×1.

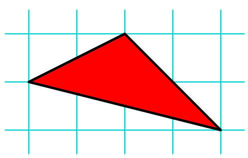
1.



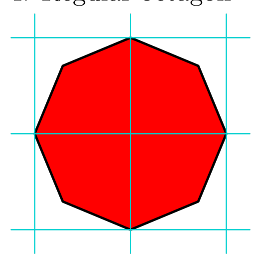
2 .



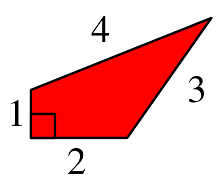
3.



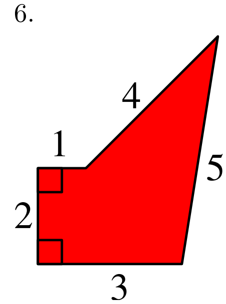
4. Regular octagon



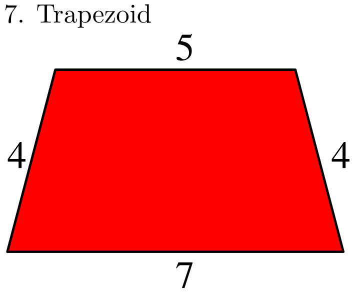
5.



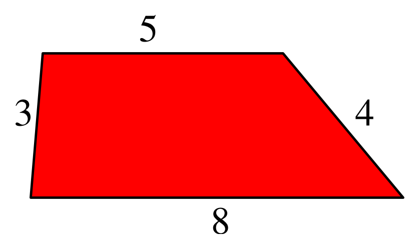
6.



7. Trapezoid

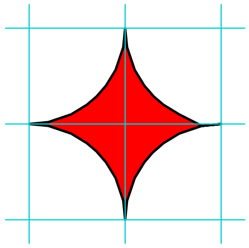


8. Trapezoid

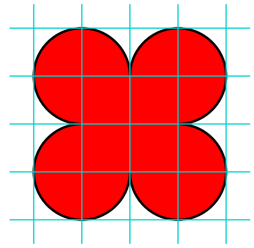


In problems 9 and 10 the curves that appear to be arcs of circles are indeed arcs of circles.

9.

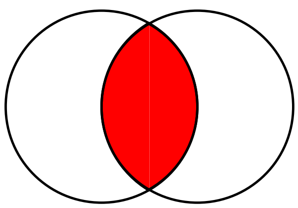


10.

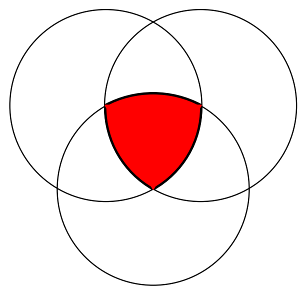


In problems 11-14 each circle has radius 1 and passes through the center of each other circle.

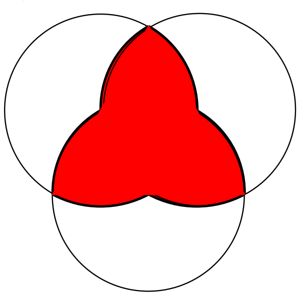
11.



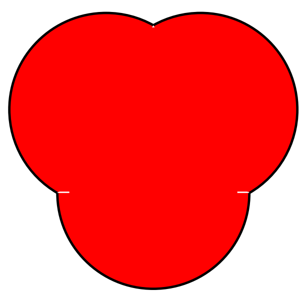
12.



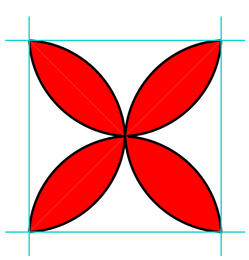
13.



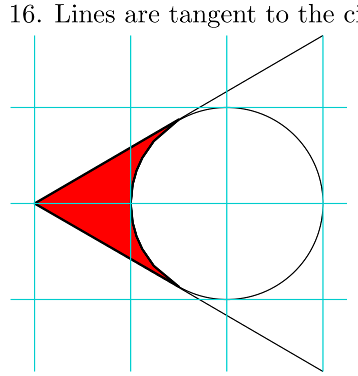
14.



15.



16. Lines are tangent to the circle

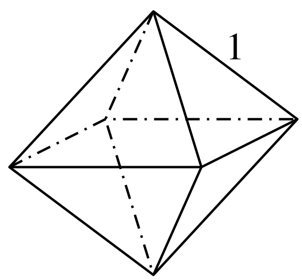


17 An open box is formed from a square of cardboard by cutting a 3 cm square from each corner and folding up the edges to form sides. If the volume of the box is 60 cubic cm, find the dimensions of the box.

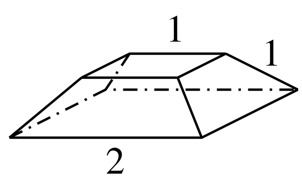
18 What are the dimensions of a cube if the volume is 1728 cubic cm and has a surface area of 888 square cm?

Compute the volumes of the following solids:

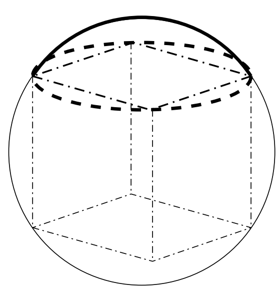
19 Regular octahedron



20 Truncated prism (square 2×2 base and 1×1 top, each side edge of length 1)



21 The solid inside the sphere of radius 1 and above the cube inscribed in it.



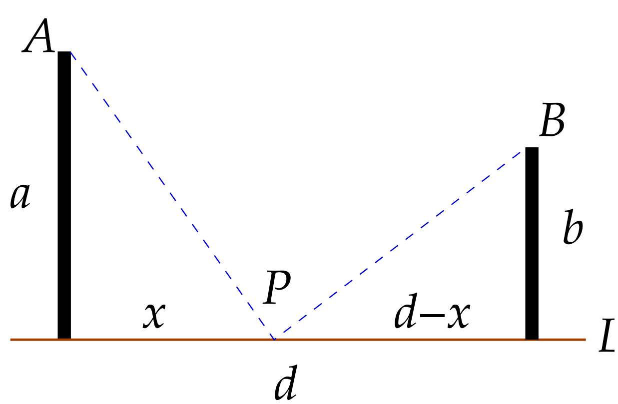
Chapter 12

Symmetry, Translations, Rotations, and Similarity

Some geometry problems can be solved algebraically. Consider the following example.

Example 12.1. There are 2 poles of heights a and b as shown below. The distance between the poles is d. Find the point on the ground equidistant from the tops of the poles.

Note. Depending on our goal, the word "find" here could mean either "calculate the location of that point", e.g. "find the distance between the point and one of the poles", or it could mean "find this point geometrically, using a ruler and a compass".



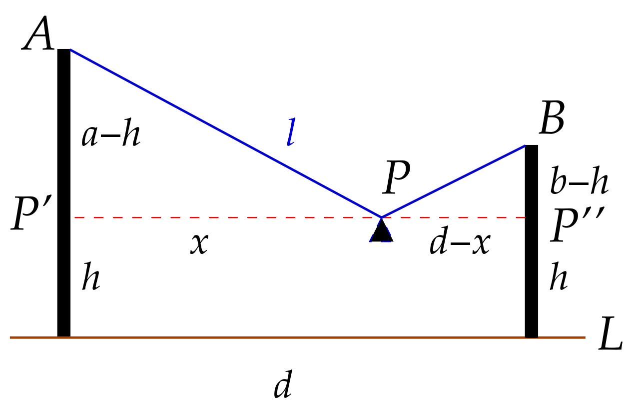
Calculation. *Let x be the distance between one of the poles and the point P we are looking for. Then the distance between the other pole and the point P is $d - x$. We use Pythagorean theorem to compute the distances between P and the tops of the poles A and B, and set the two distances equal:*

$$\sqrt{a^2 + x^2} = \sqrt{b^2 + (d - x)^2}$$
$$a^2 + x^2 = b^2 + (d - x)^2$$
$$a^2 + x^2 = b^2 + d^2 - 2dx + x^2$$
$$a^2 = b^2 + d^2 - 2dx$$
$$2dx = b^2 + d^2 - a^2$$
$$x = \frac{b^2 + d^2 - a^2}{2d}$$

Construction. *Since the point P is equidistant from the tops of the poles, it lies on the perpendicular bisector of AB. Thus all we have to do is to draw the perpendicular bisector of AB, and then P is its intersection with the line L.*

Remark. It is easy to calculate the position of the point P using the above construction. Introduce a coordinate system with the origin at the bottom of one of the poles, write an equation of the line through A and B, then write an equation of the perpendicular bisector, and find its x-intercept. Some problems are easier to solve by a geometric construction and a calculation based on the construction than by equations.

Example 12.2. A rope of length l is strung between the two pole tops and a weight is hung from a ring on the rope, which is not long enough for the weight to reach the ground. How high from the ground does the weight hang?

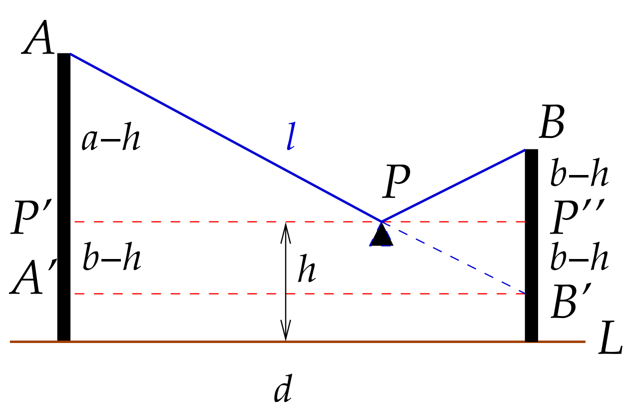


Solution. *Using Pythagorean theorem, we get* $\sqrt{(a-h)^2+x^2}+\sqrt{(b-h)^2+(d-x)^2}=l$.

Using similar triangles APP' and BPP'', we get $\dfrac{a-h}{x}=\dfrac{b-h}{d-x}$.

Thus we have a system of two equations with two unknowns. Although it is possible to solve this system, it is not easy. There is a nicer way to solve this problem.

First find the location of the weight geometrically:

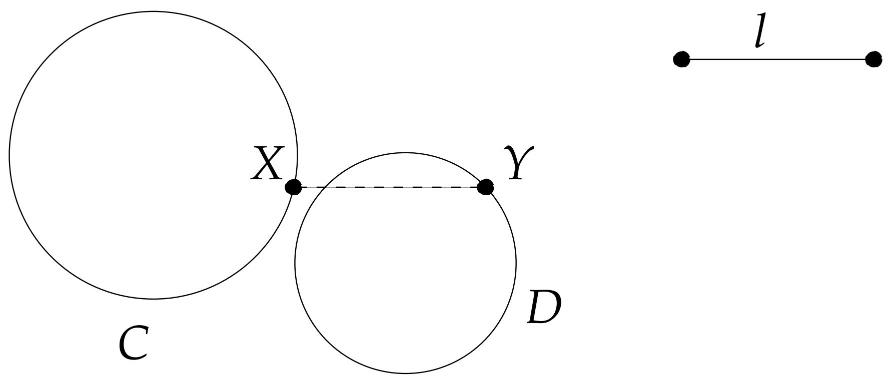


If P is the position of the weight (which we have to find), let $P'P''$ be the horizontal line through P. Reflect B about $P'P''$, let B' denote its image. Then $PB = PB'$, thus $AB' = AP + PB' = AP + PB = l$. Therefore to find B', we have to draw a circle of radius l centered at A, and then B' is its intersection point with the right pole. Once B' is found, *divide BB' into 2 equal intervals with P'' the midpoint. Draw a horizontal line through P''. Its intersection point with AB is the point P.*

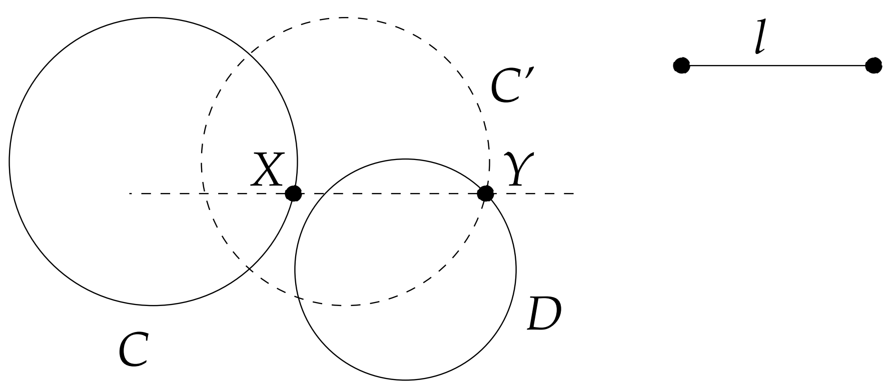
Further, let $A'B'$ be the horizontal line through B'. Look at the triangle $A'B'A$. Since $A'B' = d$, $A'A = a+b-2h$, and $AB' = l$, we have

$$d^2 + (a+b-2h)^2 = l^2$$
$$a+b-2h = \sqrt{l^2-d^2}$$
$$2h = a+b-\sqrt{l^2-d^2}$$
$$h = \frac{a+b-\sqrt{l^2-d^2}}{2}$$

Example 12.3. Two circles C and D, and a distance l ale given. Draw a horizontal segment XY of length l such that X lies on C and Y lies on D. (Assume that such a segment exists.)



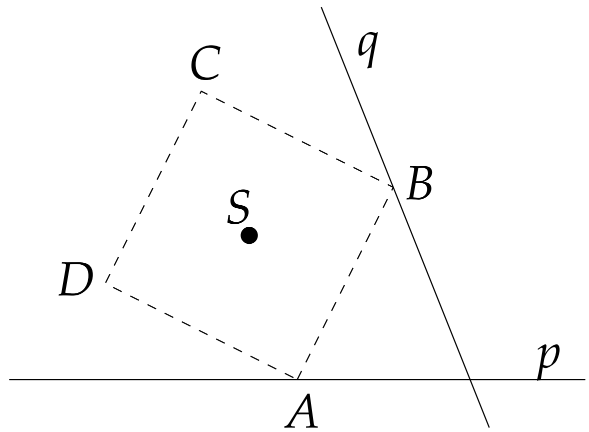
Solution. *Translate the circle C by the distance l to the right. Let's call this new circle C'. Let Y be an intersection point of C' and D if it exists. Draw a horizontal line through Y. Then X is one of the intersection points of this line and the original circle C:*



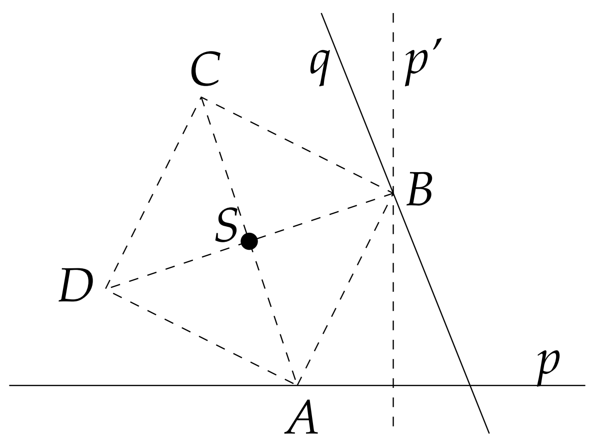
Note. *If C' and D do not intersect, translate C' to the left instead of to the right.*

Example 12.4. Two distinct lines p and q are given, and a point S. Draw a square $ABCD$ that satisfies the following conditions:

- Point S is the center of the square.
- The vertex A of the square lies on the line p.
- The vertex B, the counterclockwise neighbor-vertex of A, lies on the line q.



Solution. *Notice that the segments SA and SB must be perpendicular and of the same length. So rotate the line p through an angle of 90 degrees in the counterclockwise direction around the point S. Let p' be the new line. Let B be the intersection point of p' and q. Once we have one vertex and the center, it's easy: draw the line BS, find D such that $SD = SB$. Draw the line through S perpendicular to SB, find A and C.*



Problems

In all the problems below, "find" means "construct", or "draw". You do not have to calculate the locations of all the points. Assume that solutions exist.

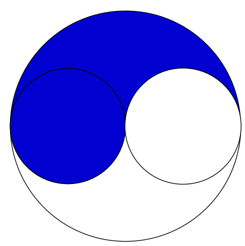
1. Two circles and a line are given. Suppose that none of them intersect. Find a point A on the first circle, a point B on the line, and a point C on the second circle such that $AB + BC$ is a minimum.
2. Show that among all rectangles with given perimeter, the square has the maximal area.
3. A circle, a line, and a distance l are given. Find a point X on the circle, and a point Y on the line, such that the segment XY is horizontal and has length l.
4. A farmer has 2400 ft of fencing and wants to fence off a rectangular field that borders a straight river. He needs no fence along the river. What are the dimensions of the field that has the largest area?

5. Two distinct lines p and q are given, and a point S. Draw a square $ABCD$ that satisfies the following conditions:

   - Point S is the center of the square.
   - The vertex A of the square lies on the line p.
   - The vertex C, the opposite of the vertex of A, lies on the line q.

6. Two lines, p and q, and a point A are given. Draw a square $ABCD$ that satisfies the following conditions:

   - The vertex A is the given point.
   - The vertex B, the counterclockwise neighbor-vertex of A, lies on the line p.
   - The vertex C, the counterclockwise neighbor-vertex of B, lies on the line q.



7. Two circles are given. Draw a line that is tangent to both circles and such that the circles lie on opposite sides of the line.

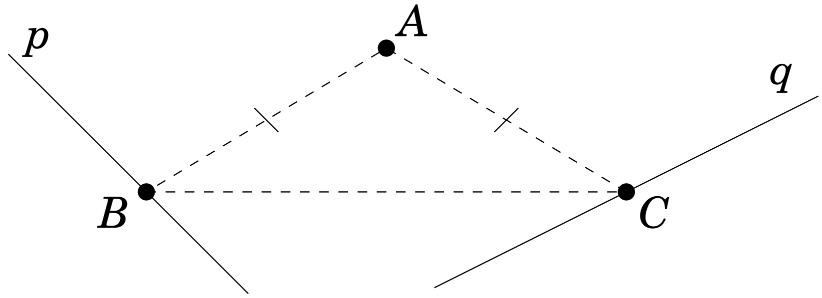
8. The shaded region in the figure is bounded by three semi-circles. Cut this region into four congruent parts, i.e. parts of equal size and shape.

9. A point A and two lines, p and q, are given. Find a point B on the line p, and a point C on the line q, such that the perimeter of the triangle ABC is a minimum.

10. (a) Two points A and C, and a line are given. Find a point B on the line such that $AB + BC$ is a minimum.

    (b) A circle, a line, and a point C are given. Find a point A on the circle and a point B on the line such that $AB + BC$ is a minimum.

11. Given a point A, and two lines p and q, find a point B on p and a point C on q such that the triangle ABC is isosceles with $AB = AC$, and the base BC is horizontal. Assume that a solution exists.



12. Two circles with centers A and D are given. Find a point C on the first circle, and a point B on the second circle, such that AB is horizontal and $AB + BC + CD$ is a minimum.

13. Two circles, S and T, and a point A are given. Find points B on S and C on T such that $\triangle ABC$ is isosceles with $AB = AC$, $\angle ABC = \angle ACB = 75°$, and $\angle BAC = 30°$. Assume that a solution exists.

14. Two circles are given. Draw a line that is tangent to both circles and such that both circles lie on one side of the line.

15. Two lines, p and q, and a point A are given. Find points B on p and C on q such that $\triangle ABC$ is isosceles with $AB = BC$, and $\angle ABC = 90°$. Assume that a solution exists.

16. Four lines, p, q, r, and s, and a distance l are given. Construct a horizontal line that intersects these lines at points A, B, C, and D respectively, and such that $AB + CD = l$.

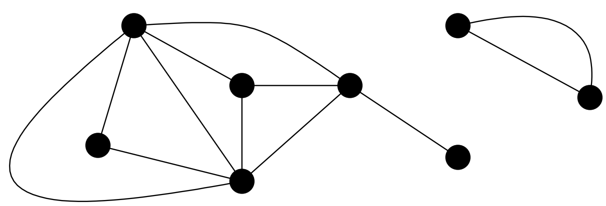
17. A length l is given, and lines p and q which intersect at an angle of $30°$. The line q is horizontal. Find points A on p and B on q such that
(1) AB is vertical (and thus $\angle CBA = 90°$ and $\angle CAB = 60°$), and
(2) The length of the bisector AD of $\angle CAB$ is equal to l.

18. Three lines are given. Find three points on these lines, one point on each line, that are vertices of an equilateral triangle. (Hint: you can choose any point on the first line as one of the vertices.)

Chapter 13

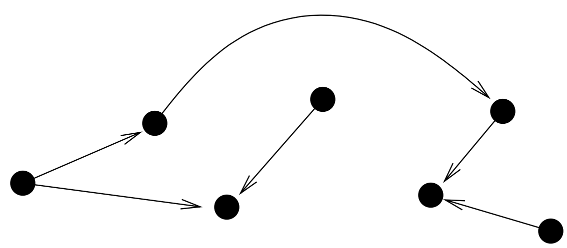
Graphs

Definition 13.1. A **graph** is an object consisting of a set of points called **vertices**, some of which are connected by lines (or arcs) called **edges**.



Definition 13.2. A graph is **simple** if any 2 vertices are connected by at most one edge and there are no loops (edges starting and ending at the same vertex).

Definition 13.3. If the edges are oriented, then we have an **oriented** or **directed** graph. An example of an oriented graph is a one-way road system.

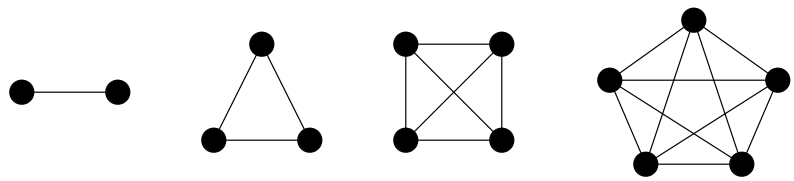


Definition 13.4. If an edge e connects the vertices v_1 and v_2, then we say that v_1 and v_2 are the **endpoints** of e. Also, we say that v_1 and v_2 are **adjacent vertices**. If two edges e_1 and e_2 share a common vertex, then we say that e_1 and e_2 are **adjacent edges**. A vertex v has **degree** m if m endpoints of edges coincide with v (a loop contributes 2 to the degree of a vertex).

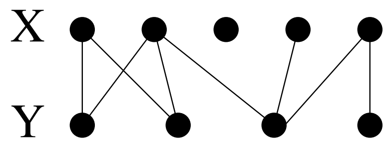
Theorem 13.5. *In any graph, the sum of the degrees of the vertices equals twice the number of the edges.*

Corollary 13.6. *In any graph, the number of vertices with odd degrees is even.*

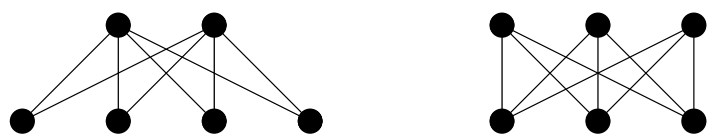
Definition 13.7. An undirected graph in which every two vertices are connected is called a **complete** graph. K_n denotes the complete graph with n vertices. The graphs K_2, K_3, K_4, and K_5 are shown below.



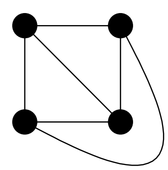
Definition 13.8. If the vertices of a graph can be separated into two parts X and Y so that for every edge in the graph, one of its endpoints belongs to X and the other belongs to Y, then we call this kind of graph a **bipartite** graph.



Definition 13.9. If every vertex in the set X is connected to every vertex in the set Y, then the graph is called a complete bipartite graph. $K_{m,n}$ denotes the complete bipartite graph with m vertices in the set X and n vertices in the set Y. The graphs $K_{2,4}$ and $K_{3,3}$ are shown below.



Definition 13.10. We say that a graph can be **embedded** into a plane if it is possible to draw it in such a way that no two edges intersect. For example, the graph K_4 can be embedded as follows:



Theorem 13.11. *The graphs K_5 and $K_{3,3}$ can not be embedded into a plane.*

Definition 13.12. A **path** is a sequence of edges e_1, e_2, …, e_n such that $e_1 = (x_0, x_1)$, $e_2 = (x_1, x_2)$, …, $e_n = (x_{n-1}, x_n)$. When there are no multiple edges in the graph, this path is denoted by its vertex sequence $x_0,\ x_1,\ \ldots, x_n$. A path that begins and ends at the same vertex is called a **cycle**. A path is **simple** if it does not contain the same edge more than once.

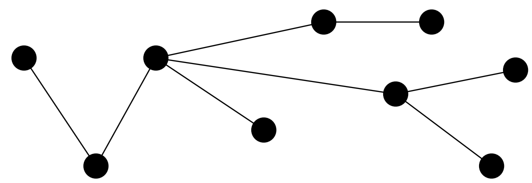
Definition 13.13. An **Euler path (resp. Euler cycle)** is a simple path (resp. cycle) containing every edge of the graph.

Theorem 13.14. *A connected graph has an Euler cycle if and only if each of its vertices has even degree.*

Definition 13.15. A **Hamilton path (resp. Hamilton cycle)** is a simple path (resp. cycle) visiting every vertex exactly once.

Definition 13.16. If all vertices of a graph can be visited by walking on edges, the graph is **connected**.

Definition 13.17. A connected graph without cycles is called a **tree**.
Here is an example of a tree:

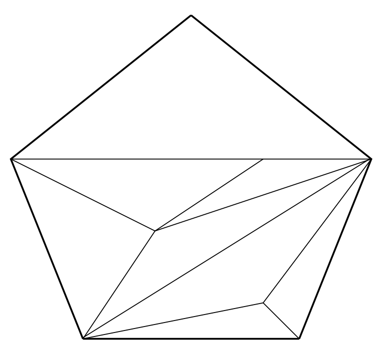


Example 13.18. Prove that in any collection of six people either three of them mutually know each other or three of them mutually do not know each other.

Solution. *Let's translate this problem into a graph theory problem. Let six vertices a, b, c, d, e, and f represent the six people. If two people know each other, then we use a red edge to join these two vertices. If two people do not know eath other, then we use a blue edge to join these two vertices. Since there are edges between every two vertices in the graph, it's a complete graph K_6 with red and/or blue edges. Now the problem has been translated into the following problem: We use red red/or blue colors to color the edges in the complete graph K_6. Prove that there must exist either 3 vertices such that the edges joining them are all red, or 3 vertices such that the edges joining them are all blue. Now, let's pick any vertex in K_6, say a. The 5 edges between this vertex and the other 5 vertices are each either red or blue. According to Dirichlet's Principle, at least 3 edges of the five have the same color. Let's assume that ab, ac, ad are red (the blue case is similar). Now consider the triangle bcd. If one of the edges bc, bd, cd is red, then we have a red triangle. Otherwise, if bc, bd, cd are all blue, then the triangle bcd is a blue triangle. This proves that there must exist a triangle all of whose edges are colored by the same color.*

Example 13.19. Is it possible to draw a triangular map inside a pentagon so that the degree of each vertex is even?

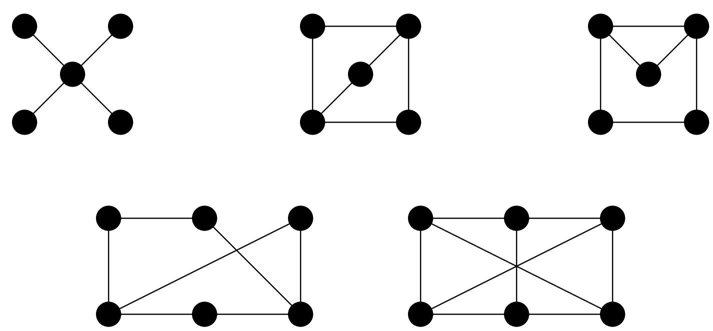
Below is an example of a triangular map (but some vertices have an odd degree):



Solution. *The answer is no. We will prove this by contradiction. Suppose such a map exists. We know (see problem 9 in chapter 4) that every map with all vertices of even degree admits a proper coloring, i.e. its regions can be colored with 2 colors so that no neighbouring regions have the same color. Color our map in blue and red so that the (infinite) region outside of the pentagon is blue. All the other regions are triangles. Each edge has a red triangle on one side and and a blue region (either a triangle or that infinite outside region) on the other side. Now, count the number of edges (boundaries) in the map in two ways: each red triangle has 3 sides, so the number of edges is a multiple of 3, say, $3n$. Each blue triangle has 3 sides, and the infinite region has 5 edges, so the number of edges is a multiple of 3 plus 5, say, $3m + 5$. Thus we have $3n = 3m + 5$. But this is impossible.*

Problems

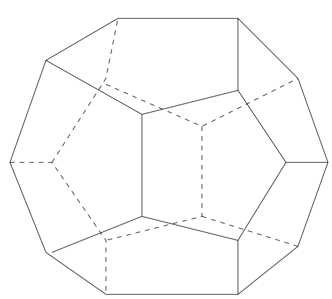
1. Explain why a graph can not have 7 vertices of degrees 4, 4, 3, 3, 3, 2, 2.
2. Can a graph have 6 vertices of degrees 4, 3, 3, 2, 2, and 1?
3. Prove that in any group of people, the number of people that are friends with an odd number of people is even.
4. How many edges does a graph have if it has vertices of degrees 4, 3, 3, 2, 2? Draw such a graph.
5. Determine which of the following graphs are bipartite:



6. Which of the graphs in problem 5 have
   (a) an Euler path?
   (b) an Euler cycle?
   (c) a Hamilton path?
   (d) a Hamilton cycle?
7. There are 8 counties in Sikinia. There are no "four corners" points (like Arizona, Colorado, New Mexico, and Utah). Each county counted the number of neighboring counties. The numbers are 5, 5, 4, 4, 4, 4, 4, 3. Prove that at least one county made a mistake.
8. Find the number of vertices and edges in K_n and $K_{n,m}$.
9. Find a necessary and sufficient condition for a graph to have an Euler path but not an Euler cycle.
10. For which values of n does K_n have
    (a) an Euler path?
    (b) a Hamilton path?
11. For what values of n and m does $K_{n,m}$ have
    (a) an Euler cycle?
    (b) an Euler path?
    (c) a Hamilton cycle?

(d) a Hamilton path?

12. A knight's tour is a sequence of legal moves by a knight starting at some square of a chessboard and visiting each square exactly once. A knight's tour is called reentrant if there is a legal move that takes the knight from the last square of the tour back to where the tour began.
    (a) Draw the graph that represents the legal moves of a knight on a 3×4 chessboard.
    (b) Show that there is no reentrant tour on a 3×4 chessboard.
    (c) Find a non-reentrant tour on a 3×4 chessboard.
13. Show that there is no reentrant knight's tour on a 4×4 chessboard.
14. Show that there is no knight's tour at all (reentrant or not) on a 4×4 chessboard.
15. There are 7 men and 7 women attending a dance. After the dance, they recall the number of people they have danced with. The numbers are as follows: 3, 3, 3, 3, 3, 3, 3, 5, 6, 6, 6, 6, 6, 6. Prove that at least one of them made a mistake. (Assume that men only danced with women, and women only danced with men.)
16. There are 10 men and 10 women at a dance. Every man knows exactly 2 women and every woman knows exactly 2 men. Prove that after suitable pairing, every man can dance with a woman he knows.
17. There are 17 scientists who communicate with each other to discuss some problems. They discuss only three topics, and each pair discusses at least one of these three. Prove that there are at least 3 scientists who are all pairwise discussing the same topic.
18. Nine mathematicians met at an international conference. They found that among any 3 of them there are at least 2 that have a language in common. If every mathematician speaks at most 3 languages, prove that at least three of the mathematicians can speak the same language.
19. Hamilton's "Round the World" puzzle: does the dodecahedron (shown below) have
    (a) a Hamilton path?
    (b) a Hamilton cycle?



20. (a) Prove that in a finite simple graph having at least 2 vertices there are always two vertices with the same degree.
    (b) Does the above hold for graphs with loops (but no multiple edges)?
    (c) Does the above hold for graphs with multiple edges (but no loops)?

21. A connected bipartite graph G has 8 vertices. Recall that the vertices of a bipartite graph can be divided into 2 groups A and B so that every edge connects a vertex in group A and a vertex in group B. Both groups for G have 4 vertices. Three of the vertices in group A have degrees 4, 2, and 2. Three of the vertices in B have degrees 3, 1, and 1. What are the degrees of the remaining vertices?

Chapter 14

Working backwards

"Working backwards" is a very powerful tool that can be used to solve many different problems.

Euclid's algorithm. Given numbers a and b, notice that if we divide a by b and obtain quotient q and remainder r, then since $a = qb + r$, the greatest common divisor of a and b is equal to the greatest common divisor of b and r. Euclid's algorithm is based on this fact:

$$\begin{array}{llll|l}
a = q_1 \cdot b + r_1, & r_1 < b, & (a,b) = (b, r_1) & & r_1 = a - q_1 \cdot b \\
b = q_2 \cdot r_1 + r_2, & r_2 < r_1, & (b, r_1) = (r_1, r_2) & & r_2 = b - q_2 \cdot r_1 \\
r_1 = q_3 \cdot r_2 + r_3, & r_3 < r_2, & (r_1, r_2) = (r_2, r_3) & & r_3 = r_1 - q_3 \cdot r_2 \\
\dots \downarrow & \dots & \dots & & \dots \uparrow \\
r_{n-2} = q_n \cdot r_{n-1} + r_n, & r_n < r_{n-1}, & (r_{n-2}, r_{n-1}) = (r_{n-1}, r_n) & & r_n = r_{n-2} - q_n \cdot r_{n-1} \\
r_{n-1} = q_{n+1} \cdot r_n, & \text{rem.} = 0, & (r_{n-1}, r_n) = r_n & &
\end{array}$$

Thus $(a, b) = r_n$.

Theorem 14.1. *If $d = (a, b)$, then there exist integer numbers x and y such that $x \cdot a + y \cdot b = d$.*

Example 14.2. Find the greatest common divisor d of $a = 115$ and $b = 80$, and find x and y such that $x \cdot a + y \cdot b = d$.

Solution. $115 = 1 \cdot 80 + 35$

$80 = 2 \cdot 35 + 10$

$35 = 3 \cdot 10 + 5$

$10 = 2 \cdot 5$

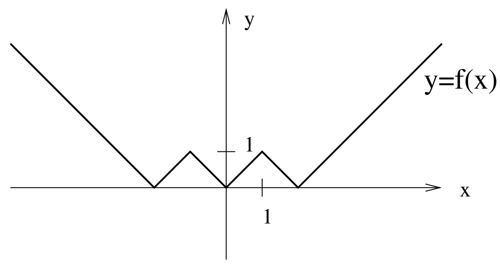
Therefore $(a, b) = 5$.

Note. To find (a, b), we could factor $115 = 5 \cdot 23$, $80 = 2^4 \cdot 5$, so $(115, 80) = 5$.

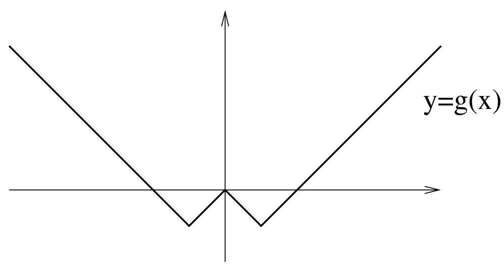
$$\begin{aligned}
5 &= 35 - 3 \cdot 10 \\
&= 35 - 3(80 - 2 \cdot 35) = 35 - 3 \cdot 80 + 6 \cdot 35 = 7 \cdot 35 - 3 \cdot 80 \\
&= 7(115 - 1 \cdot 80) - 3 \cdot 80 = 7 \cdot 115 - 7 \cdot 80 - 3 \cdot 80 = 7 \cdot 115 - 10 \cdot 80
\end{aligned}$$

Thus $x = 7$ and $y = -10$.

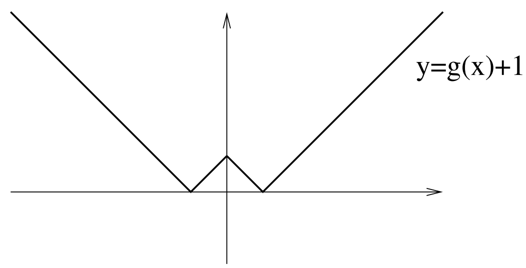
Example 14.3. Find a formula for the function whose graph is shown below.



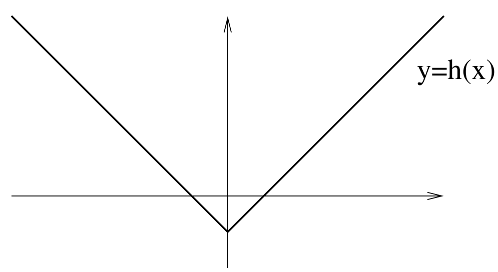
Solution. *Let $f(x)$ denote the function that we want to find. Notice that $f(x)$ is the absolute value of the function $g(x)$ whose graph is*



Record this fact: $f(x) = |g(x)|$. Here is the graph of $g(x) + 1$:



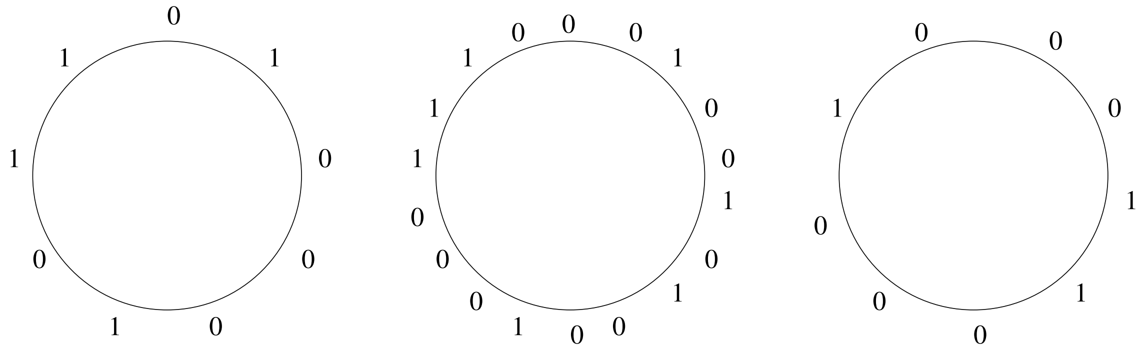
Notice that $g(x) + 1$ is the absolute value of $h(x)$ whose graph is



(So, $g(x) + 1 = |h(x)|$.) Finally, the graph of $h(x)$ is obtained from the graph of the absolute value of x by shifting it downward a distance of 1 unit, so $h(x) = |x| - 1$. Now, $g(x) + 1 = |h(x)| = ||x| - 1|$, so $g(x) = ||x| - 1| - 1$, and $f(x) = |g(x)| = |||x| - 1| - 1|$.

Example 14.4. 4 ones and 5 zeros are written along a circle. Between two equal numbers we write a one and between two distinct numbers we write a zero. Then the original numbers are wiped out. This step is repeated. Show that we can never reach 9 ones.

For example, a possible initial distribution of ones and zeros and the first step are shown below:



Solution. *Suppose the aim is attainable. Look at the first time we have 9 ones. One step before we must have 9 equal numbers. Since it was the first time we got 9 ones, one step before we must have 9 zeros. Still one step before we have 9 changes* $0-1-0-1-\ldots$*. With an odd number of integers (9), this is not possible.*

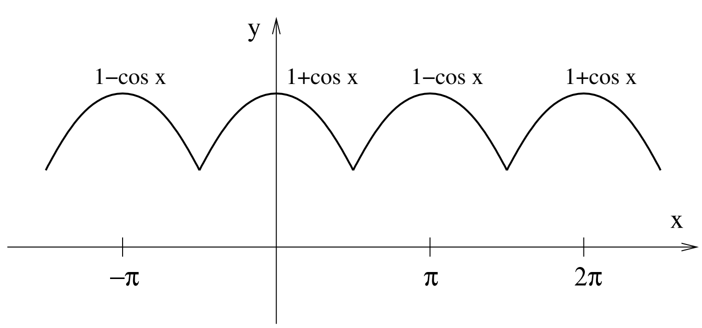
Problems

In problems 1-4, two numbers a and b are given. Use Euclid's algorithm to find their greatest common divisor $d=(a,b)$, and numbers x and y such that $xa+yb=d$.

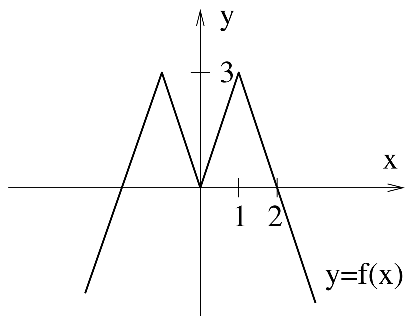
1. $a=46$, $b=32$
2. $a=24$, $b=10$
3. $a=96$, $b=54$
4. $a=219$, $b=51$
5. Find integer numbers a and b such that $6=67a+25b$.
6. Find a and b such that it will take 5 divisions to reach the greatest common divisor of a and b.
7. Find a and b such that in Euclid's algorithm $r_7=(a,b)$. Write out all the divisions.

In problems 8-12, find a formula for the function whose graph is shown.

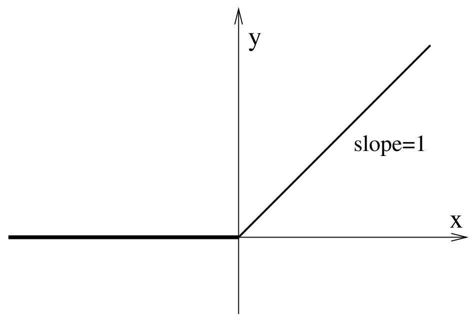
8. .



9. .

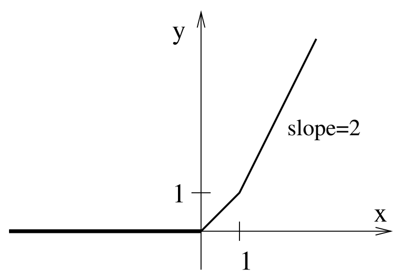


10. .



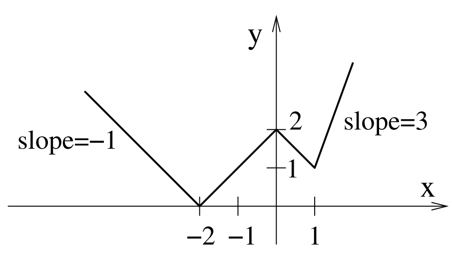
Hint: try subtracting $\dfrac{x}{2}$ from the given function.

11. .



Hint: do problem 10 first.

12. .



13. Starting with 2, 0, 0, 3, we construct the sequence 2, 0, 0, 3, 5, 8, 6, ..., where each new digit is the mod 10 sum of the preceding four terms. Prove that the 4-tuple 0, 5, 0, 5 will never occur.

14. Starting with 2, 0, 0, 3, we construct the sequence 2, 0, 0, 3, 5, 8, 6, ..., where each new digit is the mod 10 sum of the preceding four terms. Will the 4-tuple 0, 4, 0, 7 ever occur?

15. Two players play the following game.
    - Turns alternate.
    - At each turn, a player removes 1, 2, 3, or 4 counters from a pile that had initially 27 counters.
    - The game ends when all counters have been removed.
    - The player who takes the last counter loses.

    Find a winning strategy for one of the players.

16. Two players play the following game.
    - Turns alternate.
    - At each turn, a player removes 1, 2, 3, or 4 counters from a pile that had initially 27 counters.
    - The game ends when all counters have been removed.
    - The player who takes the last counter wins.

    Find a winning strategy for one of the players.

17. Two players play the following game.
    - Turns alternate.
    - At each turn, a player removes either 1 or 2 counters from a pile that had initially 10 counters.
    - The game ends when all counters have been removed.
    - The player who takes the last counter loses.

    Find a winning strategy for one of the players.

18. There are two piles of candy. One pile contains 20 pieces, and the other 21. Players take turns eating all the candy in one pile and separating the remaining candy into two (not necessarily equal) piles. (A pile may have 0 candies in it.) The player who cannot eat a candy on his/her turn loses. Which player, if either, can guarantee victory in this game?

19. Two players play the following game.
    - Turns alternate.
    - At each turn, a player removes 1, 2, 4, 8, 16, or 32 counters from a pile that had initially 50 counters.
    - The game ends when all counters have been removed.
    - The player who takes the last counter wins.

    Find a winning strategy for one of the players.

20. Starting from 1, the players take turns multiplying the current number by any whole number from 2 to 9 (inclusive). The player who first names a number greater than 1000 wins. Which player, if either, can guarantee victory in this game?

21. Suppose you are writing a calculus book. You want to find a few cubic polynomials $f(x) = ax^3 + bx^2 + cx + d$ (preferably with integer coefficients) whose critical numbers are integers. (Recall that a critical number is a value of x at which the derivative is equal to 0.) How would you find such polynomials? Use your strategy to find a couple of polynomials.

22. Suppose you want to give your high school students a system of 2 linear equations with 2 variables. You'd like the answers to be integer numbers. You could, of course, try random coefficients, say

$$\begin{cases} 2x + 3y = 4 \\ 5x - 6y = 7 \end{cases},$$

solve your systems, and hope that sooner or later you'll find a system with integer solutions, but is there a better strategy?

23. Suppose you are teaching linear algebra, and you need to find matrices with integer entries whose reduced echelon forms also have integer entries. How would you find such matrices?

24. The integers 1, 2, ..., n are placed in order, so that each value is either bigger than all preceding values or is smaller than all preceding values. In how many ways can this be done?

25. I have seven coins whose total value is $0.57. What coins do I have? And, how many of each coin do I have? (Coins being used at the time when this book is written have values 1 cent, 5 cents, 10 cents, 25 cents, and 1 dollar.)

Chapter 15

Calculus

Recall the following important definitions and theorems.

Definition 15.1. $\log_a x = y \quad \Leftrightarrow \quad a^y = x$

Theorem 15.2. *(Properties of logarithms)*

1. $\log_a(xy) = \log_a x + \log_a y$
2. $\log_a\left(\frac{x}{y}\right) = \log_a x - \log_a y$
3. $\log_a(x^r) = r \log_a x$
4. $\log_a(x) = \frac{\ln x}{\ln a}$

Definition 15.3. A function $f(x)$ is called **even** if $f(-x) = f(x)$ for all x in the domain of f.
A function $f(x)$ is called **odd** if $f(-x) = -f(x)$ for all x in the domain of f.

Definition 15.4. A function f^{-1} is called the **inverse** of f if

$$f^{-1}(y) = x \quad \Leftrightarrow \quad f(x) = y.$$

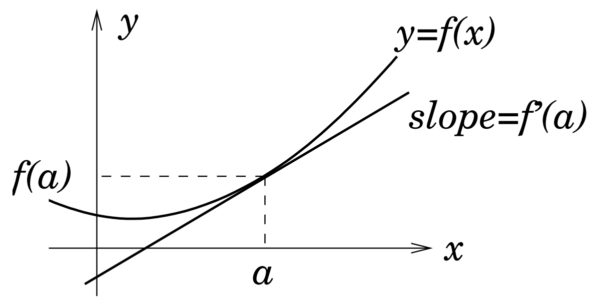
Theorem 15.5. *If f^{-1} is the inverse of $f : \mathbb{R} \to \mathbb{R}$ then the curves $y = f(x)$ and $y = f^{-1}(x)$ are symmetric about the line $y = x$.*

Theorem 15.6. *(Intermediate value theorem) Suppose $f(x)$ is continuous on $[a, b]$. Let N be any number between $f(a)$ and $f(b)$. Then there exists $c \in [a, b]$ such that $f(c) = N$.*

Definition 15.7. The derivative of $f(x)$ at a point a is

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}.$$

The derivative $f'(a)$ is the slope of the tangent line to $y = f(x)$ at $(a, f(a))$. Also, $f'(a)$ is the rate of change of $f(x)$ with respect to x at $x = a$.



Theorem 15.8. *(Important derivatives)*

$$(x^n)' = nx^{n-1}, \qquad (e^x)' = e^x, \qquad (a^x)' = (\ln a)a^x,$$

$$(c)' = 0, \qquad (\ln x)' = \frac{1}{x}, \qquad (\log_a x)' = \frac{1}{(\ln a)x},$$

$$(\sin x)' = \cos x, \qquad (\cos x)' = -\sin x, \qquad (\tan x)' = (\sec x)^2,$$

$$(\csc x)' = -\csc x \cot x, \qquad (\sec x)' = \sec x \tan x, \qquad (\cot x)' = -(\csc x)^2,$$

$$(\arcsin x)' = \frac{1}{\sqrt{1-x^2}} \qquad (\arccos x)' = -\frac{1}{\sqrt{1-x^2}}, \qquad (\arctan x)' = \frac{1}{x^2+1}$$

Theorem 15.9. *(Chain rule)* $(f \circ g)'(x) = (f(g(x)))' = f'(g(x))g'(x)$

Theorem 15.10. *If $f(x)$ is defined on some open interval containing a point c and has a local maximum or minimum c, then c is a critical number of $f(x)$ (i.e. either $f'(c) = 0$ or $f'(c)$ does not exist).*

Definition 15.11. Let $f(x)$ be continuous on an interval $[a, b]$. Divide the interval into n subintervals of equal length: $[x_0, x_1]$, $[x_1, x_2]$, …, $[x_{n-1}, x_n]$ where $x_0 = a$ and $x_n = b$. Let $\Delta x = \dfrac{b-a}{n}$ be the length of each subinterval. Then the sum

$$R_n = \sum_{i=1}^{n} f(x_i)\Delta x$$

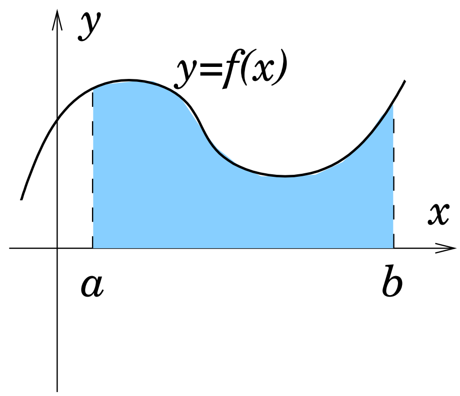
is called the **Riemann sum** of $f[x]$ on $[a, b]$ using n subintervals. It can be proved that the limit of R_n as n approaches infinity exists, and

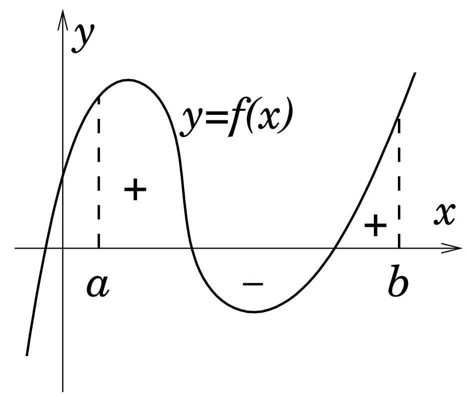
$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$$

is called the integral of $f(x)$ from a to b.

If $f(x) \geq 0$, then $\displaystyle\int_a^b f(x)dx$ is the area of the region under the curve $y = f(x)$ and above the x-axis from a to b.

If $f(x)$ takes on both positive and negative values, then $\displaystyle\int_a^b f(x)dx$ is the sum of the areas under the curve and above the x-axis minus the sum of the areas under the x-axis and above the curve.





Theorem 15.12. *(Fundamental Theorem of Calculus)*

I. $\dfrac{d}{dx}\left(\int_a^x f(t)dt\right) = f(x)$

II. If $F'(x) = f(x)$, *then* $\int_a^b f(x) = F(b) - F(a)$.

Theorem 15.13. *(Substitution Rule)*
$\int f(g(x))g'(x)dx = \int f(u)du$ *where* $u = g(x)$, $du = g'(x)dx$.

Theorem 15.14. *(Some important series)*
$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots$ *is divergent.*
$\sum_{n=0}^{\infty} q^n = 1 + q + q^2 + q^3 + \ldots = \frac{1}{1-q}$ *if* $|q| < 1$, *and divergent if* $|q| \geq 1$.
$\sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots = e^x$ *for all* x.
(*in particular, if* $x = 1$, *then* $\sum_{n=0}^{\infty} \frac{1}{n!} = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \ldots = e$.)
$\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots = \arctan x$ *for all* x.
(*in particular, if* $x = 1$, *then* $\sum_{n=0}^{\infty} (-1)^n \frac{1}{2n+1} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots = \arctan 1 = \frac{\pi}{4}$.)

Problems

1. Evaluate the integral $\int_{-4}^{2} |x+2|dx$.

2. Evaluate the integral $\int_{0}^{3\pi} |\sin x|dx$.

3. Find a number c such that the line $y = x - 1$ is tangent to the parabola $y = cx^2$.

4. (a) Show that the function $f(x) = \ln(x + \sqrt{x^2 + 1})$ is odd.
   (b) Find the inverse of $f(x)$.

5. Find a cubic polynomial $p(x) = ax^3 + bx^2 + cx + d$ that has a local maximum at $(0, 1)$ and a local minimum at $(1, 0)$.

6. Find the interval $[a, b]$ for which the value of the integral $\displaystyle\int_a^b (2+x-x^2)dx$ is a maximum.

7. Find all values of a for which the area of the region bounded by the line $y = ax$ and the parabola $y = x^2$ is equal to 1.

8. There is a line through the origin that divides the region bounded by the parabola $y = x - x^2$ and the x-axis into two regions with equal area. What is the slope of that line?

9. Find the sum of the series $\displaystyle\sum_{n=0}^{\infty} \frac{1}{2^{2n+1}} = \frac{1}{2} + \frac{1}{2^3} + \frac{1}{2^5} + \frac{1}{2^7} + \ldots$

   Hint: e.g. factor out $\frac{1}{2}$, and notice that $2^{2n} = 4^n$.

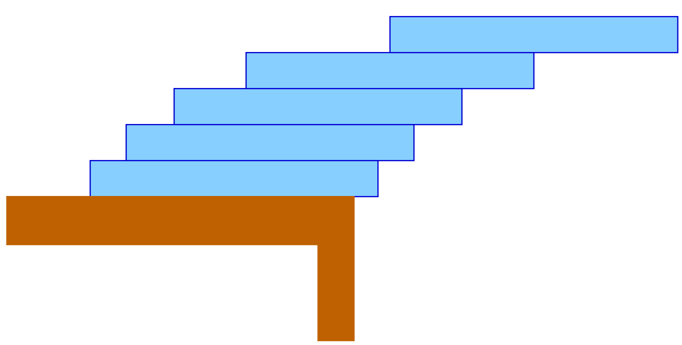
10. Find the sum of the series

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{6} + \frac{1}{8} + \frac{1}{9} + \frac{1}{12} + \frac{1}{16} + \frac{1}{18} + \ldots$$

   where the terms are the reciprocals of the positive integers whose only prime factors are 2s and 3s.

11. The parabola $y = x^2 + 2$ has two tangent lines that pass through the origin. Find their equations.

12. Suppose you have a large supply of books, all the same size, and you stack them at the edge of a table, with each book extending farther beyond the edge of the table than the one beneath it. Show that it is possible to do this so that the top book extends entirely beyond the table. In fact, show that the top book can extend any distance at all beyond the edge of the table if the stack is high enough. Try the following method of stacking: The top book extends half its length beyond the second book. The second book extends a quarter of its length beyond the third. The third extends one-sixth of its length beyond the fourth, and so on. (You could try it yourself with a deck of cards, or with tapes or CDs.) Consider centers of mass.

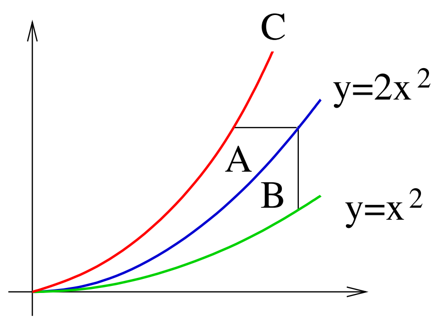


13. Find the n-th derivative of $f(x) = \dfrac{1}{x^2 + x}$.
Hint: use the partial fraction decomposition. Recall that since $x^2 + x = x(x + 1)$, the partial fraction decomposition has the form $\dfrac{A}{x} + \dfrac{B}{x+1}$.

14. Find the n-th derivative of the function $f(x) = \dfrac{x^n}{1 - x}$.

15. The parabola $y = x^2$ and the line $y = mx + 1$ are given. They have two intersection points, A and B. Find the point C on the parabola that maximizes the area of $\triangle ABC$.

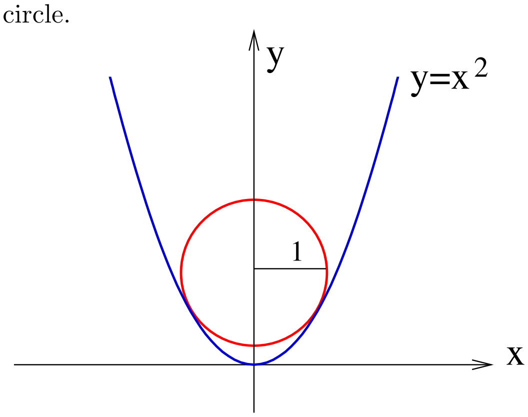
16. The figure below shows a curve C with the property that, for every point P on the middle curve $y = 2x^2$, the areas A and B are equal. Find an equation for C.



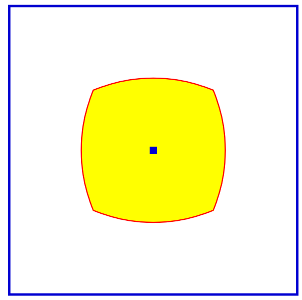
17. Find all values of a such that $x^2 + ax + 1 \geq \cos x$ for all real x.

18. For which positive numbers a is it true that $a^x \geq 1 + x$ for all x?

19. The figure below shows a circle with radius 1 inscribed in the parabola $y = x^2$. Find the center of the circle.



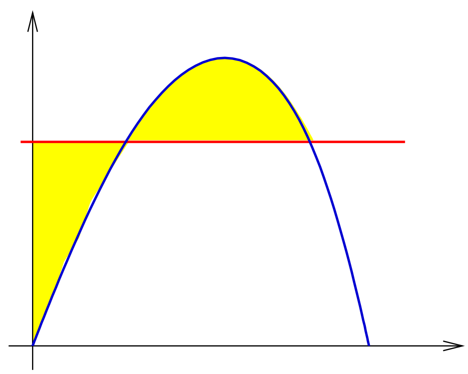
20. The figure below shows a region consisting of all points inside a square that are closer to the center than to the sides of the square. Find the area of the region.



21. Sketch the region $S = \{(x, y) \mid |x| \geq 1,\ |y| \geq 2,\ x^2 + y^2 \leq 9\}$ and find its area.

22. Find a positive continuous function f such that the area under the graph of f from 0 to t is $A(t) = t^3$ for all $t > 0$.

23. The figure below shows a horizontal line $y = c$ intersecting the curve $y = 8x - 27x^3$. Find the number c such that the areas of the shaded regions are equal.



24. Evaluate $\lim_{n\to\infty} \left(\frac{1}{\sqrt{n}\sqrt{n+1}} + \frac{1}{\sqrt{n}\sqrt{n+2}} + \ldots + \frac{1}{\sqrt{n}\sqrt{n+n}} \right)$.
Hint: interpret the sum as a Riemann sum of a function. Then the limit as n approaches infinity is the value of an integral.

25. Show that any ellipsoid (given by $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$) has a section that is a circle.
Hint: any section of the ellipsoid that passes through the origin is an ellipse.

26. Evaluate $\int \frac{1}{x^7 - x} dx$.
The straightforward approach would be to start with partial fractions, but that would be too brutal. We could reduce the power of the denominator as follows:
$\int \frac{1}{x^7 - x} dx = \int \frac{x}{x^8 - x^2} dx$, let $u = x^2$, then $du = 2xdx$, or $\frac{du}{2} = xdx$, and we have $\int \frac{x}{x^8 - x^2} dx = \frac{1}{2} \int \frac{1}{u^4 - u} du$.
$u^4 - u$ is better than $x^7 - x$, but can you find an even better substitution?

27. Let $a_1, a_2, \ldots, a_{30}$ be real numbers. Show that $a_1 \cos x + a_2 \cos(2x) + \ldots + a_{30} \cos(30x)$ cannot take on only positive values.

28. If a_0, a_1, a_2, ..., a_k are real numbers and $a_0 + a_1 + a_2 + \ldots + a_k = 0$, show that

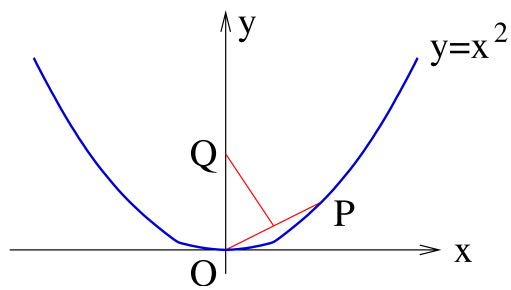
$$\lim_{n\to\infty}(a_0\sqrt{n} + a_1\sqrt{n+1} + a_2\sqrt{n+2} + \ldots a_k\sqrt{n+k}) = 0.$$

Hint: Try the special cases $k = 1$ and $k = 2$ first, and then generalize.

29. Show that, for $x > 0$,

$$\frac{x}{x^2+1} < \arctan x < x.$$

30. The figure below shows a point P on the parabola $y = x^2$ and the point Q where the perpendicular bisector of OP intersects the y-axis. As P approaches the origin along the parabola, what happens to Q? Does it have a limiting position? If so, find it.



31. Recall that the area of a circle with radius r is $A = \pi r^2$ and the circumference of the circle is $L = 2\pi r$. Notice that $(\pi r^2)' = 2\pi r$. Similarly, the volume of a ball with radius r is $V = \frac{4}{3}\pi r^3$, the surface area is $S = 4\pi r^2$, and $\left(\frac{4}{3}\pi r^3\right)' = 4\pi r^2$. Is this a coincidence? Actually, it isn't. Explain these facts. What is the ratio of the 4-dimensional volume and the usual 3-dimensional volume of its boundary (the analog of the surface area) for a 4-dimensional ball with radius 4?

32. Find the volume of a 4-dimensional unit ball.

33. Evaluate $\displaystyle\int_0^1 (\sqrt[3]{1-x^7} - \sqrt[7]{1-x^3})dx$.

34. Show that e is irrational.

35. Let $f(x) = a_1 \sin x + a_2 \sin(2x) + a_3 \sin(3x) + \ldots + a_n \sin(nx)$, where $a_1, \ldots, a_n$ are real numbers and n is a positive integer. If it is given that $|f(x)| \le |\sin(x)|$ for all x, show that $|a_1 + 2a_2 + \ldots + na_n| \le 1$.

36. Let $T(x)$ denote the temperature at the point x on Earth at some fixed time. Assuming that T is a continuous function of x, show that at any fixed time there are at least two diametrically opposite points on the equator that have the same temperature.

37. Find a curve that passes through the point $(3, 2)$ and has the property that if the tangent line is drawn at any point P on the curve, then the part of the tangent line that lies in the first quadrant is bisected by P.

Chapter 16

Various problems

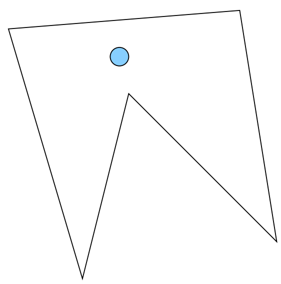
Most problems in this section can be solved in a few different ways.

Problems

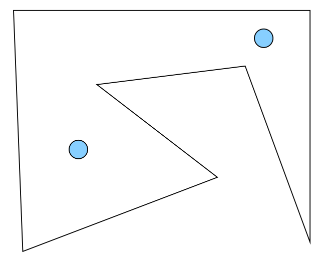
1. Show that there is no reentrant knight's tour on a 5×5 chessboard.
2. Prove that for any integer number n, $n^7 - n$ is divisible by 7.
3. A sequence $\{a_n\}$ is defined recursively by the equations
$$a_0 = a_1 = 1 \qquad n(n-1)a_n = (n-1)(n-2)a_{n-1} - (n-3)a_{n-2}.$$
Find the sum of the series $\displaystyle\sum_{n=0}^{\infty} a_n$.
4. Evaluate the integral: $\displaystyle\int_{-2}^{3} ||x| - 1|\, dx$
5. Solve the inequality: $|6 - |x| - x| + x \leq 3$.
6.
    - Find an example of a polygon and a point in its interior, so that no side of the polygon is completely visible from that point.
    - Find an example of a polygon and a point in its exterior, so that no side of the polygon is completely visible from that point.
7. A 6×6 rectangle is tiled by 2×1 dominoes. Prove that it has at least one *fault-line*, that is, a straight line cutting the rectangle without cutting any domino.
8. The plane is colored with two colors. Prove that there exist three points of the same color, which are vertices of a regular triangle.
9. The plane is colored with n colors where n is any natural number. Prove that there exist four points of the same color, which are vertices of a rectangle. (Hint: recall the "same-color-corner-rectangle" problem.)
10. Each block of a 25×25 board has either 1 or -1 written on it. Let a_i be the product of all numbers in the ith row and b_j be the product of all numbers in the jth column. Prove that $a_1 + \ldots + a_{25} + b_1 + \ldots + b_{25} \neq 0$.

11. *The Art Gallery Problem.* An art gallery has the shape of an n-gon (not necessarily a convex one). Prove that $[n/3]$ (the integer part of $n/3$) watchmen can survey the building, no matter how complicated its shape.
Note 1. The boundary of the n-gon are the only walls, there are no walls inside it.
Note 2. We assume that each watchman can turn around and watch in all directions.

For example:



pentagon
[5/3]=1
1watchman can survey the building



7−gon
[7/3]=2
2 watchmen can survey the building

12. Which natural numbers are sums of consecutive smaller natural numbers? For example, $30 = 9 + 10 + 11$ and $31 = 15 + 16$, but 32 has no such representation. Find a simple condition and prove it.

Chapter 17

Solutions and answers to selected problems

17.1 Introduction

1 Assume that each of the eleven children contributed at most \$2.72. Then the total amount cannot exceed $2.72 \cdot 11 = 29.92$ dollars. But the total amount is \$30.00. Therefore our assumption is false, thus at least one child contributed at least \$2.73. This kind of proof is called a proof by contradiction (see chapter 3). Such problems can also be solved using the Generalized Dirichlet's Principle (see chapter 5).

2 (a) Any two-digit number N can be written in the form $N = 10a + b$ where b is the units digit of the number and a is its tens digit. (For example, $27 = 10 \cdot 2 + 7$.)
Suppose that N is divisible by 3. Then $N = 3k$ for some integer k. Thus
$10a + b = 3k$
$9a + a + b = 3k$
$a + b = 3k - 9a$
$a + b = 3(k - 3a)$
Since $k - 3a$ is an integer, $a + b$ is divisible by 3.
Conversely, suppose that $a + b$ is divisible by 3. Then $a + b = 3m$ for some integer m. Thus
$9a + a + b = 9a + 3m$
$10a + b = 3(3a + m)$
$N = 3(3a + m)$
Since $3a + m$ is an integer, N is divisible by 3.

(b) Any natural number N can be written in the form

$$N = 10^n a_n + 10^{n-1} a_{n-1} + \ldots 10a_1 + a_0$$

where a_0 is the units digit of the number, a_1 is the tens digit, and so on (this is called the base 10 expansion of the number N, see chapter 6). Now,

$$\begin{aligned} N &= \underbrace{99\ldots9}_{n-1} a_n + a_n + \underbrace{99\ldots9}_{n-2} a_{n-1} + a_{n-1} + \ldots + 9a_1 + a_1 + a_0 \\ &= \underbrace{99\ldots9}_{n-1} a_n + \underbrace{99\ldots9}_{n-2} a_{n-1} + \ldots + 9a_1 + (a_n + a_{n-1} + \ldots + a_1 + a_0). \end{aligned}$$

Since all multiples of 9 are divisible by 3, an argument similar to the one in part (a) shows that N is divisible by 3 if and only if

$$a_n + a_{n-1} + \dots a_1 + a_1$$

is (also, see chapter 6 for divisibility properties).

3 False. For example, for $n = 41$, $n^2 + n + 41 = 41^2 + 41 + 41 = 41 * 43$ is not prime.

Note. You may be tempted to check a few small values of n. You will discover then that for $1 \leq n \leq 39$, the number $n^2 + n + 41$ is indeed prime. However, the above example shows that this is not the case for *all* natural values of n. Thus checking a few examples is not sufficient!

4 Choose a row with the biggest number of stars in it. Note that this row contains at least two stars since if each row contained at most one star then there would be at most 4 stars total. But there are 6 stars. (This argument is using Dirichlet's box principle, see chapter 5.) So we have the following three cases:

Case I. This row contains 4 stars. Then cross is out, and there will only be two stars left. If they are in different columns, then cross out any other row, and the two columns containing the remaining two stars. If the remaining two stars are in one column then cross out one more row, the column containing the stars, and any other column.

Case II. This row contains 3 stars. Then cross it out, and there will only be three stars left. We can eliminate three stars e.g. by crossing out the row containing one of them, and two columns containing the last two stars. (As above, if the last two stars are in one column then cross out the column containing the stars, and any other column.)

Case III. This row contains 2 stars. Cross it out, and there will only be four stars in three rows left. Therefore at least one of these rows contains two stars. Cross it out, and there will only be two stars left. As above, it is clear that we can eliminate two stars by crossing out two columns.

5 True. There are several ways to prove this. One is by induction (see chapter 4), another one is considering all possible remainders of n modulo 3 (see chapters 6 and 7). Here is a third way: $n^3 + 2n = n^3 - n + 3n = n(n^2 - 1) + 3n = n(n-1)(n+1) + 3n$. Since $n(n-1)(n+1)$ is the product of three consecutive numbers, one of them is divisible by 3 (see chapter 6), and $3n$ is clearly divisible by 3. Thus the sum is divisible by 3.

6 Yes. There are even many such tours. Below are two of them (squares are numbered in the order the knight can visit them). Notice that in the second one, it is possible to go from square number 64 back to square number 1. Such a tour is called reentrant (see chapter 13).

1	40	13	26	3	42	15	28
24	37	2	41	14	27	4	43
39	12	25	60	53	62	29	16
36	23	38	63	56	59	44	5
11	50	57	54	61	52	17	30
22	35	64	51	58	55	6	45
49	10	33	20	47	8	31	18
34	21	48	9	32	19	46	7

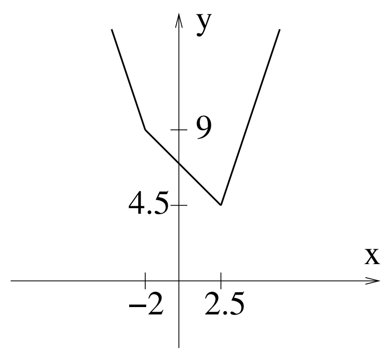
1	14	17	42	3	38	19	40
16	43	2	63	18	41	4	37
13	64	15	58	53	56	39	20
44	27	12	55	62	59	36	5
11	30	61	52	57	54	21	50
26	45	28	31	60	51	6	35
29	10	47	24	33	8	49	22
46	25	32	9	48	23	34	7

7 Since $|x+2| = \begin{cases} x+2 & \text{if } x+2 \geq 0, \text{ i.e. if } x \geq -2 \\ -x-2 & \text{if } x+2 < 0, \text{ i.e. if } x < -2 \end{cases}$ and

$|2x-5| = \begin{cases} 2x-5 & \text{if } 2x-5 \geq 0, \text{ i.e. if } x \geq 2.5 \\ -2x+5 & \text{if } 2x-5 < 0, \text{ i.e. if } x < 2.5 \end{cases}$, we have

$$f(x) = |x+2| + |2x-5| = \begin{cases} x+2+2x-5 = 3x-3 & \text{if } x \geq 2.5 \\ x+2-2x+5 = -x+7 & \text{if } -2 \leq x < 2.5 \\ -x-2-2x+5 = -3x+3 & \text{if } x < -2 \end{cases}$$

So we draw the graph of each linear function for the corresponding interval:



8 No. Consider the four regions of the town, namely the two river banks and the two islands. Each of them is connected with other regions by either 3 or 5 bridges. Suppose that if there is a tour of the town that crosses every bridge exactly once. Notice that for each intermediate region on such a tour we must come to the region by a bridge and leave the region by a bridge. So every time the tour visits a region, two bridges are crossed. This means that for every region except the one where we start and the one where we end there must be an even number of bridges connecting that region to others. But we have 4 regions with an odd number of bridges. Thus we get a contradiction. This solution can be explained in an easier and "smoother" way if we use the graph terminology discussed in chapter 13.

17.2 Logic

1 (a) Construct the truth table:

p	q	$p \to q$	$\neg q$	$\neg p$	$\neg q \to \neg p$
T	T	T	F	F	T
T	F	F	T	F	F
F	T	T	F	T	T
F	F	T	T	T	T

The columns for $p \to q$ and $\neg q \to \neg p$ are the same, thus the propositions are logically equivalent.

3 Every student at my university has a computer or has a friend who has a computer.

5 (a) True. Example: $x = 1$, $y = 2$, $1 < 2$.

(b) True. For any x, if we take $y = x + 1$ then $x < y$.

(c) False. There is no such x that for any y, $x < y$, because for any x we can take $y = x$, then $x \not< y$.

(d) False. Counterexample: $x = 2$, $y = 1$, $2 \not< 1$.

7 (a) $Q(2, 0)$
True because $2 + 0 = 2 - 0$ is true.

(b) $\forall y Q(1, y)$
False because "for every y, $1+y = 1-y$ is false: for example, if $y = 1$, then $2 \neq 0$.

(c) $\forall x \exists y Q(x, y)$
True because for any x we can take $y = 0$, then $x + 0 = x - 0$ is true.

(d) $\forall y \exists x Q(x, y)$
False because for example if $y = 1$, there is no x such that $x + 1 = x - 1$.

(e) $\exists y \forall x Q(x, y)$
True because if $y = 0$, then for any x we have $x + 0 = x - 0$.

Note. Statements (c) and (e) are not equivalent a priori! (c) says that for any x we can find a y such that $Q(x, y)$ is true. It is possible that we will find different values of y for different values of x. While (e) says that there is a value of y that works for any x.

9 The definition is as follows: the sequence $a_1, a_2, \ldots$ converges to a number L if for any positive ε there exists an index N such that for any $n \geq N$, $|a_n - L| < \varepsilon$. We rewrite this definition using quantifiers: $\exists L \forall \varepsilon ((\varepsilon > 0) \to (\exists N \forall n ((n \geq N) \to (|a_n - L| < \varepsilon))))$ (where L and ε are real numbers, and n and N are natural numbers). This can also be expressed as: $\exists L \in \mathbb{R}\ \forall \varepsilon > 0\ \exists N \in \mathbb{N}\ \forall n \geq N\ |a_n - L| < \varepsilon$.

17.3 Types of proofs

1 We will prove the statement by contrapositive: Suppose n is odd. Then $n = 2k+1$ for some integer k. Then $3n + 5 = 3(2k + 1) + 5 = 6k + 8 = 2(3k + 4)$ is even. Thus we have proved that if n is odd then $3n + 5$ is even. Therefore if $3n + 5$ is odd then n is even.

3 The statement is false. $n = 3$ is a counterexample, since $2^3 + 1 = 9$ is not prime.

5 If an odd number N is a perfect square, then $N = m^2$ where m is odd. Then m can be written in the form $m = 2k+1$. Then $N = m^2 = (2k+1)^2 = 4k^2+4k+1 = 4(k^2+k)+1$, so N is of the form $4n + 1$.

This proof is direct.

The converse is "if an odd number has the form $4n + 1$, then it is a perfect square". This is false because for example $5 = 4 \cdot 1 + 1$ but 5 is not a perfect square. Thus 5 is a counterexample.

7 (a) Let $f(x) = x^{101} + x^{51} + x + 1$. Then $f(-1) = -2 < 0$ and $f(1) = 4 > 0$. Since $f(x)$ is a continuous function, by the intermediate value theorem $f(x)$ has a root.
This proof is nonconstructive because we did not construct a root, only proved its existence.

(b) Suppose $f(x)$ has two distinct roots. By the mean value theorem there is a number c between these roots such that $f'(c) = 0$. But $f'(x) = 101x^{100} + 51x^{50} + 1 > 0$ everywhere. We get a contradiction.
This is a proof by contradiction.

9 The value $x = \frac{\pi}{6}$ is a root of the equation. This is a constructive proof since we provided an explicit example.

11 The roots of the equation $x^2 + x + 1 = 0$ can be found using the quadratic formula: $x = \frac{-1 \pm \sqrt{-3}}{2}$. Both roots are complex numbers. Since a quadratic equation has exactly roots (counting multiplicity), there are no other roots. In particular, there are no rational (or any real) solutions. This proof is direct.

13 An integer divisible by 8 has the form $8n$.
$8n = (4n^2 + 4n + 1) - (4n^2 - 4n + 1) = (2n + 1)^2 - (2n - 1)^2$.

This proof is direct and constructive: we gave an explicit example of two perfect squares whose difference is equal to $8n$.

17.4 Principle of Mathematical Induction

1 (a) We will prove this identity by Mathematical Induction.
Basis step: for $n = 1$ we have $1^2 = \frac{1 \cdot 2 \cdot 3}{6}$ which is true.
Inductive step: suppose the identity holds for $n = k$, i.e.

$$1^2 + 2^2 + 3^2 + \ldots + k^2 = \frac{k(k+1)(2k+1)}{6}.$$

Adding $(k+1)^2$ to both sides gives
$1^2 + 2^2 + 3^2 + \ldots + k^2 + (k+1)^2 = \frac{k(k+1)(2k+1)}{6} + (k+1)^2 =$

$\frac{k(k+1)(2k+1) + 6(k+1)^2}{6} = \frac{(k+1)(k(2k+1) + 6(k+1))}{6} =$

$\frac{(k+1)(2k^2 + k + 6k + 1)}{6} = \frac{(k+1)(2k^2 + 7k + 1)}{6} =$

$\frac{(k+1)(k+2)(2k+3)}{6} = \frac{(k+1)(k+2)(2(k+1)+1)}{6}.$

Thus the identity holds for $n = k + 1$.

(c) Proof by Mathematical Induction.
Basis step: for $n = 1$ we have $1 \cdot 1! = 2! - 1$, or $1 = 2 - 1$ which is true.
Inductive step: suppose the identity holds for $n = k$, i.e.

$$1 \cdot 1! + 2 \cdot 2! + \ldots + k \cdot k! = (k+1)! - 1.$$

Add $(k+1) \cdot (k+1)!$ to both sides:
$1 \cdot 1! + 2 \cdot 2! + \ldots + k \cdot k! + (k+1) \cdot (k+1)! = (k+1)! - 1 + (k+1) \cdot (k+1)! =$
$(k+1)!(1 + k + 1) - 1 = (k+2)! - 1$.
Thus the identity holds for $n = k + 1$.

3 Proof by induction on q.
If $q = 1$, then $m = 2$, and it is true that $3^m - 1 = 3^2 - 1 = 8$ is divisible by $2^{q+2} = 2^3 = 8$. Assume that the statement holds for $q = k$, i.e. $3^{2^q} - 1$ is divisible by 2^{q+2}. We want to prove that the statement holds for $q = k + 1$, i.e. $3^{2^{q+1}} - 1$ is divisible by 2^{q+3}.

We have: $3^{2^{q+1}} - 1 = 3^{2^q \cdot 2} - 1 = \left(3^{2^q}\right)^2 - 1 = \left(3^{2^q} - 1\right)\left(3^{2^q} + 1\right)$. By the induction hypothesis, $3^{2^q} - 1$ is divisible by 2^{q+2}. Clearly, $3^{2^q} + 1$ is an even number, thus it is divisible by 2. Then the product $\left(3^{2^q} - 1\right)\left(3^{2^q} + 1\right)$ is divisible by $2^{q+2} \cdot 2 = 2^{q+3}$.

5 (a) Proof by induction.
Basis step. If $n = 1$, the identity says that $F_1F_2 = F_2^2$, i.e. $1 \cdot 1 = 1^2$ which is true.
Inductive step. Assume the identity holds for $n = k$, i.e.

$$F_1F_2 + F_2F_3 + \ldots + F_{2k-1}F_{2k} = F_{2k}^2. \tag{17.1}$$

We want to prove that it holds for $n = k+1$, i.e.

$$F_1F_2 + F_2F_3 + \ldots + F_{2(k+1)-1}F_{2(k+1)} = F_{2(k+1)}^2,$$

or, equivalently,

$$F_1F_2 + F_2F_3 + \ldots + F_{2k+1}F_{2k+2} = F_{2k+2}^2.$$

Using (17.1) we have:
$F_1F_2 + F_2F_3 + \ldots + F_{2k+1}F_{2k+2} = F_1F_2 + F_2F_3 + \ldots + F_{2k-1}F_{2k} + F_{2k}F_{2k+1} + F_{2k+1}F_{2k+2} = F_{2k}^2 + F_{2k}F_{2k+1} + F_{2k+1}F_{2k+2} = F_{2k}(F_{2k} + F_{2k+1}) + F_{2k+1}F_{2k+2} = F_{2k}F_{2k+2} + F_{2k+1}F_{2k+2} = (F_{2k} + F_{2k+1})F_{2k+2} = F_{2k+2}^2$.

(c) Basis step. For $n = 1$ the identity is $F_0F_2 = F_1^2 + (-1)^1$. Since $F_0 = 0$, $F_1 = F_2 = 1$, we have $0 \cdot 2 = 1 + (-1)$ which is true.
Inductive step. Assume the identity holds for $n = k$, i.e.

$$F_{k-1}F_{k+1} = F_k^2 + (-1)^k.$$

We want to show that it then holds for $n = k+1$, i.e.

$$F_{(k+1)-1}F_{(k+1)+1} = F_{k+1}^2 + (-1)^{k+1},$$

or, equivalently,

$$F_kF_{k+2} = F_{k+1}^2 + (-1)^{k+1}.$$

We have

$$\begin{aligned} F_kF_{k+2} &= F_k(F_k + F_{k+1}) \\ &= F_k^2 + F_kF_{k+1} \\ &= F_{k-1}F_{k+1} - (-1)^k + F_kF_{k+1} \\ &= F_{k+1}(F_{k-1} + F_k) + (-1) \cdot (-1)^k \\ &= F_{k+1}^2 + (-1)^{k+1}. \end{aligned}$$

(d) Recall that multiplication of 2×2 matrices is defined by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{pmatrix}.$$

(e) Basis step. If $n = 1$, then the identity says that $F_0^2 + F_1^2 = F_1^2$, or $0^2 + 1^2 = 1^2$ which is true.
Inductive step. Assume that it holds for all $1 \leq n \leq k$. We want to prove that it holds for $n = k+1$, i.e.

$$F_{(k+1)-1}^2 + F_{k+1}^2 = F_{2(k+1)-1},$$

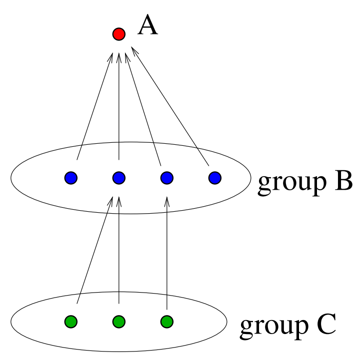
or, equivalently,

$$F_k^2 + F_{k+1}^2 = F_{2k+1}.$$

It may be easier here to work from the right hand side.
$F_{2k+1} = F_{2k} + F_{2k-1} = F_{2k-1} + F_{2k-2} + F_{2k-1} = 2F_{2k-1} + F_{2k-2} = 2F_{2k-1} + F_{2k-1} - F_{2k-3} = 3F_{2k-1} - F_{2k-3} = 3(F_{k-1}^2 + F_k^2) - (F_{(k-1)-1}^2 + F_{k-1}^2) = 3F_{k-1}^2 + 3F_k^2 - F_{k-2}^2 - F_{k-1}^2 = 2F_{k-1}^2 + 3F_k^2 - F_{k-2}^2 = 2F_{k-1}^2 + 3F_k^2 - (F_k - F_{k-1})^2 = 2F_{k-1}^2 + 3F_k^2 - F_k^2 + 2F_kF_{k-1} - F_{k-1}^2 = F_{k-1}^2 + 2F_k^2 + 2F_kF_{k-1} = F_{k-1}(F_{k-1} + F_k) + F_k(F_k + F_{k-1}) + F_k^2 = F_{k-1}F_{k+1} + F_kF_{k+1} + F_k^2 = (F_{k-1} + F_k)F_{k+1} + F_k^2 = F_k^2 + F_{k+1}^2$.
Note: the idea of the above inductive step is the following: express F_{2k+1} in terms of F_i's with i odd and less than $2k+1$, e.g. in terms of F_{2k-1} and F_{2k-3}, then use the inductive hypothesis to rewrite F_{2k-1} and F_{2k-3} as sums of squares (since we assume that the formula holds for smaller indices), and then rewrite the obtained expression in terms of F_k and F_{k+1} (because the formula we want to prove involves these terms).

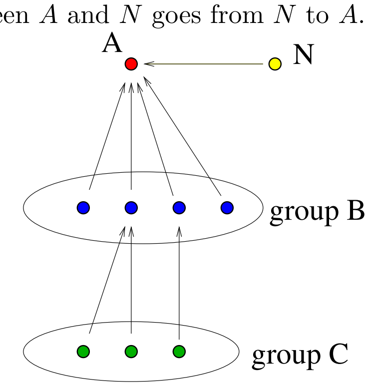
7 Basis step. For $n = 1$ city there is nothing to prove because there is no "any other city". (The step $n = 2$, in which case we have 2 cities and one road between them, so one city can be reached from the other, is also acceptable in this situation.)

Inductive step. Assume the statement is true for $n = k$, that is, for any system of roads between k cities, there is a city (let us call it city A) that can be reached from any other city either directly or via at most one other city. Let us call those cities from which there are direct roads to A group B, and the rest of the cities group C. Then from every city in group C there is a road to at least one city in group B:



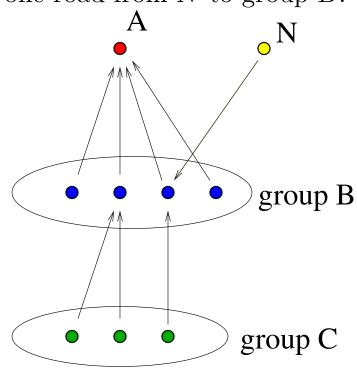
Now we add a $(k+1)$-st city, let us call it N. Consider the following 3 cases:

Case I. The road between A and N goes from N to A.



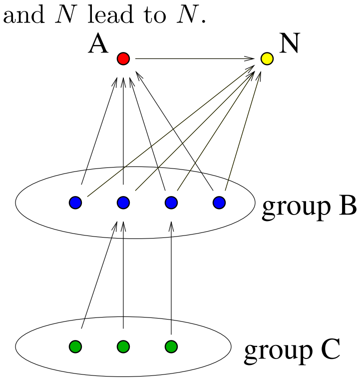
Then we put N into the group B, and A is still "a solution city".

Case II. There is at lest one road from N to group B.



Then we put N into the group C, and A is still a solution city.

Case III. None of the above: the road between A and N goes from A to N, and all the roads between group B and N lead to N.



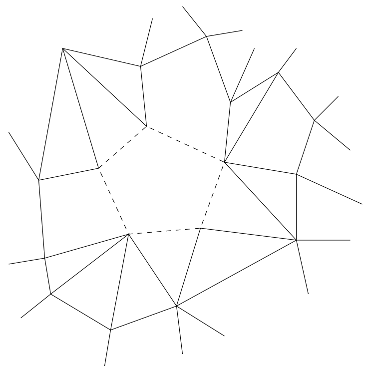
Then N is a new solution city, and A will join group B.

9 First of all, if at least one vertex has odd degree, than there is an odd number of regions around it, and it is obvious that they can not be properly colored with two colors.

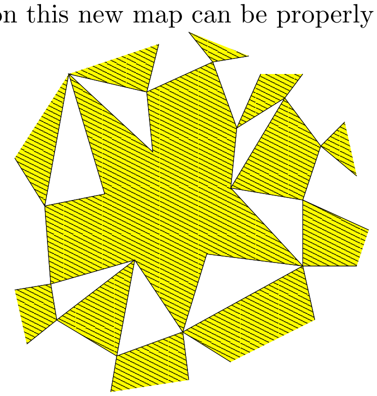
We will show that if the degree of each vertex is even, then the map can be properly colored with two colors. The induction will be on the number of boundary lines, and we will use the Strong Mathematical Induction.

Basis step. For $n = 0$ boundaries, the whole plane is one big region. We can color it with any color we like.

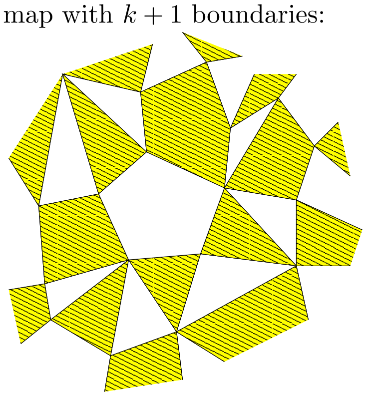
Inductive step. Suppose any map with less than or equal to k boundaries can be properly colored with two colors. We wish to show that any map with $k+1$ boundaries can be properly colored. Suppose we are given such a map. Remove temporarily all the boundaries of any one region.



We get a map with less than k boundaries, and the degree of each vertex is still even. By the inductive assumption this new map can be properly colored. Consider a coloring,



put the boundaries of our region back, and change the color inside it. We get a proper coloring for our original map with $k+1$ boundaries:



11 First we will try to estimate the sum by estimating each term. We see that

$$\frac{1}{3n+1} \le \text{each term} \le \frac{1}{n+1},$$

and we have $2n+1$ terms, therefore we have

$$\frac{2n+1}{3n+1} \le \text{sum} \le \frac{2n+1}{n+1}.$$

The left inequality doesn't help us, but from the right one we have

$$\text{sum} \le \frac{2n+1}{n+1} < \frac{2n+2}{n+1} = 2,$$

thus we do not need Math induction for this part. To show that the sum is bigger than 1, we will use Math induction.

Basis step. Check for $n = 1$: $1 < \frac{1}{2} + \frac{1}{3} + \frac{1}{4}$. We calculate $\frac{1}{2} + \frac{1}{3} + \frac{1}{4} = \frac{13}{12}$, and we see that this is bigger than 1.

Inductive step. Assume the inequality holds for $n = k$, i.e.

$$1 < \frac{1}{k+1} + \frac{1}{k+2} + \ldots + \frac{1}{3k+1}. \tag{17.2}$$

We want to prove that it holds for $n = k + 1$:

$$1 < \frac{1}{(k+1)+1} + \frac{1}{(k+1)+2} + \ldots + \frac{1}{3(k+1)+1},$$

or

$$1 < \frac{1}{k+2} + \frac{1}{k+3} + \ldots + \frac{1}{3k+1} + \frac{1}{3k+2} + \frac{1}{3k+3} + \frac{1}{3k+4}. \tag{17.3}$$

Compare (17.2) and (17.3), and notice that we lost the term $\frac{1}{k+1}$ but gained 3 terms $\frac{1}{3k+2} + \frac{1}{3k+3} + \frac{1}{3k+4}$. If we can show that we gained more than we lost, then the new sum (for $k + 1$) is bigger than 1. Thus we want to show that

$$\frac{1}{3k+2} + \frac{1}{3k+3} + \frac{1}{3k+4} > \frac{1}{k+1}.$$

The following inequalities are equivalent:

$$\frac{1}{3k+2} + \frac{1}{3k+3} + \frac{1}{3k+4} > \frac{3}{3k+3}$$

$$\frac{1}{3k+2} + \frac{1}{3k+4} > \frac{2}{3k+3}$$

$$\frac{6k+6}{(3k+2)(3k+4)} > \frac{2}{3k+3}$$

$$\frac{3k+3}{(3k+2)(3k+4)} > \frac{1}{3k+3}$$

$$(3k+3)^2 > (3k+2)(3k+4)$$

$$9k^2 + 18k + 9 > 9k^2 + 18k + 8,$$

and the last one is obviously true.

13 Calculating the determinant for first few values of n gives:

$\det A_1 = \det[2] = 2$

$$\det A_2 = \det \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} = 3$$

$$\det A_3 = \det \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix} = 4$$

$$\det A_4 = \det \begin{bmatrix} 2 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix} = 5$$

Based on the above calculation, we guess that $\det A_n = n + 1$.

We will prove this statement using Strong Mathematical Induction.

The basis step (for $n = 1$) is already shown above.

Inductive step: suppose that for some $k \geq 1$, the formula $\det A_n = n + 1$ holds for all $n \leq k$. We want to prove that $\det A_{k+1} = (k + 1) + 1$.

Expanding A_{k+1} across the first row and then expanding the second of the two obtained matrices down the first column gives

$$\det A_{k+1} = \det \begin{bmatrix} 2 & 1 & & & & \\ 1 & 2 & 1 & & 0 & \\ & 1 & 2 & & & \\ & & & \cdots & & \\ & 0 & & & 2 & 1 \\ & & & & 1 & 2 \end{bmatrix}_{(k+1)\times(k+1)} =$$

$$2 \det \begin{bmatrix} 2 & 1 & & & & \\ 1 & 2 & 1 & & 0 & \\ & 1 & 2 & & & \\ & & & \cdots & & \\ & 0 & & & 2 & 1 \\ & & & & 1 & 2 \end{bmatrix}_{k\times k} - 1 \det \begin{bmatrix} 1 & 1 & & & & \\ 0 & 2 & 1 & & 0 & \\ & 1 & 2 & & & \\ & & & \cdots & & \\ & 0 & & & 2 & 1 \\ & & & & 1 & 2 \end{bmatrix}_{k\times k} =$$

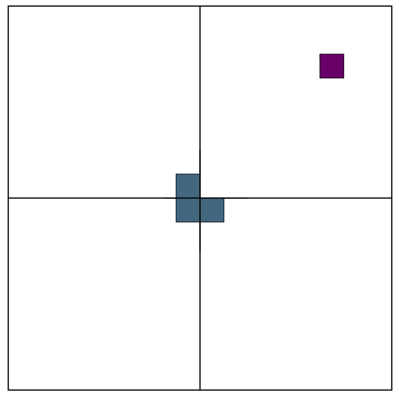
$$2 \det A_k - 1 \det \begin{bmatrix} 2 & 1 & & & & \\ 1 & 2 & 1 & & 0 & \\ & 1 & 2 & & & \\ & & & \cdots & & \\ & 0 & & & 2 & 1 \\ & & & & 1 & 2 \end{bmatrix}_{(k-1)\times(k-1)} =$$

$$2 \det A_k - \det A_{k-1} = 2(k + 1) - (k - 1 + 1) = 2k + 2 - k = k + 2.$$

15 We will prove a stronger statement: no matter which square is removed, we can cover the rest of the board by L-trominoes.

Basis step. A 2×2 board with one square removed has the shape of an L-tromino, and thus can be covered by one L-tromino.

Inductive step. Assume that a $2^k \times 2^k$ board with any square removed can be covered by L-trominoes. Now suppose we are given a $2^{k+1} \times 2^{k+1}$ board with one square removed. Divide this board into four $2^k \times 2^k$ boards. One of them has one square removed, and the three others are whole boards. Temporarily remove corner squares from those three whole boards as shown on the picture below.



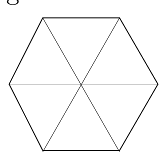
By the induction assumption, every of these four boards can be covered by L-trominoes. Now place one more L-tromino in the center to cover the 3 squares that we temporarily removed. We are done.

17.5 Dirichlet's Box Principle

1 Think of months as "boxes" and people's birth dates as "objects". Since there are more people (13) than months (12), By Dirichlet's Box Principle, at least two birth dates ("objects") are in the same month ("box").

3 There are 100 possible remainders modulo 100: 0, 1, 2, . . . , 99. Since we have 120 (more than 100) numbers, by Dirichlet's Box Principle there are at least two numbers with the same remainder. Their difference has remainder 0, and thus is divisible by 100. Therefore it ends with two zeros.

5 Divide the hexagon into 6 regions as shown in the figure below. Since we have 7 (more than 6) points, by Dirichlet's Box Principle there is a region with at least two points in it (or on its boundary). The distance between those two points is at most 1 because each region is an equilateral triangle with all sides of length 1.



6 Hint: show that we can choose 2 of the given points whose midpoint is a lattice point.

7 When we divide a number by 11, there are 11 possible remainders. Since we have 12 numbers (and thus more numbers than possible remainders), by Dirichlet's Box Principle, at least two numbers have the same remainder. Their difference is then divisible by 11. Every two-digit number that is divisible by 11, has the form aa (such numbers are 11, 22, 33, ..., 99).

9 Divide the cube into 343 small cubes with edge 1 each. Each point is inside at most one small cube (if a point is on the boundary of a small cube, then it is not inside any small cube). Since there are more small cubes than points, there is a small cube (actually, there are at least 43 of them) that doesn't contain any points inside it.

11 When we divide a number by 2, there are 2 possible remainders (namely, 0 and 1). Since we have 3 numbers, there are at least two numbers with the same remainder.

Their difference is divisible by 2, thus is of the form 2 times an integer. The other two differences are integers, thus the whole product $(x_1 - x_2)(x_1 - x_3)(x_2 - x_3)$ is of the form 2 times an integer.

13 (a) Divide the rectangle into six 1×2 rectangles. Since there are 7 points, by Dirichlet's principle at least two of them are in the same 1×2 rectangle. The distance between them is at most $\sqrt{5}$ (which is the length of the diagonal of a 1×2 rectangle).

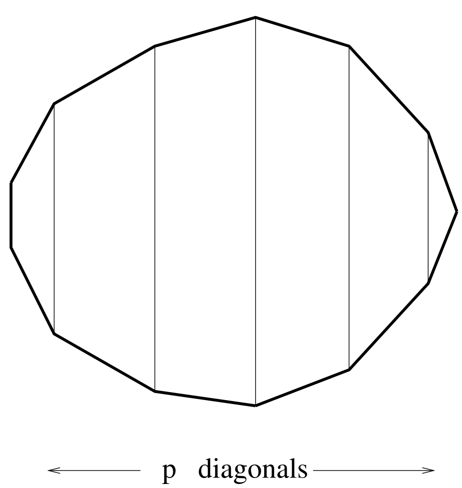
(b) Hint: divide the rectangle into 5 regions with the property that the distance between any two points in one region is at most $\sqrt{5}$. This is harder than part (a), but is possible!

15 "Make" a box for each side. There will be $2n$ boxes. We will "put" a diagonal into a box if it is parallel to the corresponding side.

box 1: diagonals parallel to side 1	...	box $2n$: diagonals parallel to side $2n$

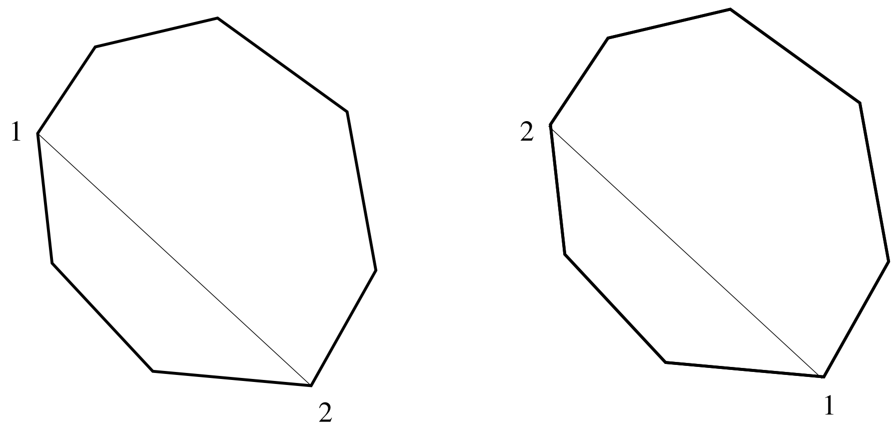
We will figure out the maximal possible number of diagonals that can be parallel to one side (and thus parallel among themselves), i.e. the maximal possible number of diagonals in each box, and we will figure out how many diagonals we have in a $2n$-gon. We will show that $2n$ times the maximal number of diagonals in each box is less than the number of diagonals in a $2n$-gon, thus there is not enough space for all the diagonals in our boxes. Therefore, there is a diagonal that is not in any box, and thus not parallel to any side.

Let p be the maximal possible number of diagonals parallel to the same side. We will find a condition on p. Notice that the vertices of these p diagonals and the 2 vertices of the side they are all parallel to, are distinct (because if 2 line segments have a common vertex, they can not be parallel). Let us draw the $2n$-gon so that all these p diagonals and the parallel side are vertical, with the side on the left. Then we must also have at least one vertex on the right (because the rightmost line segment must be inside the $2n$-gon):



Thus the number of vertices in this figure is at least $2p + 2 + 1$. We have $2n \geq 2p + 3$. Since $2n$ is an even number and $2p + 3$ is odd, we must have $2n \geq 2p + 4$. Then $2n - 4 \geq 2p$, and $n - 2 \geq p$. So there may be at most $n - 2$ diagonals in the same box.

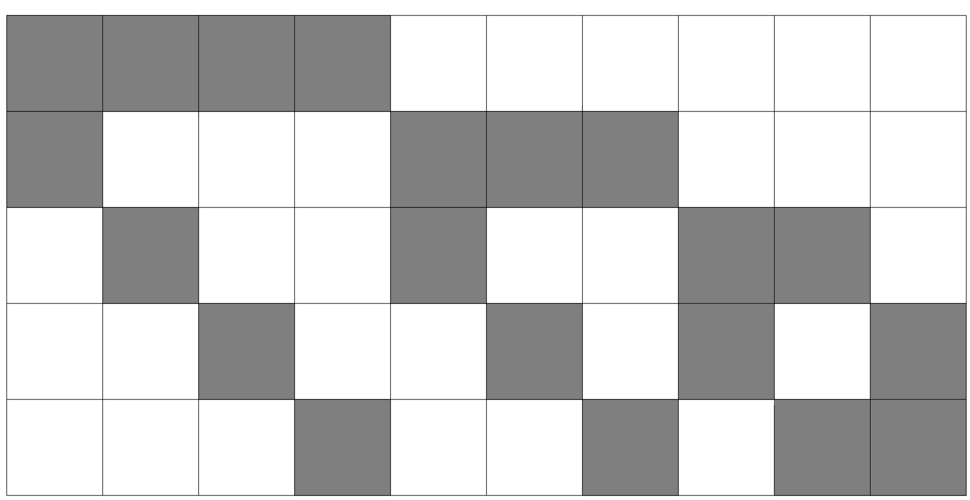
Now, there are $\dfrac{2n(2n-3)}{2} = n(2n-3)$ diagonals in a $2n$-gon because for every diagonal, there are $2n$ ways to choose the first vertex. Once the first vertex has been chosen, there are $2n-3$ ways to choose the second vertex (because the first vertex and its immediate neighbours can not be chosen as the second vertex). But this way we counted each diagonal twice:



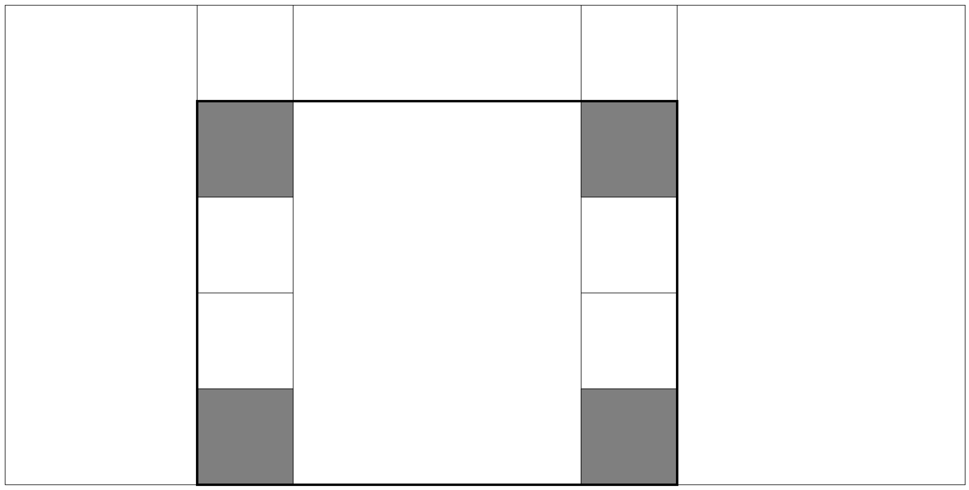
So we divide by 2.

Thus we have $2n$ boxes, at most $n-2$ diagonals may be in the same box, therefore at most $2n(n-2) = 2n^2 - 4n$ diagonals may be in the boxes. But we have $n(2n-3) = 2n^2 - 3n$ diagonals. Since $2n^2 - 3n > 2n^2 - 4n$, there is a diagonal which is not in any box, and thus not parallel to any side.

17 Since we have 5 rows and only 4 colors, every column has some color repeated. Since there are 41 columns, there is a color that is repeated (at least twice) in at least 11 columns. So each of these 11 columns contains at least 2 blocks of that color. Choose any 2. There are 10 ways to choose 2 blocks out of 5:



Since there are at least 11 columns with that color repeated, there are at least 2 columns that have the same 2 blocks of that color. Then we have a same-color-corner-rectangle:



19 Divide the numbers $\{1, 2, \ldots, 2n\}$ into n pairs of consecutive integers: $\{1, 2\}$, $\{3, 4\}$, $\ldots$, $\{2n-1, 2n\}$. Since we have $n+1$ integers, at least two of them are consecutive. Their greatest common divisor is 1 because if a number p divides both k and $k+1$ then p divides their difference $(k+1)-k=1$, so $p=1$. Thus our two consecutive numbers are relatively prime.

21 (a) Since there are only 10 different digits, by Dirichlet's principle, among 11 integers there are at least two with the same last digit. Their difference ends with 0.

(b) No. Example: 11, 110, 209, 308, 407, 506, 605, 704, 803, 902, 1001, 1100. These numbers are of the form $100k + (11-k)$ for $0 \le k \le 11$. The difference of two such numbers is $(100k+11-k)-(100n+11-n) = 100(k-n)-(k-n)$ (assume $k > n$). Since $0 < k-n < 12$, the tens digit of the difference is 8 or 9.

23 Divide the rectangle into twenty-five 4×3 rectangles. Since there are 25 such small rectangles and 26 points, by Dirichlet's principle there is a small rectangle containing at least 2 points. The distance between these points is less than or equal to the diagonal of the 4×3 rectangle which is $\sqrt{4^2+3^2} = 5$.

17.6 Number theory

1 Suppose $\sqrt[3]{25}$ is rational. Then it can be wtitten as an irreducible quotient:
$\sqrt[3]{25} = \dfrac{m}{n}$, $m, n \in \mathbb{Z}$, $(m, n) = 1$.
$25 = \dfrac{m^3}{n^3}$
$25n^3 = m^3$
Now there are several ways to get a contradiction.

Way 1: From the last equation, $5|m$, so $m = 5a$ for some integer a.
$25n^3 = (5a)^3$
$25n^3 = 125a^3$
$n^3 = 5a^3$
Now $5|n$. Thus both m and n are divisible by 5, which contradicts the condition $(m, n) = 1$.

Way 2: If $n = 1$, then $25 = m^3$ which is impossible.
If $n > 1$, then $n|m$ which contradicts $(m, n) = 1$.

Way 3: We have $5 \cdot 5 \cdot n^3 = m^3$. Both n and m cab be written as products of primes. Since n and m are cubed, the number of 5's on the left is 2 plus a multiple of 3, and the number of 5's on the right is a multiple of 3. This contradicts the fundamental theorem of arithmetic.

3 (a) A number $N = \underline{a_n a_{n-1} \ldots a_1 a_0}$ (with digits a_n, a_{n-1}, …, a_1, a_0) can be written as

$$N = a_n \cdot 10^n + a_{n-1} \cdot 10^{n-1} + \ldots + a_1 \cdot 10 + a_0 = \sum_{k=0}^{n} a_k \cdot 10^k.$$

The sum of its digits is

$$S = a_n + a_{n-1} + \ldots + a_1 + a_0 = \sum_{k=0}^{n} a_k.$$

We have $10 \equiv 1 \pmod 9$
$10^k \equiv 1 \pmod 9$
$a_k \cdot 10^k \equiv a_k \pmod 9$
$\displaystyle\sum_{k=0}^{n} a_k \cdot 10^k \equiv \sum_{k=0}^{n} a_k \pmod 9$
$N \equiv S \pmod 9$
Thus n is divisible by 9 if and only if S is divisible by 9.

(b) If the sum of the digits of a number is 66, then the number is divisible by 3 but not divisible by 9. But if a pefect square is divisible by 3 then it must be divisible by 9. Therefore a number with the digital sum 66 cannot be a perfect square.

5 (a) First notice that if k is the last digit of m, then the last digit of m^2 is that of k^2 because $m = 10n + k$ for some n, and $m^2 = (10n + k)^2 = 100n^2 + 20nk + k^2 = (10n^2 + 2nk) \cdot 10 + k^2$. So we consider all possible last digits and compute their squares: $0^2 = 0$, $1^2 = 1$, $2^2 = 4$, $3^2 = 9$, 4^2 ends with 6, 5^2 ends with 5, 6^2 ends with 6, 7^2 ends with 9, 8^2 ends with 4, and 9^2 ends with 1. Thus the last digit of a perfect square can be 0, 1, 4, 5, 6, or 9.

(b) Since 3 is not listed above, a number ending with 3 cannot be a perfect square.

7 No. Assume $n = a^2$ is a perfect square that ends with 65, then it is divisible by 5. Then a is divisible by 5, and therefore n is divisible by 25. Any number divisible by 25 ends with 00, 25, 50, or 75. Thus it cannot end with 65. We get a contradiction.

9 Since $2^{100} \equiv (2^2)^{50} \equiv 4^{50} \equiv (-1)^{50} \equiv 1 \pmod 5$, the remainder is 1.
There are many other ways to obtain this answer, e.g. $2^{100} \equiv (2^4)^{25} \equiv 16^{25} \equiv 1^{25} \equiv 1 \pmod 5$.

11 If the units digit of n is 3 then n can be written in the form $n = 10k+3$ for some integer k. Then $n^2+1 = (10k+3)^2+1 = 100k^2+60k+9+1 = 100k^2+60k+10 = 5(20k^2+12k+2)$ is divisible by 5.

13 If n is composite, then $n = ab$ for some $1 < a, b < n$. Then

$$2^n - 1 = (2^a)^b - 1^b = (2^a - 1)((2^a)^{b-1} + \ldots + 2^a + 1).$$

Both multiples are bigger than 1: $2^a - 1 > 2^1 - 1 = 1$, and $(2^a)^{b-1} + \ldots + 2^a + 1 > 1$, so $2^n - 1$ is composite.

If n is neither prime nor composite (1, 0, or negative), $2^n - 1$ is neither prime nor composite (namely, 1, 0, or noninteger).

15 Rewrite the equation as $p^2 - 1 = 2q^2$, and then factor the left hand side:

$$(p-1)(p+1) = 2q^2.$$

There are different ways to proceed from here.

Solution 1.

Since q is prime, the only ways to factor the right hand side are $1 \cdot 2q^2$, $2 \cdot q^2$, $q \cdot 2q$. Since $p - 1 < p + 1$, we get the following three cases:

Case 1. $p - 1 = 1$, $p + 1 = 2q^2$. The first equation gives $p = 2$, and then the second equation becomes $3 = 2q^2$ which has no integer solutions. Therefore this case is not possible.

Case 2. $p - 1 = 2$, $p + 1 = q^2$. Then $p = 3$ and $4 = q^2$, so $q = 2$. We get a solution:

$$p = 3, \quad q = 2.$$

Case 3. $p - 1 = q$, $p + 1 = 2q$. Substituting $p - 1$ for q into the second equation gives $p + 1 = 2(p - 1)$, or $p + 1 = 2p - 2$, or $p = 3$. Then again $q = 2$. We get the same solution as above.

Solution 2.

Case 1. $p = 2$. Then $(p-1)(p+1) = 3$, but $3 = 2q^2$ has no integral solutions. Thus this case is not possible.

Case 2. p is odd. Then $p - 1$ and $p + 1$ are even. Since the prime factorization of the right-hand side is $2 \cdot q \cdot q$, it must also be the prime factorization of $(p-1)(p+1)$. Since $p + 1 \geq 4$ (it must be at least 3 and it is even), it can be factored as 2 times an integer. Therefore $p - 1$ can't be factored, so it's a prime. Thus $p - 1 = 2$. Then $p = 3$, and solving for q gives $q = 2$.

17 Consider the equation modulo 3. The number x can be congruent to 0, 1, or 2 modulo 3. Then x^2 is congruent to either 0 or 1 modulo 3 (because $0^2 = 0$, $1^2 = 1$, and $2^2 = 4 \equiv 1 \pmod 3$). The number $3y^2$ is congruent to 0 modulo 3 since $3y^2$ is divisible by 3. Thus the left hand side is congruent to either 0 or 1 modulo 3. But the right hand side is congruent to 2 modulo 3. This is impossible, so the the equation has no integral solutions.

19 Notice first of all that y must be even because $3y = 100 - 2x$ and the right hand side is even. Second, y must be positive. Third, y cannot exceed 32 because if $y \geq 34$ then $3y \geq 102$ and then x would have to be negative. But if y satisfies all the above conditions, namely, y is even and $2 \leq y \leq 32$, then $3y$ is even and $6 \leq 3y \leq 96$, so $100 - 3y$ is even and $4 \leq 100 - 3y \leq 94$, so there exists a positive integer x such that $2x = 100 - 3y$. Thus for any even y such that $2 \leq y \leq 32$ we have a unique solution pair (x, y). There are 16 even numbers satisfying $2 \leq y \leq 32$, thus 16 pairs are solutions to the given equation.

23 Computing the first few powers of 2 and 3 modulo 5 gives

$2^1 = 2$, $2^2 = 4$, $2^3 = 8 \equiv 3 \pmod 5$, $2^4 = 16 \equiv 1 \pmod 5$;

$3^1 = 3$, $3^2 = 9 \equiv 4 \pmod 5$, $3^3 = 27 \equiv 2 \pmod 5$, $3^4 = 81 \equiv 1 \pmod 5$.

Therefore $2^{457} + 3^{457} \equiv 2^{456+1} + 3^{456+1} \equiv 2^{456} \cdot 2 + 3^{456} \cdot 3 \equiv 2^{4 \cdot 114} \cdot 2 + 3^{4 \cdot 114} \cdot 3 \equiv (2^4)^{114} \cdot 2 + (3^4)^{114} \cdot 3 \equiv 1 \cdot 2 + 1 \cdot 3 \equiv 0 \pmod 5$.

17.7 Case study

1 (a) Consider the following cases:

Case I: $n \equiv 0 \pmod 3$. Then $n^3 + 5n \equiv 0^3 + 5 \cdot 0 \equiv 0 \pmod 3$.

Case II: $n \equiv 1 \pmod 3$. Then $n^3 + 5n \equiv 1^3 + 5 \cdot 1 \equiv 6 \equiv 0 \pmod 3$.

Case III: $n \equiv 2 \pmod 3$. Then $n^3 + 5n \equiv 2^3 + 5 \cdot 2 \equiv 18 \equiv 0 \pmod 3$.

We see that in all cases $n^3 + 5n \equiv 0 \pmod 3$.

2 (a) Case I: $x = 1$.

Case II: $x \neq 0$, $x^2 - 7x + 12 = 0$ gives 2 roots: $x = 3$ and $x = 5$.

Case III: $x = -1$, $x^2 - 7x + 12$ is even. Since $x^2 - 7x + 12$ is indeed even when $x = -1$, this is also a solution.

Answer: 1, 3, 5, -1.

(c) First check $x = 0$. This is not a root because 0^0 is undefined. If $x \neq 0$, we can divide both sides of the equation by x^2. We get $x^{x^2-2} = 1$. Consider 3 cases:

Case I: $x = 1$, this is a root since (as is easy to check) it satisfies the original equation.

Case II: $x \neq 0$, $x^2 - 2 = 0$ gives $x = \pm\sqrt{2}$. Both satisfy the original equation.

Case III: $x = -1$, $x^2 - 2$ is even. This has no solutions because if $x = -1$ then $x^2 - 2 = -1$ which is not even.

Answer: 1, $\sqrt{2}$, $-\sqrt{2}$.

(e) First rewrite the equation as $x^{\frac{x+1}{2}} = x^{\sqrt{x+1}}$. Check $x = 0$. This is a root since $0^{\frac{1}{2}} = 0^1$. If $x \neq 0$ then we can divide both sides by $x^{\sqrt{x+1}}$. We get $x^{\frac{x+1}{2} - \sqrt{x+1}} = 1$.

Case I: $x = 1$.

Case II: $x \neq 0$, $\dfrac{x+1}{2} - \sqrt{x+1} = 0$. Solve the second equation:

$\dfrac{x+1}{2} = \sqrt{x+1}$

$x + 1 = 2\sqrt{x+1}$

$(x+1)^2 = 4(x+1)$

$(x+1)^2 - 4(x+1) = 0$

$(x+1)(x+1-4) = 0$

$(x+1)(x-3) = 0$

$x = -1$, $x = 3$. Both roots are nonzero.

Case III: $x = -1$, $\dfrac{x+1}{2} - \sqrt{x+1}$ is even. Since $\dfrac{-1+1}{2} - \sqrt{-1+1} = 0$ and thus is even, -1 satisfies this case too.

Answer: 0, 1, -1, 3.

(g) Case I: $x - 3 = 1$, then $x = 4$.
Case II: $x^2 - 8x + 15 = 0$, $x - 3 \neq 0$. The first equation holds when $x = 3$ or $x = 5$, but since $x - 3 \neq 0$, we have only one solution in this case: $x = 5$.
Case III: $x - 3 = -1$, $x^2 - 8x + 15$ is even. However, the only solution of the first equation is $x = 2$, and for this value $x^2 - 8x + 15$ is not even. Therefore there are no solutions in this case.
Answer: $x = 4$, $x = 5$.

3 (a) First consider the second equation. There are 3 cases:
Case I: $x = 1$. Then the first equation gives $1^2 = y + 1$, so $y = 0$. Check again that $(1, 0)$ satistfies both equations.
Case II: $x \neq 0$, $y = 0$. The first equation then becomes $x^{2x} = 1$. Consider 3 cases here:

Case I: x=1 gives the same solution as the one we found above.
Case II: $x \neq 0$, $2x = 0$ has no solutions.
Case III: $x = -1$, $2x$ is ok, and then check again that $(-1, 0)$ satisfies both equations.

Case III: $x = -1$, y even. The first equation then becomes $(-1)^{-2} = y + 1$, so $y = 0$. This gives $(-1, 0)$ again.
Answer: $(1, 0)$ and $(-1, 0)$.

4 (a) Case I: $2x - 2 \geq 0$. Then $|2x - 2| = 2x - 2$, and the equation becomes
$x^2 + 2x - 2 = 1$
$x^2 + 2x - 3 = 0$
$(x - 1)(x + 3) = 0$
$x = 1$, $x = -3$
Since only $x = 1$ satisfies the condition $2x - 2 \geq 0$, we have only this one root in this case.
Case II: $2x - 2 < 0$. Then $|2x - 2| = -(2x - 2)$, and the equation becomes
$x^2 - (2x - 2) = 1$
$x^2 - 2x + 1 = 0$
$(x - 1)^2 = 0$
$x = 1$
However, $x = 1$ does not satisfy the condition $2x - 2 < 0$, so there is no root in this case.
Answer: $x = 1$

(c) Here we have to consider two cases for each of the expressions inside an absolute value: $2x + 3$ and x. Thus we have four cases total:
Case I. $2x + 3 \geq 0$, $x \geq 0$. Then $|2x + 3| = 2x + 3$ and $|x| = x$, so the equation becomes
$2x + 3 - x = 3$
$x = 0$
This root satisfies both of the above conditions.
Case II. $2x + 3 \geq 0$, $x < 0$. Then $|2x + 3| = 2x + 3$ and $|x| = -x$, so the equation becomes
$2x + 3 + x = 3$

$3x = 0$
$x = 0$
This root does not satisfy the condition $x < 0$.
Case III. $2x+3 < 0$, $x \geq 0$. Then $|2x+3| = -(2x+3)$ and $|x| = x$, so the equation becomes
$-(2x+3) - x = 3$
$-3x = 6$
$x = -2$
This root does not satisfy the condition $x \geq 0$.
(Note: actually, we could notice that if $x \geq 0$ then $2x+3$ cannot be negative, so no real number would satisfy both of these conditions. So we could disregard this case from the very beginning.)
Case IV. $2x+3 < 0$, $x < 0$. Then $|2x+3| = -(2x+3)$ and $|x| = -x$, so the equation becomes
$-(2x+3) + x = 3$
$-x = 6$
$x = -6$
This root satisfies both of the above conditions.
Answer: 0 and -6.

(e) Similarly to part (c), we have 4 cases:
Case I. $x - 5 \geq 0$, $2x - 4 \geq 0$. Then $|x-5| = x-5$ and $|2x-4| = 2x-4$, so the inequality becomes
$x - 5 + 2x - 4 \leq 6$
$3x \leq 15$
$x \leq 5$
The above conditions are equivalent to: $x \geq 5$, $x \geq 2$. The only value of x that is less than or equal to 5 and greater than or equal to 5 at the same time is $x = 5$, and it satisfies $x \geq 2$.
Thus in this case we have one solution: $x = 5$.
Case II. $x - 5 \geq 0$, $2x - 4 < 0$. These conditions are equivalent to: $x \geq 5$, $x < 2$. There are no values of x that satisfy both of these, so we disregard this case.
Case III. $x - 5 < 0$, $2x - 4 \geq 0$. Then $|x-5| = -(x-5)$ and $|2x-4| = 2x-4$, so the inequality becomes
$-(x-5) + 2x - 4 \leq 6$
$-x + 5 + 2x - 4 \leq 6$
$x \leq 5$
The above conditions are equivalent to $x < 5$ and $x \geq 2$. Since the intersection of the intervals $(-\infty, 5]$, $(-\infty, 5)$, and $[2, +\infty)$ is the interval $[2, 5)$, the solution set in this case is $[2, 5)$.
Case IV. $x - 5 < 0$, $2x - 4 < 0$. Then $|x-5| = -(x-5)$ and $|2x-4| = -(2x-4)$, so the inequality becomes
$-(x-5) - (2x-4) \leq 6$
$-x + 5 - 2x + 4 \leq 6$
$-3x \leq -3$
$x \geq 1$

The above conditions are equivalent to $x < 5$ and $x < 2$. Since the intersection of the intervals $[1, +\infty)$, $(-\infty, 5)$, and $(-\infty, 2)$ is the interval $[1, 2)$, the solution set in this case is $[1, 2)$.

Answer: the union of the solution sets in all cases, i.e. $[1, 5]$.

(g) Case I: $x + 1 \geq 0$

$|x + 1| = x + 1$, so the inequality becomes

$x + 1 + 5 - x^2 \geq 0$

$x + 6 - x^2 \geq 0$

$x^2 - x - 6 \leq 0$

$(x - 3)(x + 2) \leq 0$

$-2 \leq x \leq 3$

The condition $x + 1 \geq 0$ implies $x \geq -1$, so the solution set in this case is $[-1, 3]$.

Case II: $x + 1 < 0$

$|x + 1| = -(x + 1)$, so the inequality becomes

$-(x + 1) + 5 - x^2 \geq 0$

$-x + 4 - x^2 \geq 0$

$x^2 + x - 4 \leq 0$

$$\left(x - \frac{-1 + \sqrt{17}}{2}\right)\left(x - \frac{-1 - \sqrt{17}}{2}\right) \leq 0$$

$$\frac{-1 - \sqrt{17}}{2} \leq x \leq \frac{-1 + \sqrt{17}}{2}$$

The condition $x + 1 < 0$ implies $x < -1$, so the solution set in this case is $\left[\frac{-1 - \sqrt{17}}{2}, -1\right]$.

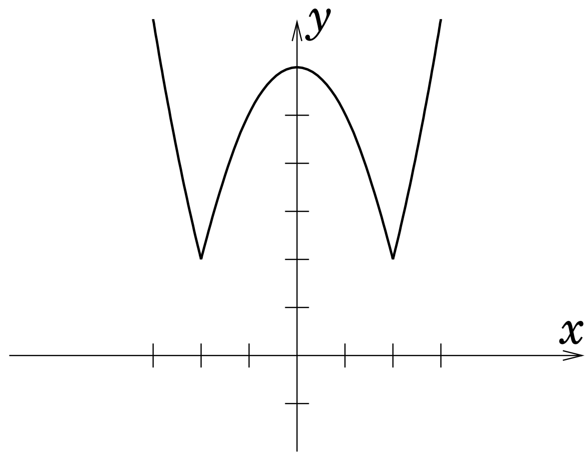
Answer: $\left[\frac{-1 - \sqrt{17}}{2}, 3\right]$.

5 (a) Case I: $x^2 - 4 \geq 0$, i.e. $x \geq 2$ or $x \leq -2$. Then $|x^2 - 4| = x^2 - 4$, and the function becomes $f(x) = x^2 - 4 + 2 = x^2 - 2$.

Case II: $x^2 - 4 < 0$, i.e. $-2 < x < 2$. Then $|x^2 - 4| = -(x^2 - 4)$, and the function becomes $f(x) = -x^2 + 4 + 2 = -x^2 + 6$.

Thus $f(x) = \begin{cases} x^2 - 2 & \text{if } x \geq 2 \text{ or } x \leq -2 \\ -x^2 + 6 & \text{if } -2 < x < 2 \end{cases}$.

The graph of this functions is shown below.



(c) First sketch the graph of $y = x + |x + 2|$.

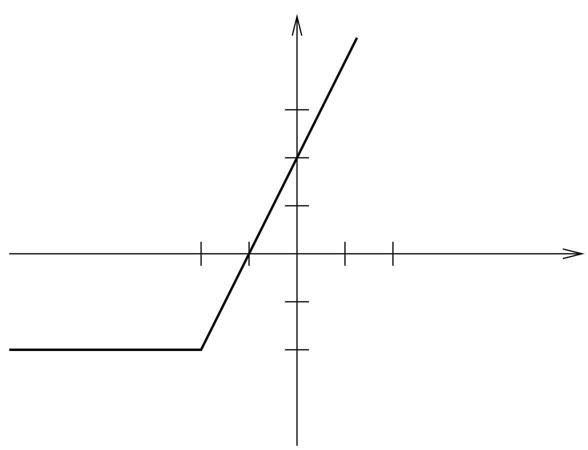
Case I: $x + 2 \geq 0$, or $x \geq -2$.

Then $y = x + x + 2 = 2x + 2$, so we draw the line $y = 2x + 2$ on the interval $[-2, \infty)$.

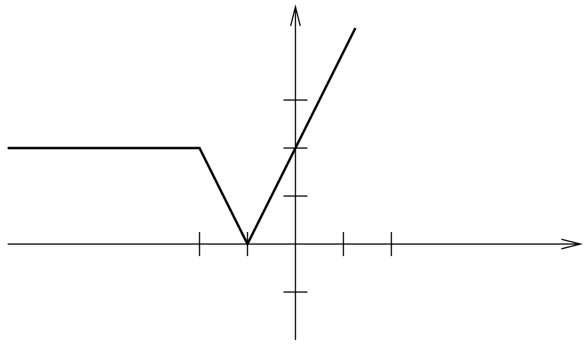
Case II: $x + 2 < 0$, or $x < -2$.

Then $y = x - (x + 2) = x - x - 2 = -2$, so we draw the horizontal line $y = -2$ on the interval $(-\infty, -2)$.

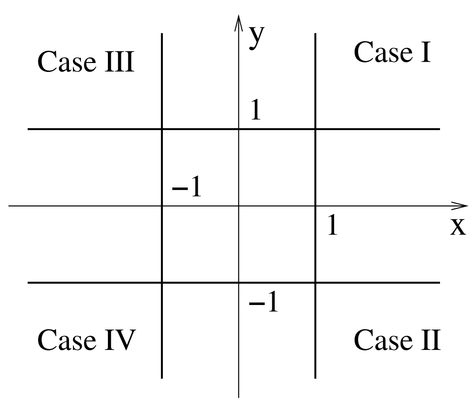
Thus we have the graph of $y = x + |x + 2|$:



Now we take the absolute value of the whole expression, and obtain the graph of $y = |x + |x + 2||$:



(e) Case I: $x \geq 0$, $y \geq 0$ (first quadrant). In this case the equation becomes
$x + y = 1 + xy$
$xy - x - y + 1 = 0$
$(x - 1)(y - 1) = 0$
$x = 1$ or $y = 1$
Case II: $x \geq 0$, $y < 0$ (fourth quadrant).
$x - y = 1 - xy$
$xy + x - y - 1 = 0$
$(x - 1)(y + 1) = 0$
$x = 1$ or $y = -1$
Case III: $x < 0$, $y \geq 0$ (second quadrant).
$-x + y = 1 - xy$
$xy - x + y - 1 = 0$
$(x + 1)(y - 1) = 0$
$x = -1$ or $y = 1$
Case IV: $x < 0$, $y < 0$ (third quadrant).
$-x - y = 1 + xy$
$xy + x + y + 1 = 0$
$(x + 1)(y + 1) = 0$
$x = -1$ or $y = -1$
Here is the graph:



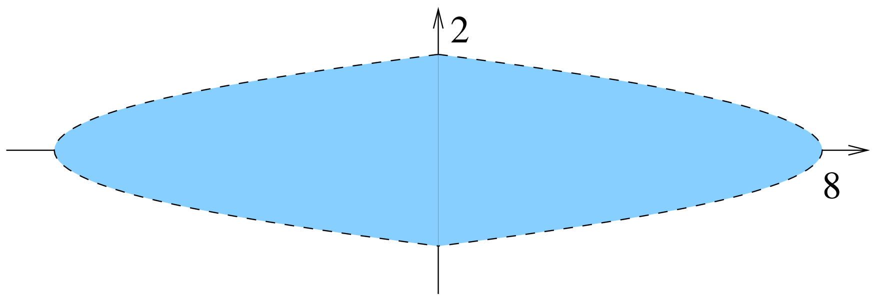
6 (a) Case I: $x \geq 0$, $y \geq 0$, then $x + y^3 < 8$, or $x < 8 - y^3$.

Case II: $x \geq 0$, $y < 0$, then $x - y^3 < 8$, or $x < 8 + y^3$.

Case III: $x < 0$, $y \geq 0$, then $-x + y^3 < 8$, or $x > y^3 - 8$.

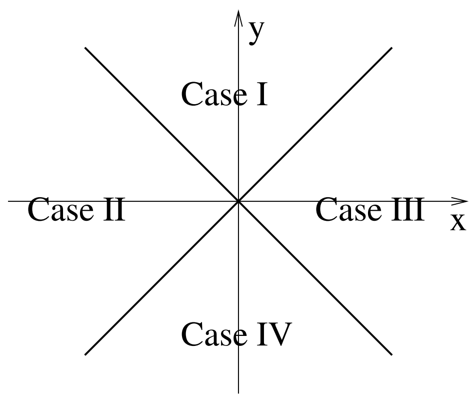
Case IV: $x < 0$, $y < 0$, then $-x - y^3 < 8$, or $x > -8 - y^3$.

Now we draw the corresponding region in each quadrant, and we get the following figure:



Note: since the inequality is strict, the boundary if the region is excluded.

(c) Consider 4 cases: $y - x$ positive or negative, and $x + y$ positive or negative. In each case, get rid of the absolute value, and solve for y.



Case I. $y - x \geq 0$, $x + y \geq 0$.

$2y - 2x + y + x \leq 1$

$3y \leq 1 + x$

$y \leq \dfrac{1}{3} + \dfrac{1}{3}x$

Case II. $y - x \geq 0$, $x + y < 0$.

$2y - 2x - y - x \leq 1$

$y \leq 1 + 3x$

Case III. $y - x < 0$, $x + y \geq 0$.

$-2y + 2x + y + x \leq 1$

$-y \leq 1 - 3x$

$y \geq 3x - 1$

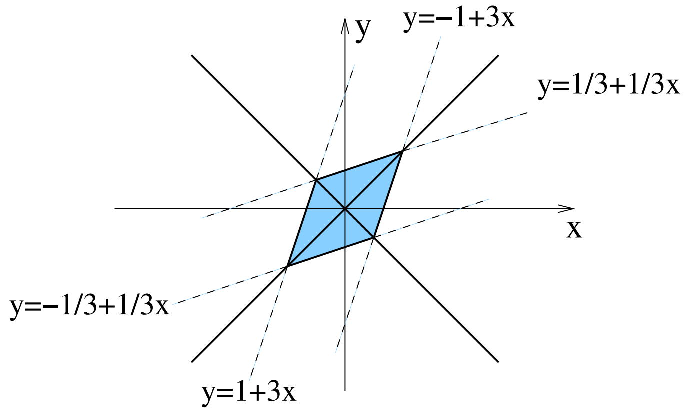
Case IV. $y - x < 0$, $x + y < 0$.

$-2y + 2x - y - x \leq 1$

$-3y \leq 1 - x$

$y \geq -\dfrac{1}{3} + \dfrac{1}{3}x$

Now sketch the region in each case:



7 (a) First of all, $b > 0$ because otherwise $a^b = 1$ or -1 or is not an integer.
So $a^b = \underbrace{a \cdot a \cdot \ldots \cdot a}_{b\ times}$. Since $625 = 5 \cdot 5 \cdot 5 \cdot 5$, b cannot be 3 or larger than 4. So we have 3 possible cases:
Case I: $b = 1$, then $a = 625$.
Case II: $b = 2$, then $a^2 = 625$ has 2 solutions: $a = 25$ and $a = -25$.
Case III: $b = 4$, then $a^4 = 625$ has 2 solutions: $a = 5$ and $a = -5$.
Thus we have 5 pairs: $(625, 1)$, $(25, 2)$, $(-25, 2)$, $(5, 4)$, and $(-5, 4)$.

17.8 Finding a pattern

1 (a) $a_n = n - 2$

(b) $a_n = 2n + 3$

(c) $a_n = \begin{cases} \dfrac{3n-1}{2} & \text{if } n \text{ is odd} \\ \dfrac{3n}{2} & \text{if } n \text{ is even} \end{cases}$

Also, $a_n = \left[\dfrac{3n}{2}\right]$

(d) $a_n = \dfrac{n}{2^n}$

3 $A_1 = 1$
$A_2 = 1 + 3 = 4$
$A_3 = 1 + 3 + 5 = 9$
$A_4 = 1 + 3 + 5 + 7 = 16$

We guess that $A_n = n^2$.

Proof by Mathematical Induction:

Basis step: for $n = 1$ we have $A_1 = 1^2 = 1$ which is true.

Inductive step: suppose that the formula hold for $n = k$, i.e. $A_k = 1+3+\ldots+(2k-1) = k^2$. Then $A_{k+1} = 1+3+\ldots+(2k-1)+(2k+1) = A_k+(2k+1) = k^2+2k+1 = (k+1)^2$, thus the formula holds for $n = k + 1$.

5 Find a formula for $f_n(x)$ for the first few values of n:
$f_1(x) = 2x + 1$
$f_2(x) = 2(2x + 1) + 1 = 4x + 3$
$f_3(x) = 2(4x + 3) + 1 = 8x + 7$
$f_4(x) = 2(8x + 7) + 1 = 16x + 15$
It appears that $f_n = 2^n x + 2^n - 1$, so we will try to prove this by Mathematical Induction.
Basis step: if $n = 1$, then our formula gives $f_1 = 2x + 1$ which is true.
Inductive step: suppose the formula holds for $n = k$, i.e. $f_k = 2^k x + 2^k - 1$. Then $f_{k+1} = f_1 \circ f_k = 2(2^k x + 2^k - 1) + 1 = 2^{k+1}x + 2^{k+1} - 2 + 1 = 2^{k+1}x + 2^{k+1} - 1$.
Thus the formula holds for $n = k + 1$.

7 First of all, notice that the last digit of a positive number is its remainder upon division by 10.

Solution 1.
Find the last digit of 107^n for some small values of n:
$107^1 \equiv 7 \pmod{10}$
$107^2 = 107 \cdot 107 \equiv 7 \cdot 7 \equiv 49 \equiv 9 \pmod{10}$
$107^3 = 107^2 \cdot 107 \equiv 9 \cdot 7 \equiv 63 \equiv 3 \pmod{10}$
$107^4 = 107^3 \cdot 107 \equiv 3 \cdot 7 \equiv 21 \equiv 1 \pmod{10}$
$107^5 = 107^4 \cdot 107 \equiv 1 \cdot 7 \equiv 7 \pmod{10}$
$107^6 = 107^5 \cdot 107 \equiv 7 \cdot 7 \equiv 49 \equiv 9 \pmod{10}$
We see that the last digits start repeating. As we keep multiplying our number by 7, the 4-tuple of last digits 7, 9, 3, and 1 will keep repeating. More precisely,

$$107^n \equiv \begin{cases} 7 \pmod{10} & \text{if } n \equiv 1 \pmod 4 \\ 9 \pmod{10} & \text{if } n \equiv 2 \pmod 4 \\ 3 \pmod{10} & \text{if } n \equiv 3 \pmod 4 \\ 1 \pmod{10} & \text{if } n \equiv 0 \pmod 4 \end{cases}.$$

This formula can be proved by Strong Mathematical Induction.
Basis step: if $n = 1$, then $107^n \equiv 7 \pmod{10}$ is true.
Inductive step: Suppose the formula holds for all $1 \le n \le k$. We want to prove that it holds for $n = k + 1$.
Case I. If $k + 1 = 2$, then $k + 1 \equiv 2 \pmod 4$, and $107^2 \equiv 9 \pmod{10}$ is true.
Case II. If $k + 1 = 3$, then $k + 1 \equiv 3 \pmod 4$, and $107^3 \equiv 3 \pmod{10}$ is true.
Case III. If $k + 1 = 4$, then $k + 1 \equiv 0 \pmod 4$, and $107^4 \equiv 1 \pmod{10}$ is true.
Case IV. If $k + 1 \ge 5$, then $k - 3 = (k + 1) - 4 \ge 1$, and we assumed that the formula above was true for $k - 3$.
Since $k + 1 \equiv k - 3 \pmod 4$, the formula suggests that $107^{k+1} \equiv 107^{k-3} \pmod{10}$, so this is what we have to prove in this case.
Well, it is true that $107^{k+1} \equiv 107^{k-3} \cdot 107^4 \equiv 107^{k-3} \pmod{10}$ since we know that $107^4 \equiv 1 \pmod{10}$. Thus the formula holds for $n = k + 1$.
Now, since $107 \equiv 3 \pmod 4$, the last digit of $107^{107} \equiv 3 \pmod{10}$, so the last digit of 107^{107} is 3.

Solution 2. Since $107^4 \equiv 1 \pmod{10}$ and $107^3 \equiv 3 \pmod{10}$ (as we saw above), $107^{107} \equiv 107^{104} \cdot 107^3 \equiv (107^4)^{26} \cdot 107^3 \equiv 1^{26} \cdot 107^3 \equiv 1 \cdot 3 \equiv 3 \pmod{10}$.

9 The last two digits of a positive number are determined by its remainder upon division by 100.

Solution 1.
Find the last two digits of 7^n for some small values of n:

$7^1 = 7$
$7^2 = 49$
$7^3 = 343 \equiv 43 \pmod{100}$
$7^4 = 7^3 \cdot 7 \equiv 43 \cdot 7 \equiv 301 \equiv 1 \pmod{100}$
$7^5 = 7^4 \cdot 7 \equiv 1 \cdot 7 \equiv 7 \pmod{100}$
$7^6 = 7^5 \cdot 7 \equiv 7 \cdot 7 \equiv 49 \pmod{100}$
We see that the last two digits 01, 07, 49, 43 will repeat. So

$$7^n \equiv \begin{cases} 7 \pmod{100} & \text{if } n \equiv 1 \pmod 4 \\ 49 \pmod{100} & \text{if } n \equiv 2 \pmod 4 \\ 43 \pmod{100} & \text{if } n \equiv 3 \pmod 4 \\ 1 \pmod{100} & \text{if } n \equiv 0 \pmod 4 \end{cases}.$$

As in problem 7, this can be proved by Strong Mathematical Induction. Since $50 \equiv 2 \pmod 4$, the last two digits of 7^{50} are 49.

Solution 2.
Since $7^4 \equiv 1 (\text{mod } 100)$, we have $7^{50} \equiv 7^{48} \cdot 7^2 \equiv (7^4)^{12} \cdot 49 \equiv 1^{12} \cdot 49 \equiv 49 \pmod{100}$.

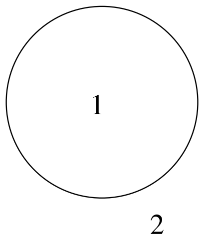
11 (a) If $f(x) = \sin(x)$, then the first few derivatives are:
$f'(x) = \cos(x)$
$f''(x) = -\sin(x)$
$f'''(x) = -\cos(x)$
$f^{(4)}(x) = \sin(x)$
$f^{(5)}(x) = \cos(x)$
We got $\cos(x)$ again, so the derivatives $\cos(x)$, $-\sin(x)$, $-\cos(x)$, $\sin(x)$ will repeat. Therefore

$$f^{(n)}(x) = \begin{cases} \cos(x) & \text{if } n \equiv 1 \pmod 4 \\ -\sin(x) & \text{if } n \equiv 2 \pmod 4 \\ -\cos(x) & \text{if } n \equiv 3 \pmod 4 \\ \sin(x) & \text{if } n \equiv 0 \pmod 4 \end{cases}.$$

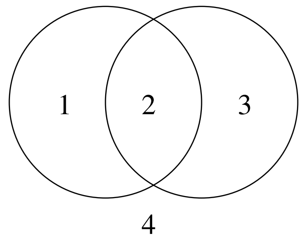
This formula can be proved by Strong Mathematical Induction (the proof is similar to that in problem 7).

(c) If $h(x) = 2e^{5x}$, then the first few derivatives are:
$h'(x) = 2 \cdot 5e^{5x}$
$h''(x) = 2 \cdot 5 \cdot 5e^{5x}$
$h'''(x) = 2 \cdot 5 \cdot 5 \cdot 5e^{5x}$
We guess that $h^{(n)}(x) = 2 \cdot 5^n e^{5x}$, and prove this formula by Mathematical Induction.
Basis step: $h'(x) = 2 \cdot 5e^{5x}$ is true.
Inductive step: suppose $h^{(k)}(x) = 2 \cdot 5^k e^{5x}$, then $h^{(k+1)}(x) = 2 \cdot 5^k \cdot 5e^{5x} = 2 \cdot 5^{k+1} e^{5x}$.

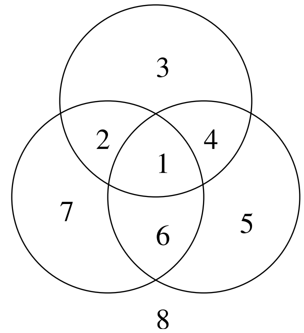
13 First find the number of regions for some small n:



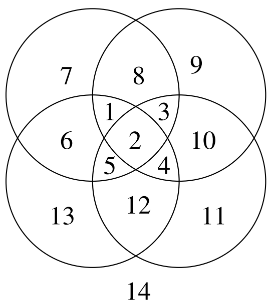
$n = 1$
2 regions



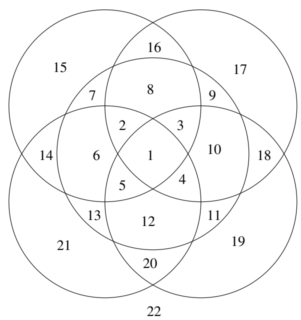
$n = 2$
4 regions



$n = 3$
8 regions



$n = 4$
14 regions



$n = 5$
22 regions

The sequence is 2, 4, 8, 14, 22, ... The differences between consecutive terms are 2, 4, 6, 8, ... We guess that the differences are increasing consecutive even numbers, so $a_n = 2 + 2 + 4 + 6 + ... + 2(n-1) = 2 + 2(1 + 2 + 3 + ... + (n-1)) = 2 + 2\frac{(n-1)n}{2} = 2 + (n-1)n = n^2 - n + 2$.

Now we will prove this formula by Mathematical Induction.

Basis step: If $n = 1$, the formula gives 2, and it is true that there are 2 regions.

Inductive step: Suppose the formula is true for k circles. We add a $(k+1)$-th circle. This new circle intersects the old k circles in $2k$ points. Thus the intersection points divide the new circle into $2k$ arcs. Therefore, the number of regions increases by $2k$ (each arc divides an old region into 2). Then, if k circles divided the plane into $k^2 - k + 2$ regions, $k+1$ circles will divide it into $k^2 - k + 2 + 2k = k^2 + 2k + 1 - k - 1 + 2 = (k+1)^2 - (k+1) + 2$ regions, and thus the formula holds for $k + 1$.

14 We compute the first few Fibonacci numbers: 0, 1, 1, 2, 3, 5, 8, 13, 21, and notice that every third of them is even. More precisely, F_n is even if and only if $n \equiv 0 \pmod 3$. Therefore exactly third of $F_1, F_2, \ldots, F_{99}$ is even which gives 33 numbers, and F_0 is even, thus we have 34 even numbers total.

Note. The pattern described above can be proved by Strong Mathematical Induction.

Basis step. If $n = 0$, $F_0 = 0$ is even.

Inductive step. Suppose the statement "F_n is even if and only if $n \equiv 0 \pmod 3$" holds for $0 \leq n \leq k$. We will prove that the statement holds for $n = k + 1$.

Case I. $k + 1 = 1$. Then $k + 1 \not\equiv 0 \pmod 3$, and F_1 is odd.

Case II. $k+1=2$. Then $k+1 \not\equiv 0 \pmod 3$, and F_2 is odd.

Case III. $k+1 \geq 3$. Then we consider all possible cases of $k+1$ modulo 3.

Case IIIA. $k+1 \equiv 0 \pmod 3$. Then by the inductive hypothesis F_k is odd and F_{k-1} is odd (since $k \equiv 2 \pmod 3$ and $k-1 \equiv 1 \pmod 3$), so $F_{k+1} = F_k + F_{k-1}$ is even.

Case IIIB. $k+1 \equiv 1 \pmod 3$. Then by the inductive hypothesis F_k is even and F_{k-1} is odd (since $k \equiv 0 \pmod 3$ and $k-1 \equiv 2 \pmod 3$), so $F_{k+1} = F_k + F_{k-1}$ is odd.

Case IIIC. $k+1 \equiv 2 \pmod 3$. Then by the inductive hypothesis F_k is odd and F_{k-1} is even (since $k \equiv 1 \pmod 3$ and $k-1 \equiv 0 \pmod 3$), so $F_{k+1} = F_k + F_{k-1}$ is odd.

15 Let $A_n = \displaystyle\prod_{i=1}^{2n-1} \left(1 - \frac{(-1)^i}{i}\right)$, then

$$A_1 = 1 - \frac{-1}{1} = 2$$

$$A_2 = \left(1 - \frac{-1}{1}\right)\left(1 - \frac{1}{2}\right)\left(1 - \frac{-1}{3}\right) = 2 \cdot \frac{1}{2} \cdot \frac{4}{3} = \frac{4}{3}$$

$$A_3 = \left(1 - \frac{-1}{1}\right)\left(1 - \frac{1}{2}\right)\left(1 - \frac{-1}{3}\right)\left(1 - \frac{1}{4}\right)\left(1 - \frac{-1}{5}\right) = 2 \cdot \frac{1}{2} \cdot \frac{4}{3} \cdot \frac{3}{4} \cdot \frac{6}{5} = \frac{6}{5}$$

Guess: $A_n = \dfrac{2n}{2n-1}$.

Proof by Mathematical Induction:

Basis step. If $n=1$, we have $A_1 = 2 = \dfrac{2}{1}$ as shown above.

Inductive step. Suppose $A_k = \dfrac{2k}{2k-1}$.

We want to prove that $A_{k+1} = \dfrac{2(k+1)}{2(k+1)-1} = \dfrac{2k+2}{2k+1}$.

Using the inductive hypothesis, we have:

$$A_{k+1} = \left(1 - \frac{-1}{1}\right)\left(1 - \frac{1}{2}\right)\left(1 - \frac{-1}{3}\right)\ldots\left(1 - \frac{-1}{2(k+1)-1}\right) =$$

$$\left(1 - \frac{-1}{1}\right)\left(1 - \frac{1}{2}\right)\left(1 - \frac{-1}{3}\right)\ldots\left(1 - \frac{-1}{2k-1}\right)\left(1 - \frac{1}{2k}\right)\left(1 - \frac{-1}{2k+1}\right) =$$

$$A_k\left(1 - \frac{1}{2k}\right)\left(1 - \frac{-1}{2k+1}\right) = \frac{2k}{2k-1} \cdot \frac{2k-1}{2k} \cdot \frac{2k+2}{2k+1} = \frac{2k+2}{2k+1}.$$

17.9 Invariants

1 Proof 1.
When we change the signs of 2 numbers, the possibilities are:
pos, pos → neg, neg
pos, neg → neg, pos
neg, neg → pos, pos.
We see that the number of positive numbers either does not change or changes by 2. Thus the parity of the number of positive numbers is an invariant. We start with the set containing 3 positive numbers. It is not possible to reach 6 positive numbers because 3 is odd but 6 is even.

Proof 2.
When 2 numbers are multiplied by -1, the product of all the numbers does not change. Initially the product is -36. It is not possible to make it 36.

3 An odd number times 3 is an odd number, and an even number times 3 is an even number. So multiplication by 3 does not change the parity of the number. Also, an odd number minus 2 is an odd number, and an even number minus 2 is an even number. So neither of these operations changes the parity of the number. The initial set consists of four odd numbers. Thus the four numbers will always be odd. It is not possible to reach 2 odd and 2 even numbers.

5 When we replace a by $a + 2b$ or $a - 2b$, we do not change its parity (if a is even, then $a \pm 2b$ is even, and if a is odd, then $a \pm 2b$ is odd). Thus the parity of each number will always be the same. Initially we had 2 even and 2 odd numbers. It is not possible to make all of the numbers even.

7 When we add 1 to all numbers in any row or column, we increase the sum of all 9 numbers by 3. When we subtract 1 from all numbers in any row or column, we decrease the sum by 3. Therefore the sum does not change mod 3. The sum of the original numbers is 1. The sum of nine 1's is 9. Since $1 \not\equiv 9 \pmod 3$, it is not possible to reach nine 1's.

9 The parity of the number of $-$ signs does not change:

- if two $+$'s are replaced by $+$, then the number of $-$'s does not change,
- if two $-$'s are replaced by $+$, then the number of $-$'s is decreased by 2,
- if $+$ and $-$ are replaced by $-$, then the number of $-$'s does not change.

Therefore if we had an even number of $-$ signs then a $+$ will remain in the end, and if we had an odd number of $-$ signs then a $-$ will remain in the end.

11 We have seen that any number is conguent to the sum of its digits mod 9. Thus when we replace a number by the sum of its digits, its remainder mod 9 does not change. Thus the question is equivalent to whether there are more numbers congruent to 1 or congruent to 2 mod 9 (among 1, 2, ..., 10^6). Remainders mod 9 are 1, 2, 3, ..., 8, 0, and they repeat. The last number is $10^6 \equiv 1 \pmod 9$, thus there will be more 1's.

13 When we replace a and b (let $a > b$) by $a - b$, the sum of all the numbers changes by

$$-a - b + (a - b) = -2b \equiv 0 \pmod 2.$$

So the parity of the sum does not change. Initially the sum is

$$1 + 2 + \ldots + (4n - 1) = \frac{(4n - 1)4n}{2} = (4n - 1)2n$$

which is even. Thus the sum of the numbers is always even. Therefore an even number will remain in the end.

15 Proof 1.
The sum of the numbers does not change since a, b, c, d, ... are replaced by $2b - a$, $2c - b$, $2d - c$, ... The sum of the original numbers is 45. But the sum of ten 5's is 50. Therefore it is not possible to reach ten 5's.

Proof 2.
Since $2b - a \equiv a \pmod 2$, $2c - b \equiv b \pmod 2$, etc., and we start with 5 even and 5 odd numbers, we'll always have 5 even and 5 odd numbers. Therefore it is not possible to reach ten 5's.

17 Let the integers be a_1, a_2, a_3, a_4, a_5, and a_6. The sets $\{a_1, a_2, a_3, a_4, a_5, a_6\}$ and $\{0, 1, 2, 3, 4, 5\}$ are equal. Thus the sum of all the a_i's is

$$a_1 + a_2 + a_3 + a_4 + a_5 + a_6 = 1 + 2 + \ldots + 6 = 21.$$

We add its place number to each integer and get

$$a_1 + 1,\ a_2 + 2,\ a_3 + 3,\ a_4 + 4,\ a_5 + 5,\ a_6 + 6.$$

Then the sum of these sums is

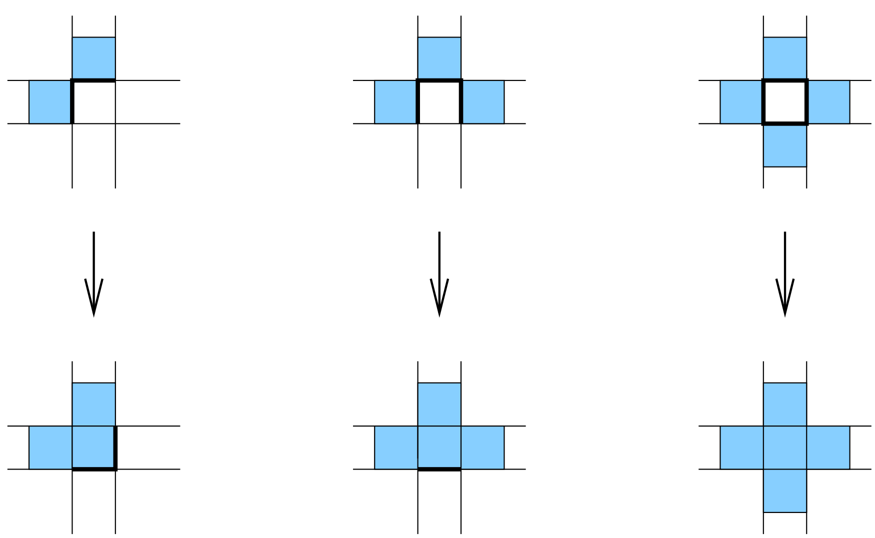
$$\begin{aligned}&(a_1 + 1) + (a_2 + 2) + (a_3 + 3) + (a_4 + 4) + (a_5 + 5) + (a_6 + 6)\\&= (a_1 + a_2 + a_3 + a_4 + a_5 + a_6) + (1 + 2 + 3 + 4 + 5 + 6) = 21 + 21 = 42.\end{aligned}$$

If all the sums $a_1 + 1$, $a_2 + 2$, $a_3 + 3$, $a_4 + 4$, $a_5 + 5$, and $a_6 + 6$ have different remainders mod 6, then the remainders are a permutation of the set $\{0, 1, 2, 3, 4, 5\}$ whose sum is

$$0 + 1 + 2 + 3 + 4 + 5 = 15 \equiv 3 \pmod 6.$$

Since $42 \not\equiv 3 \pmod 6$, we get a contradiction.

19 Consider squares with 2, 3, or 4 infected neigbors. Notice that when the infection spreads to such a square, the perimeter of the contaminated area cannot increase (but it may decrease). Namely (look at the picture below), when a square with 2 infected neigbors becomes infected, the perimeter of the contaminated area does not change. When a square with 3 infected neigbors becomes infected, the perimeter decreases by 2. When a square with 4 infected neigbors becomes infected, the perimeter decreases by 4. Initially the perimeter is at most $4 \cdot 9 = 36$. It cannot become 40.



21 First we divide the Parliament into two houses randomly. We will say that a Parliament member is unsatisfied with his placement if he has two or more enemies in his house.

If there are unsatisfied members, we'll choose any one of them and move him to the other house. Now there is at most one enemy in his house. By this move we reduced the number of hostile pairs (because the member moved was in two hostile pairs and now he is in at most one hostile pair, and no pairs not containing that member were affected by his move). If any unsatisfied members remained, then again we'll choose one of them and move him, thus reducing the number of hostile pairs again. And so on. Since it is not possible for the number of hostile pairs to become negative, sooner or later there will be no unsatisfied members.

23 After experimenting with a couple of examples, we notice that the parity of the number of 1's is always the same. Here is a proof.

If we replace two 1's by -1 then the number of 1's decreases by 2, so its parity is the same as before.

If we replace two -1's by -1 then the number of 1's does not change, so its parity is the same as before.

If we replace 1 and -1 by 1 then the number of 1's does not change, so its parity is the same as before.

We started with an odd number (seven) of 1's, therefore an odd number of 1's should remain at the end. Thus 1 will remain.

24 When we divide one number by 2 and multiply another number by 2, we do not change the product of all five numbers. Thus the product is an invariant. However, the product of the numbers in the set $\{512, 32, 16, 16, 2\}$ is not equal to the product of the numbers in the initial set $\{1, 4, 32, 128, 256\}$.

(This can be shown in different ways, e.g.:)

$1 \cdot 4 \cdot 32 \cdot 128 \cdot 256 = 2^0 \cdot 2^2 \cdot 2^5 \cdot 2^7 \cdot 2^8 = 2^{0+2+5+7+8} = 2^{22}$,

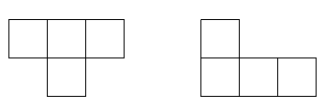
$512 \cdot 32 \cdot 16 \cdot 16 \cdot 2 = 2^9 \cdot 2^5 \cdot 2^4 \cdot 2^4 \cdot 2^1 = 2^{9+5+4+4+1} = 2^{23}$;

(or $\dfrac{1 \cdot 4 \cdot 32 \cdot 128 \cdot 256}{512 \cdot 32 \cdot 16 \cdot 16 \cdot 2} = \dfrac{1 \cdot 4 \cdot 32 \cdot 128 \cdot 256}{2 \cdot 16 \cdot 16 \cdot 32 \cdot 512} = \dfrac{1}{2} \cdot \dfrac{4}{16} \cdot \dfrac{32}{16} \cdot \dfrac{128}{32} \cdot \dfrac{256}{512} = \dfrac{1}{2} \cdot \dfrac{1}{4} \cdot 2 \cdot 4 \cdot \dfrac{1}{2} = \dfrac{1}{2} \neq 1$.)

Therefore it is not possible to reach the set $\{512, 32, 16, 16, 2\}$.

17.10 Coloring

1 A 14×14 board has 196 squares. Since each T-tetrominoe covers 4 squares, we must use $\dfrac{196}{4} = 49$ T-tetrominoes. Color the board using the standard chessboard coloring. Then it has 98 black squares and 98 white squares. Eeach T-tetromino covers either 3 black and 1 white or 1 black and 3 white squares. Suppose there are n T-tetrominoes covering 3 black squares. Then there are $49 - n$ T-tetrominoes covering 1 black square. Then all 49 tetrominoes cover $3n + (49 - n) = 2n + 49$ black squares. They must cover 98, so $2n + 49 = 98$, or $2n = 49$. This equation has no integer solutions since 49 is not divisible by 2.

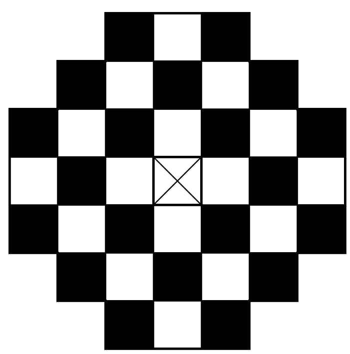


3 Proof 1. Consider the chessboard coloring. Every T-tetromino covers an odd number (namely, either 3 or 1) of black squares. Then 15 (which is an odd number) of them must cover an odd number of black squares. Every L-tetromino covers exactly 2 black squares, therefore 10 of them must cover 20. Then 15 T-tetrominoes and 10 L-tetrominoes must cover an odd number (odd $+20$) of black squares. But the board has an even number (namely, 50) of black squares. Therefore a covering is not possible.

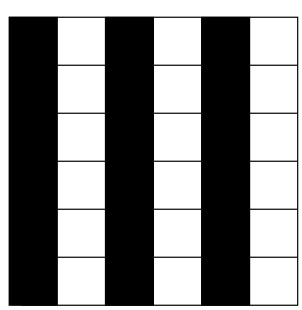
Proof 2. Suppose such a covering is possible. As in proof 1, consider the chessboard coloring, and notice that every T-tetromino covers either 3 or 1 black squares, and every L-tetromino covers 2 black squares. Suppose that n T-tetrominos cover 3 black squares. Then $15-n$ T-tetrominos cover 1 black square. The board has 50 black squares. Thus $3n+15-n+20=50$. This gives $2n=15$ which has no integer solutions. Contradiction.

5 Consider the traditional coloring of the chessboard. A T-tetromino covers either 1 or 3 (i.e. an odd number of) black squares. Every other tetromino covers 2 (i.e. an even number of) black squares. If the number of T-tetrominoes were odd, then they would cover an odd number of black squares, and the other tetrominoes cover an even number of black squares. Thus all tetrominoes together cover an odd number of black squares (because the sum of an odd number of odd numbers is odd). But the chessboard has 32 black squares, and 32 is even. Contradiction.

7 There are 36 squares, and each domino covers 2, so we need 18 dominoes. Color the figure as a chessboard. It has 20 black and 16 white squares. Since each domino covers one black and one white square, 18 dominoes must cover 18 black and 18 white squares while we have 20 and 16. So it is not possible to cover the figure with dominoes.



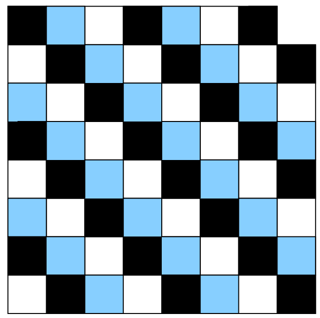
9 Cover the board as shown below.



There are 18 black squares. The rest of the argument is the same as in problem 1. Each L-tetromino covers either 1 or 3 black squares. Let n tetrominoes cover 3 black squares, then $9-n$ tetrominoes cover 1 black square, and all 9 together cover $3n+(9-n) = 2n+9$. Therefore $2n + 9 = 18$, or $2n = 9$, but this equation has no integer solutions.

11 Suppose the top right corner has been removed.

Proof 1. Color the board diagonally using 3 colors as shown below.



The board contains 21 white, 22 black, and 20 blue squares. Since each straight tromino must cover 1 white, 1 black, and 1 blue square, the board cannot be covered by straight trominoes.

Proof 2. Color the board using horizontal stripes of 3 colors, say, from top to bottom: white, blue, black. Then there are 23 white squares (3 rows minus one removed square), 24 blue squares (3 full rows), and 16 black squares (2 full rows). Each tromino can either cover 3 squares of one color, or 1 square of each color. Let a be the number of trominoes covering 3 white squares, b the number of trominoes covering 3 blue squares, c the number of trominoes covering 3 black squares, and d the number of trominoes covering one square of each color. Then, for the total number of squares of each color, we have: $3a + d = 23$, $3b + d = 24$, and $3c + d = 16$. Subtracting e.g. the first equation from the second we have $3b-3a = 1$, or $3(b-1) = 1$. Since the left-hand side is divisible by 3 and the right-hand side is not, the system does not have integer solutions.

13 Suppose such a covering exists. Color the board using the stripe coloring using two colors, say, white and black, starting with black (as shown in the first picture after example 10.2. Then there are 12 black columns and 11 white columns, so there are 23 more black squares than white ones. Each 2×2 tile covers 2 black and 2 white squares, so if there are n of such tiles they cover $2n$ black and $2n$ white squares. Each 3×3 tile covers either 6 or 3 black squares and, respectively, either 3 or 6 white squares. Let m be the number of 3×3 tiles that cover 6 black and 3 white squares, and let k be the number of 3×3 tiles that cover 3 black and 6 white squares. Then the total number of black squares covered by 3×3 tiles is $6m + 3k$, and the total number of white squares covered by 3×3 tiles is $3m + 6k$. Thus the total number of black squares covered by all tiles is $2n + 6m + 3k$, and the total number of white squares covered by all tiles is $2n + 3m + 6k$. Since there are 23 more black squares than white squares, we have $(2n+6m+3k)-(2n+3m+6k) = 23$, or $3m-3k = 23$, where n, m, and k are integers. However, we see that the left-hand side is divisible by 3, but the right-hand side is not. Therefore this equation has no integer solutions. Contradiction.

15 Let a be the number of rows and let b be the number of columns. If $n|a$ then $a = nk$ for some integer k, and each column contains nk squares. Thus we can cover each column by k "vertical" $1 \times n$ tiles. Similarly, if $n|b$ then $b = nk$ for some integer k, and each row contains nk squares. Thus we can cover each row by k "horizontal" $1 \times n$ tiles.

Now suppose that $n \nmid a$ and $n \nmid b$ but an $a \times b$ board can be covered by $1 \times n$ tiles. Color the board diagonally using n colors. Each tile must cover exactly one square of each color. Therefore each color must appear the same number of times. (We will show below that this is not possible, thus obtaining a contradiction.)

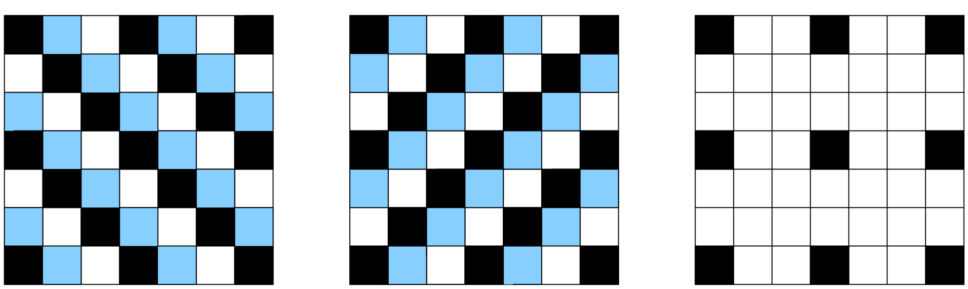
If $a > n$, then in the first n rows each color appears exactly b times (because each color apears exactly once in each column of length n). Therefore if we throw these first n rows away, each color must still appear the same number of times. Similarly, we can throw away the next set of n consecutive rows, and so on, until less than n rows remain. Similarly for the columns. So now we reduced our board to, say, a $c \times d$ board where $c < n$ and $d < n$, and each color must appear the same number of times. Without loss of generality we can assume that $c \leq d$. This $c \times d$ piece is colored diagonally, and we can renumber our colors so that they appear in the increasing order as shown in the picture below.

$\longleftarrow$ d $\longrightarrow$

c							
	1	2	3				d
	2	3				d	d+1
	3				d	d+1	
				d	d+1		

Since $d < n$, the number of colors is at least $d + 1$, so the first $d + 1$ diagonals are of different colors. Since only $c - 2 \leq d - 2 < d < n$ diagonals remain, colors d and $d + 1$ will not repeat. Therefore in this piece there are c squares of color d but only $c - 1$ squares of color $d + 1$. Thus the colors are not distributed evenly. We get a contradiction.

17 Color the board diagonally with 3 colors. You'll get 16 squares of one color, 16 squares of another color, and 17 squares of the third color (white, blue, and black respectively on the picture below). Since each 3×1 tile covers one square of each color, the 1×1 tile must cover one of the 17 squares of the third color. Now, notice that we could color the board diagonally in the other direction. The 1×1 tile must cover one of the 17 squares for the second coloring (black again). Therefore it must be in the intersection of the two sets. The intersection consists of 9 squares (4 corners, 4 midpoints of edges, and the center). In each case, it is easy to find a covering by sixteen 3×1 and one 1×1 tiles. Thus the set of permissible positions of the 1×1 tile consists of those 9 squares.

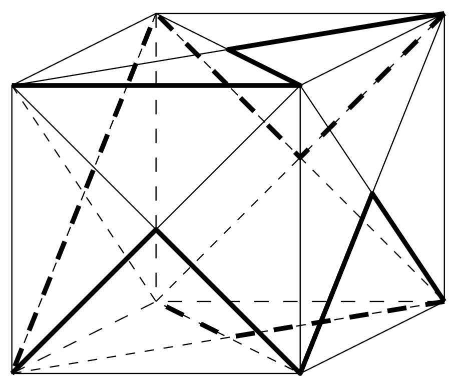


19 (a) Notice that every piece of a face diagonal connects a vertex and a face midpoint. Thus if we only use face diagonals, vertices and midpoints must alternate. But there are 8 vertices and 6 midpoints, so there is no way to make them alternate (there are too many vertices).

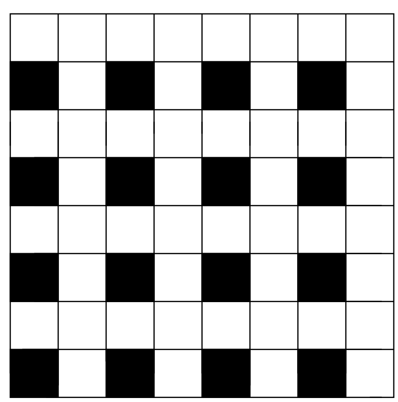
Note. We could color all the marked points: let vertices be black, and let midpoints be white... then black and white points must alternate, but there are 8 black points and 6 white points, so that's impossible.

(b) If one edge is allowed, then we could have two vertices in the beginning, after which we would be left with 6 midpoints and 6 vertices, and we can make them alternate. Again, let vertices be black and midpoints white, then a path could be e.g. *bbwbwbwbwbwbwb*.

Here is an example. (But there are many other such paths.)



20 "Color" the small (i.e. $1 \times 1 \times 1$) cubes of the $8 \times 8 \times 8$ cube as follows. Color each other row as in the picture below, and each other row all white.

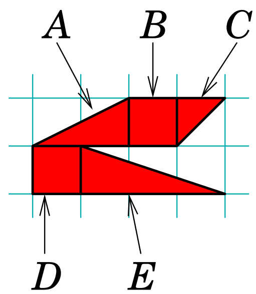


Then the $8 \times 8 \times 8$ cube contains 64 (which is even) black cubes. Each $4 \times 1 \times 1$ brick fills either 0 or 2, so, an even number of black cubes. Thus the $2 \times 2 \times 2$ cubes must cover an even number of remaining black cubes. Each $2 \times 2 \times 2$ cube fills exactly one black cube, therefore the number of $2 \times 2 \times 2$ cubes must be even.

21 Color the board using the traditional chessboard coloring. Then there are 32 black squares and 32 white squares. Each T-tetromino covers either 1 or 3 black squares, therefore seven T-tetrominoes must cover an odd number of black squares. Each L-tetromino covers 2 black squares, therefore nine L-tetrominoes must cover 18 (i.e. and odd number of) black squares. Thus all sixteen tiles together must cover an (odd + even=) odd number of black squares. However, the board has an even number (32) of black squares, therefore a covering is not possible.

17.11 Areas and Volumes

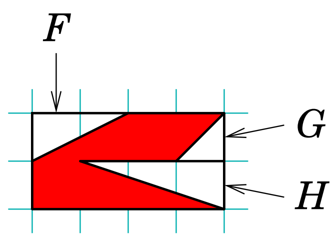
1 Solution 1. Divide the region into smaller regions whose areas are easy to find, for example:



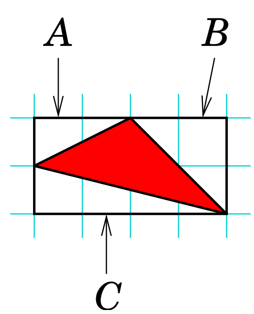
The areas of these regions are: $A = 1$, $B = 1$, $C = \frac{1}{2}$, $D = 1$, $E = \frac{3}{2}$

Then the total area is the sum of these areas: $1 + 1 + \frac{1}{2} + 1 + \frac{3}{2} = 5$

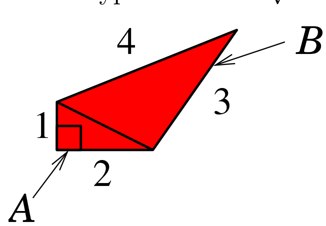
Solution 2. Consider the 2×4 rectangle containing the region. Its area is 8. The area of the complement is $F + G + H = 1 + \frac{1}{2} + \frac{3}{2} = 3$, thus the area of the region is $8 - 3 = 5$.



3 Consider the 2×4 rectangle containing the triangle. Its area is 8. The area of the complement is $A + B + C = 1 + 2 + 2 = 5$, thus the area of the trianle is $8 - 5 = 3$.



5 Divide the region into two triangles. One of them is a right triangle and has area $A = 1$. By the Pythagorean theorem its hypothenuse is $\sqrt{1^2 + 2^2} = \sqrt{5}$.



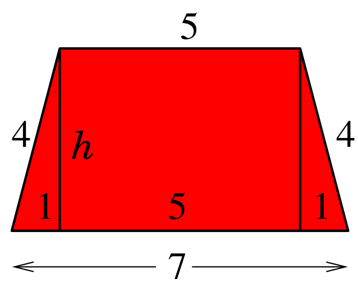
Therefore the area of the second trianle is $B = \sqrt{\frac{p}{2}\left(\frac{p}{2}-a\right)\left(\frac{p}{2}-b\right)\left(\frac{p}{2}-c\right)}$ where a, b, and c are its sides and p is its perimeter: $a = 3$, $b = 4$, $c = \sqrt{5}$, $p = 7+\sqrt{5}$. Thus we have $B = \sqrt{\frac{7+\sqrt{5}}{2}\left(\frac{7+\sqrt{5}}{2}-3\right)\left(\frac{7+\sqrt{5}}{2}-4\right)\left(\frac{7+\sqrt{5}}{2}-\sqrt{5}\right)} =$

$$\sqrt{\frac{7+\sqrt{5}}{2}\cdot\frac{1+\sqrt{5}}{2}\cdot\frac{\sqrt{5}-1}{2}\cdot\frac{7-\sqrt{5}}{2}} = \sqrt{\frac{(7+\sqrt{5})(1+\sqrt{5})(\sqrt{5}-1)(7-\sqrt{5})}{16}} =$$

$$\sqrt{\frac{(49-5)(5-1)}{16}} = \sqrt{\frac{44\cdot 4}{16}} = \sqrt{11}$$

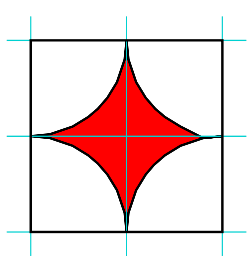
Therefore the total area is $A + B = 1 + \sqrt{11}$.

7 Draw two heights of the trapezoid:

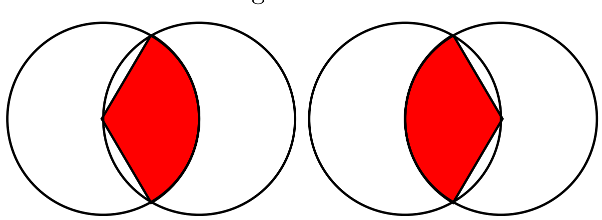


Since the two triangles are congruent, we calculate that the heights divide the base into segments of length 1, 5, and 1. Then the height is $h = \sqrt{4^2 - 1^2} = \sqrt{15}$, thus the area of the trapezoid is $A = \frac{1}{2}(5+7)\sqrt{15} = 6\sqrt{15}$.

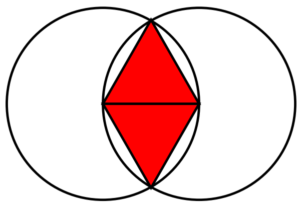
9 Consider the 2×2 square containing the region. Its area is 4. The complement of the region in the square consists of four quarters of a circle with radius 1, therefore the total area of the complement is the same as the area of the circle, i.e. π. Thus the area of the region is $4 - \pi$.



11 Consider the following sectors:

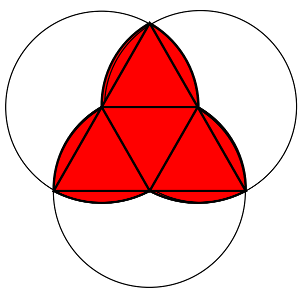


Each of the above sectors is one-third of a full circle of radius 1, thus has area $\frac{\pi}{3}$. Their total area is $\frac{2\pi}{3}$, and they cover our region, but they overlap. We must subtract the area of the overlap from $\frac{2\pi}{3}$ to obtain the area of the region. The overlap consists of two equilateral triangles with side 1:



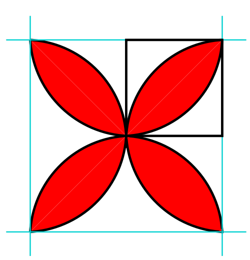
Using the Pathegorean theorem it is easy to find that the height of such a triangle is $\frac{\sqrt{3}}{2}$, and thus the area is $\frac{\sqrt{3}}{4}$. The total area of two triangles is then $\frac{\sqrt{3}}{2}$, and the area of the original region is $\frac{2\pi}{3} - \frac{\sqrt{3}}{2}$.

13 Divide the region into ten regions (four equilateral triangles and 6 thin pieces around them) as follows:



The area of each triangle (see problem 11) is $\frac{\sqrt{3}}{4}$. The area of each thin piece is $\frac{\pi}{6} - \frac{\sqrt{3}}{4}$ (the area of one-sixth of the circle minus the area of the triangle). Thus the total area of the region is $4 \cdot \frac{\sqrt{3}}{4} - 6\left(\frac{\pi}{6} - \frac{\sqrt{3}}{4}\right) = \frac{10\sqrt{3}}{4} - \pi = 2.5\sqrt{3} - \pi$.

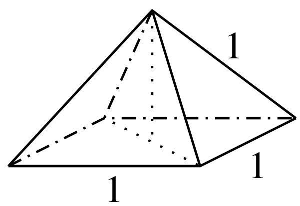
15 Consider one of the four leaves in the figure. It is the overlap of two sectors, one quarter of a circle (of radius $\frac{1}{2}$) each. The area of a quarter of a circle is $\frac{\pi}{16}$, therefore the area of the overlap is $2 \cdot \frac{\pi}{16} - \frac{1}{4} = \frac{\pi}{8} - \frac{1}{4}$.



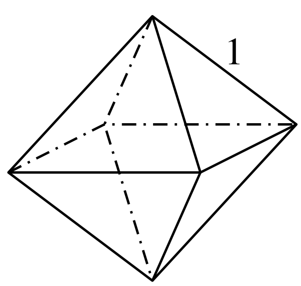
The total area of the region is $4\left(\frac{\pi}{8} - \frac{1}{4}\right) = \frac{\pi}{2} - 1$.

17 Clearly, the base of the box is a square. Let the base be a cm $\times a$ cm. Since the height of the box is 3, its volume is $3a^2$. We are given that the volume is 60 cubic cm, so we have $3a^2 = 60$, so $a^2 = 20$, and $a = \sqrt{20} = 2\sqrt{5}$. Thus the box has dimensions $2\sqrt{5}$ cm $\times 2\sqrt{5}$ cm $\times 3$ cm.

19 Consider the top half of the octahedron.

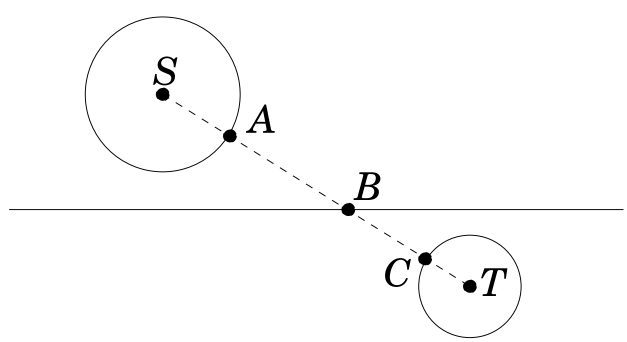


It is a pyramid with a square 1×1 base (whose area is 1). Its diagonal has length $\sqrt{2}$, therefore the distance from the center of the octahedron to any vertex is $\frac{\sqrt{2}}{2}$. Thus the height of the pyramid is $\frac{\sqrt{2}}{2}$, and then its volume is $\frac{1}{3} \cdot 1 \cdot \frac{\sqrt{2}}{2} = \frac{\sqrt{2}}{6}$. Since the octahedron consists of two such pyramids, its total volume is $2 \cdot \frac{\sqrt{2}}{6} = \frac{\sqrt{2}}{3}$.



17.12 Symmetry, Translations, Rotations, and Similarity

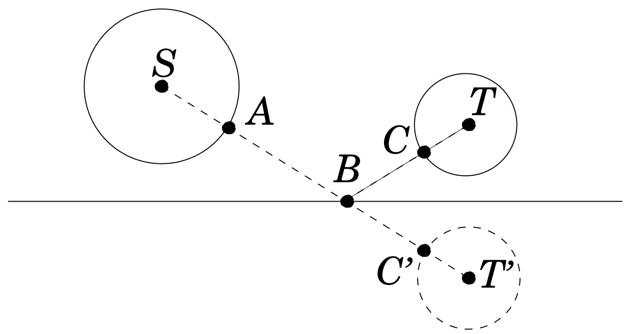
1 Case 1: the circles lie on the opposite sides of the line.



Let S and T be the centers of the given circles, and let r_1 and r_2 be their radii. Draw a line through the centers of the circles (let's call it L). Let A be the intersection point (closest to the given line) of L and the first circle, let B be the intersection point of L and the given line, and let C be the intersection point (closest to the given line) of L

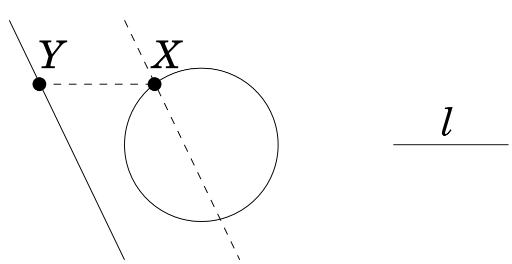
and the second circle. This choice of A, B, and C minimizes $SA + AB + BC + CT$ because the shortest path from S to T is the straight line. Since $SA = r_1$ and $CT = r_2$ are fixed, this choice of A, B, and C minimizes $AB + BC$.

Case 2: the circles lie on the same side of the line.

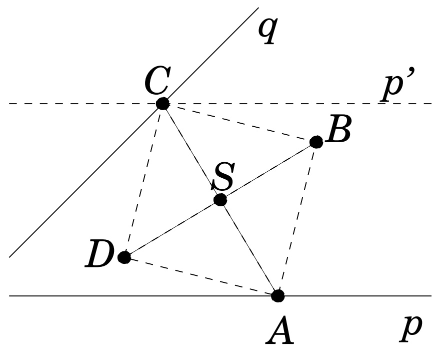


Reflect the second circle about the given line, and draw a straight line through the center of the first circle and the center of the new one. Let the intersection points A, B, and C' be as above. Reflect the point C' about the given line, get a point C on the second (old) circle. This choice of A, B, and C minimizes $AB + BC$ because $AB + BC = AB + BC'$, and we have seen above that this choice of points minimized $AB + BC'$.

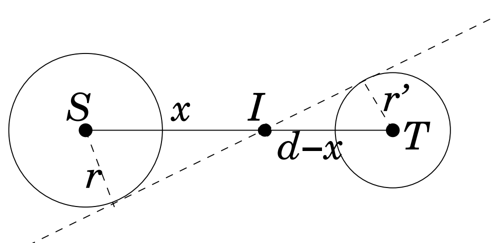
3 Move the line a distance l horizontally (to the right or to the left depending on where the circle is). Let X be an intersection point of the new line and the circle. Move X back, get the point Y on the old line. Then XY is horizontal and has length l. (Note: if the new line and the circle do not intersect, then there is no solution.)



5 Rotate the line p through an angle of 180 degrees around S. Let's call the new line p'. Let C be the intersetion point of p' and q. (Note: if p' and q do not intersect, then there is no solution.) Rotate C back - get a point A on the old line p. Thus we have that A, S, and C lie on the same line, and $SA = SC$. Now to find the remaining vertices of the square, rotate A through 90 degrees around S (in both directions).



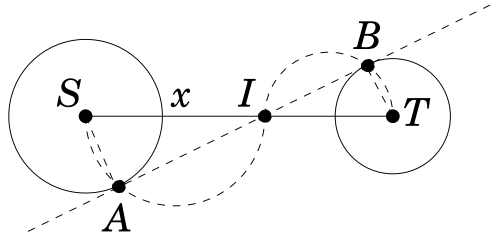
7 Let the centers of the circles be S and T, their radii r and r', and the distance between their centers d. Draw a line through the centers of the circles. We know that it must cross the common tangent line that we are looking for. Let's find the location of the intersection point I. Let the distance between the center of the first circle and the intersetion point be x, then the distance between the center of the second circle and the intersection point is $d - x$.



From similar triangles we see that $\dfrac{x}{r} = \dfrac{d-x}{r'}$.

Solving this equation for x gives $x = \dfrac{r'd}{r+r'}$.

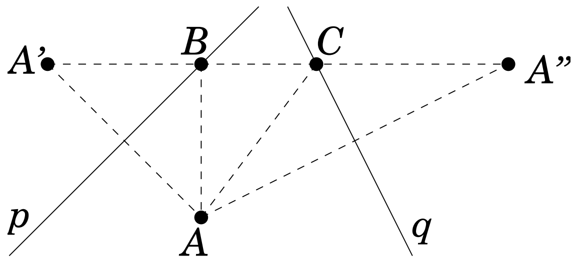
Once we have this intersection point, we draw semicircles with diameters x and $d - x$, and find the intersection points A and B of these semicircles with the given circles. These are the points where the tangent line touches the circles, so we draw a line through these points.



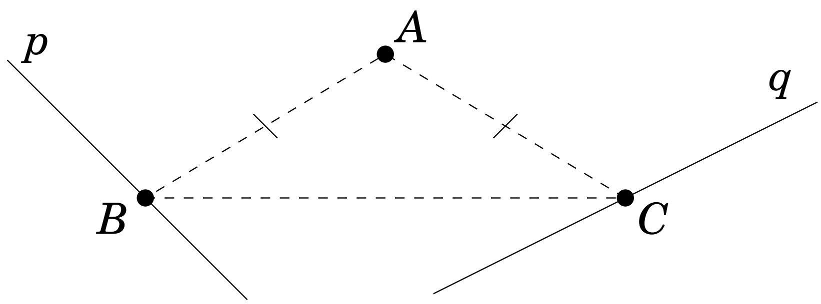
(We know that an angle inscribed in a semicircle is 90 degrees, so both SA and AI are perpendicular, and TB and BI are perpendicular, thus AB is the common tangent.)

9 Reflect the point A about p, get a point A'. Reflect A about q, get a point A''. We want to find B and C on p and q respectively so that to minimize the perimeter of the triangle ABC. Since $AB + BC + CA = A'B + BC + CA''$, the problem is equivalent to minimizing $A'B + BC + CA''$. But this is minimimized when A', B, C, and A'' line on one line (because the shortest path from A' to A'' is the straight line). Thus we

connect A' and A'', and let B and C be the intersection points of $A'A''$ and the lines p and q respectively.



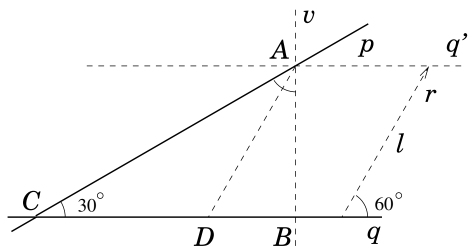
11 Draw a vertical line l through A. Reflect the line p about l, get p'. Let C be the intersection point of p' and q. Now reflect the point C about l, get a point B on p. Since B and C are symmetric about the vertical line l, we have that BC is horizontal and $AB = AC$.



13 Rotate the circle S through an angle of 30° around A (toward circle T). Call the image S'. Let C be an intersection point of S' and T. Rotate the point C back - get a point B on the original circle S. Then $AB = AC$ and $\angle BAC = 30^\circ$. The other angles are as required because $\angle ABC = \angle ACB$ since $\triangle ABC$ is isosceles, and the sum of all the angles in a triangle is 180°.

15 Draw a solution. Notice that $AC = \sqrt{2}AB$ and $\angle BAC = 45^\circ$. This means that if we rotate the point B through 45° around A, let's call the image B', then C lies on the line AB' and $AC = \sqrt{2}AB'$. Thus to find such points, we have to rotate the line p through 45° around A, let's call the image p', and then draw a line p'' parallel to p' and such that the distance from A to p'' is $\sqrt{2}$ times the distance from A to p'. (To do this, pick any point X' on p', draw the line AX', find X'' on AX' such that $AX'' = \sqrt{2}AX'$, and draw the line p'' through X'' and parallel to p'.) Let C be the intersection point of p'' and q. Draw the line AC. Let B' be the intersection point of p' and AC. Rotate B' through 45° around A to get a point B on the original line p. Now we have $\angle BAC = 45^\circ$ and $AC = \sqrt{2}AB' = \sqrt{2}AB$.

17 <u>Solution 1.</u> Since $\angle CAB = 60^\circ$ and AD bisects $\angle CAB$, $\angle DAB = 30^\circ$. Then $\angle ADB = 60^\circ$. Thus the problem is equivalent to finding points A and D such that $\angle ADB = 60^\circ$ and $|AD| = l$. To do this, translate the line q a distance l in the direction of the ray r such that the angle between r and q is 60° (see picture below). Call the new line q'. Let A be the intersection point of p and q'. Translate A back - get point D on the original line q. Now draw a vertical line v through A. Let B be the intersection point of v and q. By our construction, $|AD| = l$, AB is vertical, $\angle ADB = 60^\circ$, thus $\angle DAB = 30^\circ$, and thus AD bisects $\angle CAB$.

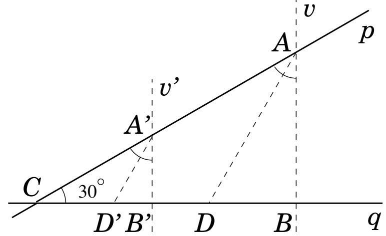


Solution 2. Since $\angle DAB = 30^\circ$, $\dfrac{|AB|}{|AD|} = \cos 30^\circ = \dfrac{\sqrt{3}}{2}$. Then $|AB| = \dfrac{\sqrt{3}}{2}|AD| = \dfrac{\sqrt{3}}{2}l$. Thus we have to translate q a distance $\dfrac{\sqrt{3}}{2}l$ upward (this is doable with a ruler and a compass), and let A be the intersection point of the new line q' and p. (Note: the new line q' here is, of course, the same as in solution 1.) Then draw a vertical line through A to find B.

Solution 3. As in solution 2, $|AB| = \dfrac{\sqrt{3}}{2}l$. Now, since $\angle ACB = 30^\circ$, $\dfrac{|AB|}{|CB|} = \tan 30^\circ = \dfrac{1}{\sqrt{3}}$. Then $|CB| = \sqrt{3}|AB| = \dfrac{3}{2}l$. Thus we just have to find B on q such that $|CB| = \dfrac{3}{2}l$, and then draw a vertical line through B to find A.

Solution 4. Since we need $\angle CAD = 30^\circ = \angle DCA$, $\triangle CAD$ must be isosceles. Thus we need $|CD| = |AD| = l$. Therefore, we have to find D on q such that $|CD| = l$, then draw the line AD such that $\angle ADC = 120^\circ$, and, finally, draw the vertical line AB.

Solution 5. Choose any point A' on p. Draw a vertical line v' through A', and let B' be the intersection point of v' and q. Draw the bisector $A'D'$ of $\angle CA'B'$. The length of $A'D'$ is probably not equal to l. So we have to adjust our construction (proportionally) to make this distance equal to l. Namely, let $A'D' = l'$. Since we need $AD = l$, $\triangle CAB$ we are looking for is similar to $\triangle CA'B'$ with coefficient of similarity $\dfrac{l}{l'}$. Thus, we find A on p such that $|CA| = \dfrac{l}{l'}|CA'|$ (again, this is doable with a ruler and a compass; please ask me if you are not sure how to do this). Then draw a vertical line through A to find B as before, and, finally, draw the bisector AD. From similar triangles, we have $\dfrac{|AD|}{|A'D'|} = \dfrac{|CA|}{|CA'|} = \dfrac{l}{l'}$. Then $|AD| = \dfrac{l}{l'}|A'D'| = \dfrac{l}{l'}l' = l$ as desired.



Note. Solution 4 works only because $\triangle ACD$ must be isosceles which uses the fact that the given angle is 30° and AB must be vertical. Solutions 2 and 3 may work for some other angles and directions of AB, but they still use the fact that all the angles are nice. Solutions 1 and 5 would work for any angle and any direction of AB.

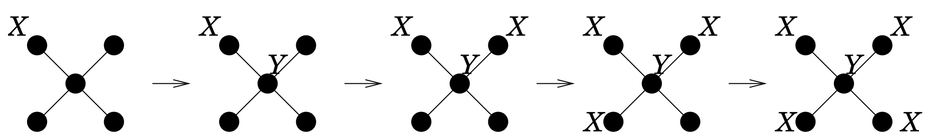
Note. The idea of solution 5 is very useful for many problems.

17.13 Graphs

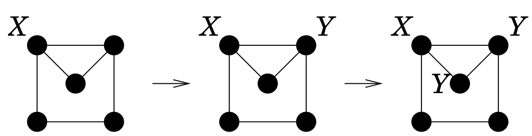
1 By corollary 13.6, in any graph, the number of vertices of odd degree is even. Here there are 3 vertices of degree 3, so there is no such graph.

3 Let vertices represent people, and edges represent friendship (two vertices are connected if and only if the corresponding people are friends). Then the degree of each vertex is the number of friends of the corresponding person. Since in any graph the number of vertices of odd degree is even, we have that the number of people with an odd number of friends is even.

5 We can start with any vertex and assume it's in set X. Then consider any vertex connected with the first one, and if the graph is bipartite, this vertex must be in the other set, say, Y. Then consider any vertex connected with the first or second one, and so on. If we ever run into a situation when two vertices in one set are connected, the graph is not bipartite. If not, we'll have a division of the set of vertices into two sets X and Y such that there are no edges within one set, and hense the graph is bipartite.

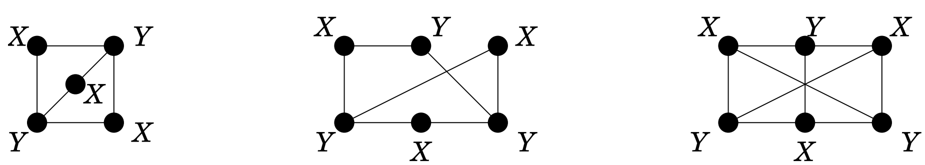


bipartite



contradiction, so not bipartite

Similarly:



bipartite   not bipartite   bipartite

7 If this were possible, consider the following graph with 8 vertices: each vertex represents a county, and two vertices are connected if and only if the corresponding counties are neighbors. Then the degree of each vertex is the number of the neighbors of that county. Thus we would have a graph with 8 vertices of degrees 5, 5, 4, 4, 4, 4, 4, 3. But in any graph, the sum of the degrees of all the vertices is even. The sum $5 + 5 + 4 + 4 + 4 + 4 + 4 + 3 = 33$ is odd. Contradiction.

9 A graph has an Euler path but not an Euler cycle iff it is connected and has two vertices of odd degree.

11 First of all, recall that $K_{n,m}$ has 2 groups of vertices, n vertices in group A, m vertices in group B, and every vertex in group A is connected to every vertex in group B.

(a) We know that there exists an Euler cycle if and only if the degree of each vertex is even. The graph $K_{n,m}$ has n vertices of degree m and m vertices of degree n. Since all the degrees must be even, both m and n must be even.

(b) By problem 9, an Euler path exists if and only if the graph has at most 2 vertices of odd degree. So consider the following cases:

(1) No vertices of odd degree, i.e. all the degrees are even. Then both m and n are even.

(2) 2 vertices of odd degree, both in group A of n vertices. Since all the vertices in this group have the same (odd) degree, and we can have at most 2 vertices of odd degree, there are only 2 vertices in this group, thus $n = 2$. Since their degree is odd, m is odd. Thus we have $n = 2$ and m is odd.

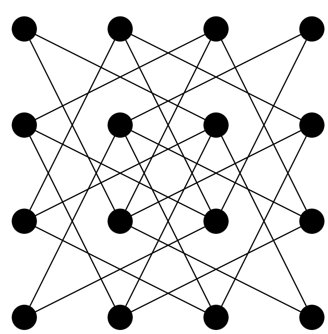
(3) 2 vertices of odd degree, both in group B of m vertices. This case is similar to case (2), only n and m are switched. Thus $m = 2$ and n is odd.

(4) 2 vertices of odd degree, one in group A and the other in group B. Then both m and n are odd, thus all the degrees are odd, but we can have at most 2 odd degrees, so $n = m = 1$.

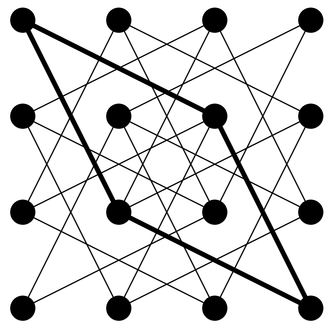
(d) A Hamilton cycle is a cycle that visits every vertex exactly once. If a Hamilton cycle starts at a vertex in group A, then its second vertex belongs to group B, the next one belongs to group A, the fourth one belongs to group B, and so on, i.e. A and B will alternate. It must eventually come back to the original vertex, therefore the number of vertices in group A must be equal to the number of vertices in group B. Thus $m = n$.

(c) A path does not return to the starting point, thus in addition to the case $m = n$ (in this case a path has the form ABAB...AB), we have $m = n - 1$ (then we can find a path of the form ABAB...ABA), and $m = n + 1$ (then we can find a path of the form (BABA...BAB).

13 First draw the graph representing all possible moves of a knight:



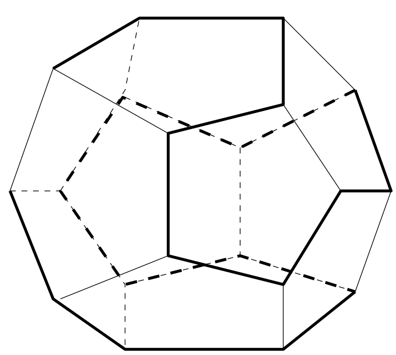
A reentrant tour is a Hamilton cycle. Thus we have to show that this graph has no Hamilton cycle. Notice that there are 4 vertices of degree 2, and in order to visit a vertex of degree 2 we have to use both its edges. Consider the upper left corner vertex and the lower right corner vertex. We must use both edges at each of them. But then we get a cycle. There is no way of adding anything to this cycle (because if we add more edges, we'll have to go through some vertex more than once). But this cycle misses many vertices. Thus there is no Hamilton cycle.



15 If nobody made a mistake, we would be able to draw a bipartite graph with 14 vertices, 7 vertices representing men and 7 vertices representing women, and such that 2 vertices are connected if and only if the corresponding people shared a dance. Then the sum of the degrees of the 7 vertices representing men should be equal to the sum of the degrees of the 7 vertices representing women (both sums being equal to the number of edges). But it is not possible to divide the given 14 numbers into 2 groups such that the sums are equal because one group must contain the 5, and the other group must consist of 3's and 6's. The sum of the numbers in the first group is congruent to 2 mod 3, and the sum of the numbers in the second group is congruent to 0 mod 3, so the sums cannot be equal.

17 Let scientists be represented by vertices. Connect all vertices. We get the complete graph K_{17}. Color the edges using three colors, say, blue, red, and green, according to the topic discussed by the scientists these vertices represent. We have to prove that there are three vertices connected (pairwise) by 3 edges of the same color. Choose any vertex, say, vertex A. It is connected with 16 other vertices. Among the 16 edges connecting vertex A with other vertices at least 6 are of the same color, say, blue. Look at those 6 vertices. If at least two of them are connected by a blue edge then we have a blue triangle. If not, look at the K_6 graph for those 6 vertices. All its edges are red and green. By example 13.18, it contains at least one triangle with all 3 sides of the same color, either red or green.

19 (a) Yes. See the picture below.



(b) Yes. In the above picture, we can connect the two ends of the path.

21 Let the degrees of the remaining vertices be a (in group A) and b (in group B). The sum of degrees of vertices in the first group must be equal to the sum of degrees of vertices in the second group. Thus $4+2+2+a = 3+1+1+b$, or $3+a = b$. Since the graph is connected, the degree of each vertex is at least 1. Thus $a \geq 1$. Now, it is easy

to see that for every pair a, b satisfying $3 + a = b$ and $a \geq 1$, there exists a graph with vertices of such degrees. (Because we can have multiple edges between the vertices of degrees a and b.) Draw a few such graphs!

17.14 Working backwards

1 $46 = 1 \cdot 32 + 14$
$32 = 2 \cdot 14 + 4$
$14 = 3 \cdot 4 + 2$
$4 = 2 \cdot 2$

Thus $d = (46, 32) = 2$.

$$\begin{aligned} 2 &= 14 - 3 \cdot 4 \\ &= 14 - 3(32 - 2 \cdot 14) = 7 \cdot 14 - 3 \cdot 32 \\ &= 7(46 - 1 \cdot 32) - 3 \cdot 32 = 7 \cdot 46 - 10 \cdot 32 \end{aligned}$$

Thus $x = 7$ and $y = -10$.

3 $96 = 1 \cdot 54 + 42$
$54 = 1 \cdot 42 + 12$
$42 = 3 \cdot 12 + 6$
$12 = 2 \cdot 6$

Thus $d = (96, 54) = 6$.

$$\begin{aligned} 6 &= 42 - 3 \cdot 12 \\ &= 42 - 3(54 - 1 \cdot 42) = 4 \cdot 42 - 3 \cdot 54 \\ &= 4(96 - 1 \cdot 54) - 3 \cdot 54 = 4 \cdot 96 - 7 \cdot 54, \text{ so } x = 4 \text{ and } y = -7. \end{aligned}$$

5 Use Euclid's algorithm to find $1 = 67 \cdot 3 - 25 \cdot 8$. Then $6 = 67 \cdot 18 - 25 \cdot 48$, thus $a = 18$ and $b = -48$ work.

Note. Of course, this is not the only pair of such integers.

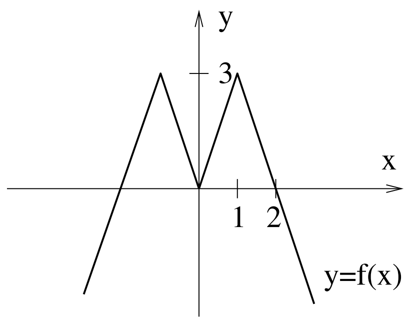
7 We want to find a and b such that

$a = q_1 \cdot b + r_1$
$b = q_2 \cdot r_1 + r_2$
$r_1 = q_3 \cdot r_2 + r_3$
$r_2 = q_4 \cdot r_3 + r_4$
$r_3 = q_5 \cdot r_4 + r_5$
$r_4 = q_6 \cdot r_5 + r_6$
$r_5 = q_7 \cdot r_6 + r_7$
$r_6 = q_8 \cdot r_7$

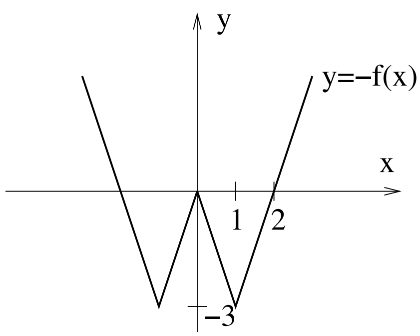
Choose any numbers for r_7 and all the quotients q_i, and work backwards to find all the numbers r_i, b, and a. For example,

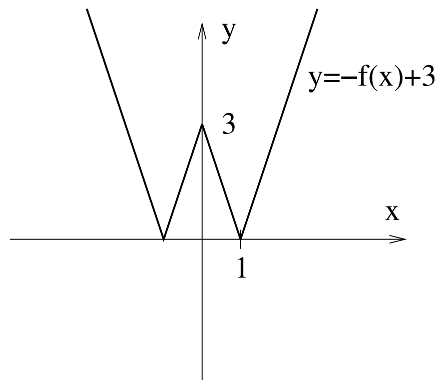
$a = q_1 \cdot b + r_1$	...	$220 = 1 \cdot 127 + 93$
$b = q_2 \cdot r_1 + r_2$	...	$127 = 1 \cdot 93 + 34$
$r_1 = q_3 \cdot r_2 + r_3$	...	$93 = 2 \cdot 34 + 25$
$r_2 = q_4 \cdot r_3 + r_4$	$\uparrow$	$34 = 1 \cdot 25 + 9$
$r_3 = q_5 \cdot r_4 + r_5$	$25 = 2 \cdot 9 + 7$	$25 = 2 \cdot 9 + 7$
$r_4 = q_6 \cdot r_5 + r_6$	$9 = 1 \cdot 7 + 2$	$9 = 1 \cdot 7 + 2$
$r_5 = q_7 \cdot r_6 + r_7$	$7 = 3 \cdot 2 + 1$	$7 = 3 \cdot 2 + 1$
$r_6 = q_8 \cdot r_7$	$2 = 2 \cdot 1$	$2 = 2 \cdot 1$

9 .

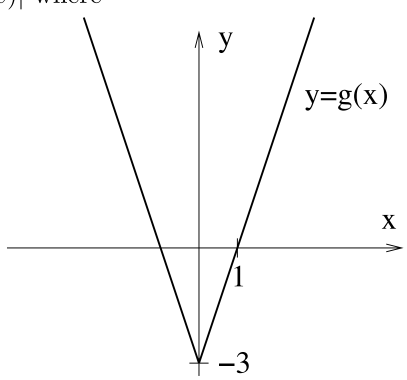


Reflect the given graph about the x-axis (i.e. multiply the function by -1) and shift 3 units upward (i.e. add 3).





Then $-f(x)+3=|g(x)|$ where



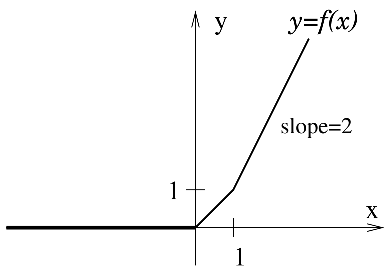
Shifting the graph of $g(x)$ 3 units upward will give the graph of $|3x|$, therefore

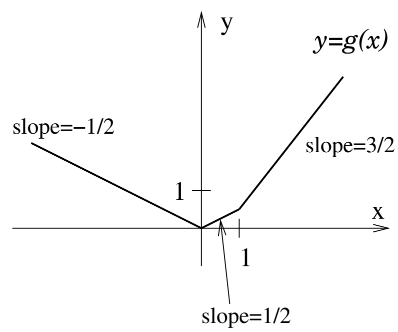
$g(x)+3=|3x|$
$g(x)=|3x|-3$
$-f(x)+3=||3x|-3|$
$f(x)=3-||3x|-3|$

11 Let's denote the function whose graph is given by $f(x)$. Let $g(x)=f(x)-\dfrac{x}{2}$ (using the hint given in problem 10). It's easiest to sketch the graph of $g(x)$ if we write piece-wise linear formulas for $f(x)$ and $g(x)$ first:

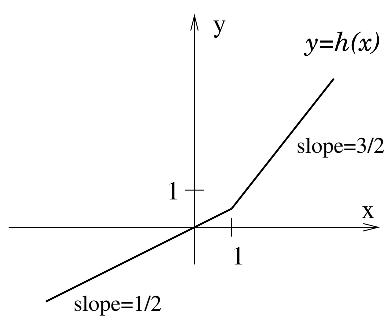
$$f(x) = \begin{cases} 0 & \text{if } x < 0 \\ x & \text{if } 0 \le x < 1 \\ 2x - 1 & \text{if } x \ge 1 \end{cases}, \qquad \text{so} \qquad g(x) = \begin{cases} -x/2 & \text{if } x < 0 \\ x/2 & \text{if } 0 \le x < 1 \\ 3x/2 - 1 & \text{if } x \ge 1 \end{cases}.$$

Then the graphs are as follows:

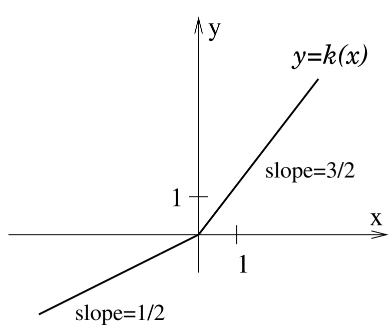




We see that $g(x) = |h(x)|$ where the graph of $h(x)$ is:

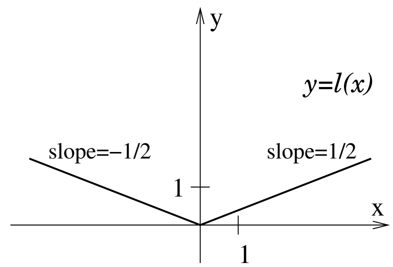


Now we will move the graph of $h(x)$ so that the vertex is at the origin, let's call the new function $k(x)$:



Sinc $y = h(x)$ can be obtained from $y = k(x)$ by shifting it 1 unit to the right and $\frac{1}{2}$ unit upward, $h(x) = k(x-1) + \frac{1}{2}$.

Subtracting x from $k(x)$ gives $l(x) = k(x) - x$:



We see that $l(x) = \left|\frac{x}{2}\right|$.

The equation $l(x) = k(x) - x$ implies $k(x) = l(x) + x = \left|\frac{x}{2}\right| + x$.

Then $h(x) = k(x-1) + \frac{1}{2} = \left|\frac{x-1}{2}\right| + (x-1) + \frac{1}{2} = \left|\frac{x-1}{2}\right| + x - \frac{1}{2}$.

Further, $g(x) = |h(x)| = \left|\left|\frac{x-1}{2}\right| + x - \frac{1}{2}\right|$.

Finally, $g(x) = f(x) - \frac{x}{2}$ implies $f(x) = g(x) + \frac{x}{2} = \left|\left|\frac{x-1}{2}\right| + x - \frac{1}{2}\right| + \frac{x}{2}$.

13 Solution 1.

Suppose the 4-tuple 0, 5, 0, 5 does occur. Then before it we must have the digit 0, and before that 5, and before that another 0... In fact, all the digits in our sequence must be 0's and 5's.

Proof: Solving $a_n \equiv a_{n-4} + a_{n-3} + a_{n-2} + a_{n-1} \pmod{10}$ for a_{n-4} gives

$a_{n-4} \equiv a_n - a_{n-3} - a_{n-2} - a_{n-1} \pmod{10}$.

This implies $a_{n-4} \equiv a_n - a_{n-3} - a_{n-2} - a_{n-1} \pmod{5}$.

Thus if four consecutive digits are divisible by 5, then all the digits in the sequence are divisible by 5.

But the starting sequence 2, 0, 0, 3 contains 2 and 3 which are not divisible by 5. Contradiction.

Solution 2.

If the 4-tuple 0, 5, 0, 5 does occur, look at the very first time it occurs. Then working backwards, determine a few digits before these 4:

$$\begin{array}{ll}
2,0,0,3,\ldots & \ldots,\underline{0,5,0,5},\ldots \\
2,0,0,3,\ldots & \ldots,0,\underline{0,5,0,5},\ldots \\
2,0,0,3,\ldots & \ldots,5,0,\underline{0,5,0,5},\ldots \\
2,0,0,3,\ldots & \ldots,0,5,0,\underline{0,5,0,5},\ldots \\
2,0,0,3,\ldots & \ldots,5,0,5,0,\underline{0,5,0,5},\ldots \\
2,0,0,3,\ldots & \ldots,\underline{0,5,0,5},0,\underline{0,5,0,5},\ldots
\end{array}$$

We see that we have this 4-tuple in the sequence again, hence the one we started with was not the first occurence. Contradiction.

15 We want to force our opponent to take the last counter. Thus we have to leave 1 counter on our last turn. To ensure that we'll be able to do that, we'll leave 6 counters on our next to last turn (then if our opponent takes 1, we take 4 and leave 1; if our

opponent takes 2, we take 3; if they take 3, we take 2; if they take 4, we take 1). On the turn before the next to last we'll leave 11... and so on. Thus we have to go first, take 1 counter and leave 26. Then no matter how our opponent plays we'll be able to leave 21, 16, 11, 6, 1.

17 On our last turn we want to leave one counter. Then our opponent will have to take it, and they will lose. Notice that no matter how our opponent plays, we can always play in such a way that the number of counters our opponent takes plus the number of counters we take is equal to 3 (namely, if they take 1, we can take 2; if they take 2, we can take 1). Thus on our next to last turn we'll leave 4 (then no matter how they play, we'll be able to leave 1). On the turn before that we want to leave 7. This means that we should let our opponent go first. Then, if they take x, we take $3 - x$, and leave 7. Then we leave 4, then we leave 1, and we win.

19 We will work backwards. We want to take the last counter. How many counters should we leave on our next to last turn so that our opponent cannot take the last counter? We can't leave 1 because they will take it. We can't leave 2 either, since they can take 2. But we can leave 3. Then they can take either 1 or 2, that will leave 2 or 1 (respectively), and we will take them. So, we must leave 3 counters.
How many should we leave on the turn before that so that our opponent cannot take all or leave 3? We can't leave 4 (they may take all of them). We can't leave 5 (they may take 2 and leave 3). How about 6? Let's see what choices our opponent has then. If they take 1 and leave 5, that's good - we'll take 2 then. If they take 2 and leave 4, also good for us - we'll take 1. They can also take 4 and leave 2. That's good too, we will then take the last 2. So, we must leave 6.
The turn before that: we don't want to give our opponent an opportunity to leave 0 or 3 or 6. 7 is bad (they may take 1 and leave 6), 8 is bad (they may take 2 and leave 6). How about 9? Our opponent may take 1 (and leave 8) or take 2 (and leave 7) or take 4 (and leave 5) or take 8 (and leave 1). In each case, we'll be able to leave 6 or 3 or 0. So, we must leave 9.
Now notice that the numbers 3, 6, 9 are multiples 3. Does this mean that leaving multiples of 3 is a winning strategy? Let's see... Suppose we leave a multiple of 3. Our opponent will take a power of 2. Since a power of 2 is not divisible by 3, they will leave a number not divisible by 3. Then we can take the remainder, and leave a multiple of 3 again. Thus we have to go first, take 2 counters, and leave 48 (or take 8 and leave 42, or take 32 and leave 18). Then, each time we'll be able to leave a multiple of 3. Thus sooner or later we'll leave 0, and we'll win.

21 The derivative of a cubic polynomial is a quadratic polynomial. We want this quadratic polynomial to have integer roots. Instead of trying random coefficients a, b, c, and d, let's choose the roots of the quadratic polynomial (the derivative of f), and then find f:

Choose the roots, e.g. $r_1 = 3$ and $r_2 = 5$.

$(x-3)(x-5) = x^2 - 8x + 15$.

Now $f(x)$ can be any antiderivative of this polynomial, say, $\dfrac{1}{3}x^3 - 4x^2 + 15x - 3$. However, we want it to have integer coefficients, so let's mulitply this function by 3:

$f(x) = x^3 - 12x^2 + 45x - 9$. (Then $f'(x) = 3x^2 - 24x + 45 = 3(x^2 - 8x + 15) = 3(x-3)(x-5)$ has integer roots.)

Here is another choice of roots and the constant d:

$r_1 = -3$, $r_2 = 4$, $(x+3)(x-4) = x^2 - x - 12$, an antiderivative is $\frac{1}{3}x^3 - \frac{1}{2}x^2 - 12x - \frac{1}{6}$, multiply by 6:

$f(x) = 2x^3 - 3x^2 - 72x - 1$. (Then $f'(x) = 6x^2 - 6x - 72 = 6(x^2 - x - 12) = 6(x+3)(x-4)$ has integer roots.)

23 Start with a matrix in reduced echelon form with integer entries, and perform a few operations (i.e. work backwards in the reducing algorithm) to modify some (or all) coefficients.

For example:

$$\begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix} \leftarrow \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & -2 \\ 0 & -2 & 4 \end{bmatrix} \leftarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & 3 & 6 \\ 0 & -2 & 4 \end{bmatrix} \leftarrow \begin{bmatrix} 1 & 1 & 1 \\ -2 & 1 & 4 \\ 0 & -2 & 4 \end{bmatrix} \leftarrow$$
$$\begin{bmatrix} 1 & 1 & 1 \\ -2 & 1 & 4 \\ 3 & 1 & 7 \end{bmatrix} \leftarrow \begin{bmatrix} 4 & 4 & 4 \\ -2 & 1 & 4 \\ 3 & 1 & 7 \end{bmatrix}$$

25 We will use the following names of coins: penny (1 cent), nickel (5 cents), dime (10 cents), quarter (25 cents).

Based on the amount we see that coins of some value must be present. For example, at least two pennies must be used to make 57 cents. Instead of adding the values of coins, we will subtract the value of known coins from the initial amount and keep track of the number of other coins.

So, since at least two pennies must be present, the value of the other 5 coins is 55 cents. These 5 coins cannot all be less than a quarter, because 5 coins whose value is at most 10 cents would not add up to 55, and no coins can be larger than a quarter since the next available value is 1 dollar, so at least one quarter must be present. Thus the value of the remaining 4 coins is $55 - 25 = 30$. Here no pennies or coins larger than 10 cents can be used, and four dimes would be too much, so at least one nickel is present. The value of the 3 remaining coins is 25, so again at least one nickel must be present, and the value of the remaining 2 coins is 20. The only possibility here is two dimes.

Thus I have two pennies, two nickels, two dimes, and one quarter.

Note. There are other ways to derive the answer.

17.15 Calculus

1 Since $|x+2| = \begin{cases} x+2 & \text{if } x+2 \geq 0, \text{ i.e. } x \geq -2 \\ -(x+2) & \text{if } x+2 \leq 0, \text{ i.e. } x \leq -2 \end{cases}$, we have:

$$\int_{-4}^{2} |x+2|dx = \int_{-4}^{-2} |x+2|dx + \int_{-2}^{2} |x+2|dx = -\int_{-4}^{-2} (x+2)dx + \int_{-2}^{2} (x+2)dx =$$
$$-\left(\frac{x^2}{2} + 2x\right)\Bigg|_{-4}^{-2} + \left(\frac{x^2}{2} + 2x\right)\Bigg|_{-2}^{2} = -(2-4) + (8-8) + (2+4) - (2-4) = 10$$

Note: another way to do this problem is to interpret the integral in terms of areas.

3 Let the given line be tangent to the parabola at the point $(a, a-1)$. Then first, the parabola passes through $(a, a-1)$, thus

$$a - 1 = ca^2.$$

Second, the line and the parabola have the same slope at this point:

$$1 = 2ca.$$

From the second equation we have $c = \dfrac{1}{2a}$. Substitute this for c in the first equation:

$$a - 1 = \frac{a^2}{2a} \quad \Rightarrow \quad a - 1 = \frac{a}{2} \quad \Rightarrow \quad 2a - 2 = a \quad \Rightarrow \quad a = 2 \quad \Rightarrow \quad c = \frac{1}{4}$$

5 We need the polynomial to pass through the given points, and have slope (which is $p'(x) = 3ax^2 + 2bx + c$) equal to 0 at both points.
The value at 0: $d = 1$.
The value at 1: $a + b + c + d = 0$.
The slope at 0: $c = 0$.
The slope at 1: $3a + 2b + c = 0$.
Since $d = 1$ and $c = 0$, the second and fourth equations become $a + b = -1$ and $3a + 2b = 0$. Then $b = -\frac{3}{2}a$, and $a - \frac{3}{2}a = -1$. This gives $a = 2$. Then $b = -3$.
$p(x) = 2x^3 - 3x^2 + 1$.

7 To find the intersection points of the line $y = ax$ and the parabola $y = x^2$, solve $ax = x^2$.

The roots are $x = 0$ and $x = a$, thus the intersection points are $(0, 0)$ and (a, a^2).

If $a > 0$, the area is $\displaystyle\int_0^a (ax - x^2)dx = \left(a\frac{x^2}{2} - \frac{x^3}{3}\right)\Big|_0^3 = \frac{a^3}{2} - \frac{a^3}{3} = \frac{a^3}{6}$.

We want the area to be equal to 1, so $\dfrac{a^3}{6} = 1 \quad \Rightarrow \quad a^3 = 6 \quad \Rightarrow \quad a = \sqrt[3]{6}$

If $a < 0$, then the area is $\displaystyle\int_a^0 (ax - x^2)dx = -\frac{a^3}{6} \quad \Rightarrow \quad a = -\sqrt[3]{6}$.

9 $\displaystyle\sum_{n=0}^{\infty} \frac{1}{2^{2n+1}} = \frac{1}{2} + \frac{1}{2^3} + \frac{1}{2^5} + \frac{1}{2^7} + \ldots = \frac{1}{2}\left(1 + \frac{1}{2^2} + \frac{1}{2^4} + \frac{1}{2^6} + \ldots\right)$
$\displaystyle = \frac{1}{2}\left(1 + \frac{1}{4} + \frac{1}{4^2} + \frac{1}{4^3} + \ldots\right) = \frac{1}{2} \cdot \frac{1}{1 - \frac{1}{4}} = \frac{2}{3}$

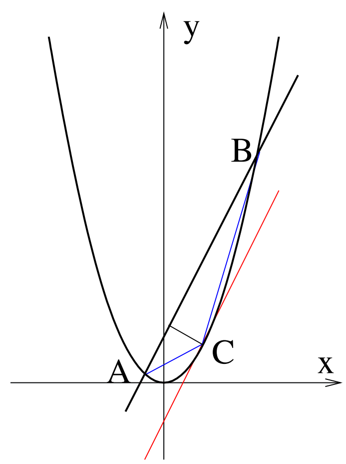
11 Draw a picture so that you see what's going on. Let the slope of such a tangent line be m, then its equation is $y = mx$. Let (a, ma) be the touching point. Since this point lies on the parabola, $ma = a^2 + 2$. The slope of the parabola at the touching point must be m, therefore $2a = m$. Substituting this into the first equation gives $2a^2 = a^2 + 2$. Then $a = \pm\sqrt{2}$, and $m = \pm 2\sqrt{2}$. Thus the equations of the tangent lines are $y = 2\sqrt{2}x$ and $y = -2\sqrt{2}x$.

13 First find the partial fraction decomposition, i.e. A abd B such that
$\dfrac{1}{x^2 + x} = \dfrac{A}{x} + \dfrac{B}{x + 1}$.
Multiply both sides by $x^2 + x = x(x + 1)$:
$1 = A(x + 1) + Bx$
$1 = (A + B)x + A \quad \Rightarrow \quad A + B = 0$ and $A = 1$, then $B = -1$.
Thus $f(x) = \dfrac{1}{x^2 + x} = \dfrac{1}{x} - \dfrac{1}{x + 1} = x^{-1} - (x + 1)^{-1}$.

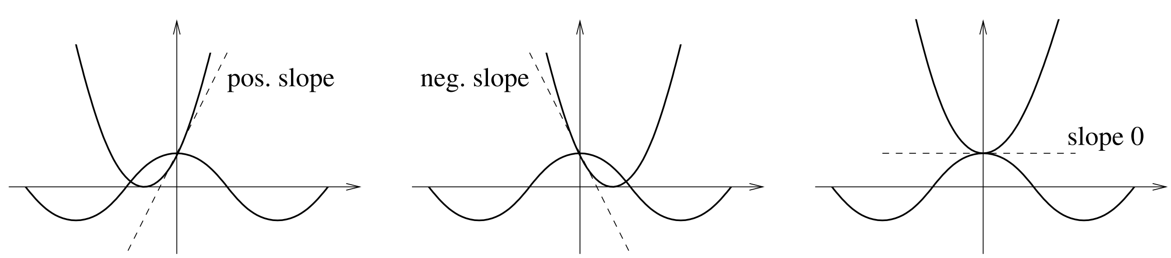
$f'(x) = -x^{-2} + (x+1)^{-2}$
$f''(x) = 2x^{-3} - 2(x+1)^{-3}$
$f'''(x) = -2 \cdot 3x^{-4} + 2 \cdot 3(x+1)^{-4}$
$\ldots$
$f^{(n)}(x) = (-1)^n n! x^{-n-1} - (-1)^n n!(x+1)^{-n-1}$

Note: this formula can be proved by Mathematical Induction.

15 Since A and B are given, the length of AB is given. Now, to maximize the area of $\triangle ABC$, we have to maximize the height h_c. To do this, the point C must lie on the tangent line parallel to the given line. Thus the slope of the parabola at C must be equal to m. Then the x-coordinate of C is $\frac{m}{2}$ (since the slope is $2x$). The y-coordinate of C is then $\frac{m^2}{4}$.



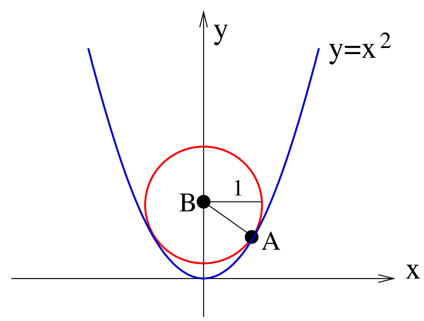
17 Consider the graphs of $f(x) = x^2 + ax + 1$ and $g(x) = \cos x$. Both graphs pass through the point $(0, 1)$. The graph of $g(x) = \cos x$ has slope 0 at that point. If the slope of $f(x) = x^2 + ax + 1$ at $(0, 1)$ is positive, then for some small negative values of x we will have $x^2 + ax + 1 < \cos x$. If the slope of $f(x) = x^2 + ax + 1$ at $(0, 1)$ is negative, then for some small positive values of x we will have $x^2 + ax + 1 < \cos x$. The only case in which $x^2 + ax + 1 \geq \cos x$ for all real x is when the slope of $f(x) = x^2 + ax + 1$ at $(0, 1)$ is 0. The derivative of $f(x)$ is $f'(x) = 2x + a$, thus the slope at $(0, 1)$ is $f'(0) = a$. Therefore $a = 0$ is the only such value of a.



19 Let the point A (with positive x-coordinate) where the circle touches the parabola be (a, a^2), and the center B of the circle be $(0, b)$. Then the distance between these points

is 1, thus

$$a^2 + (b - a^2)^2 = 1.$$



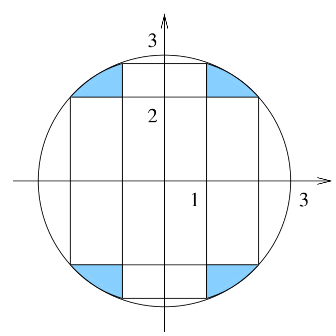
The slope of the parabola at the point (a, a^2) is $2a$ (the derivative of x^2 at $x = a$), then the slope of AB is $-\dfrac{1}{2a}$ (since AB and the parabola are orthogonal at (a, a^2)). Thus we have

$$\frac{b - a^2}{0 - a} = -\frac{1}{2a}.$$

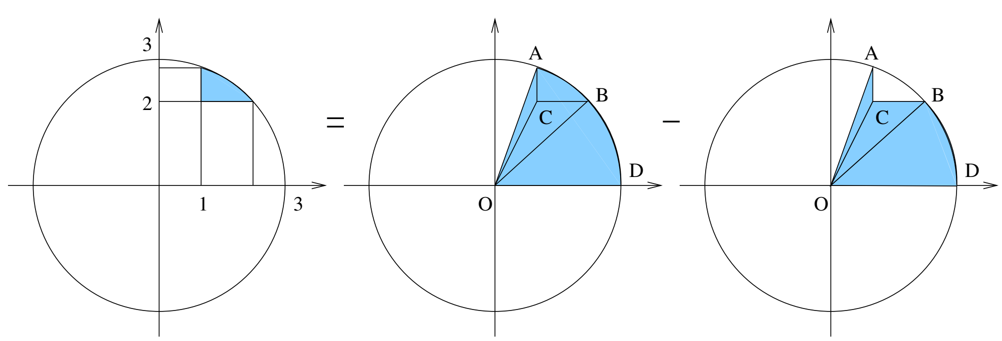
The last equation gives $b - a^2 = \dfrac{1}{2}$, then from the first equation we have

$a^2 + \dfrac{1}{4} = 1 \quad \Rightarrow \quad a^2 = \dfrac{3}{4}$. Then $b = a^2 + \dfrac{1}{2} = \dfrac{3}{4} + \dfrac{1}{2} = \dfrac{5}{4}$.

21 Our region consists of 4 parts of equal area.



The total area of our region is 4 times the area of each part.



The area of each part is the area of sector OAD minus the area of sector OBD minus the area of triangle OBC minus the area of triangle OAC.

A has coordinates $(1, \sqrt{8})$, thus $\text{Area}_{OAD} = \dfrac{9\arccos(1/3)}{2}$.

B has coordinates $(\sqrt{5}, 2)$, thus $\text{Area}_{OBD} = \dfrac{9\arcsin(2/3)}{2}$.

OBC has base $BC = \sqrt{5} - 1$ and height $h_{BC} = 2$, thus $\text{Area}_{OBC} = \dfrac{2(\sqrt{5}-1)}{2}$.

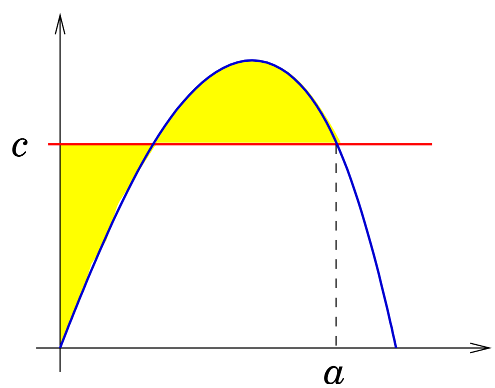
OAC has base $AC = \sqrt{8} - 2$ and height $h_{AC} = 1$, thus $\text{Area}_{OAC} = \dfrac{\sqrt{8}-2}{2}$.

Then $\text{Area}_{ABC} = \dfrac{9\arccos(1/3) - 9\arcsin(2/3) - 2(\sqrt{5}-1) - (\sqrt{8}-2)}{2}$

$= \dfrac{9\arccos(1/3) - 9\arcsin(2/3) - 2\sqrt{5} - \sqrt{8} + 4}{2}$

The total area is then $2(9\arccos(1/3) - 9\arcsin(2/3) - 2\sqrt{5} - \sqrt{8} + 4)$

23 Let the right intersection point have coordinates (a, c). Then $c = 8a - 27a^3$.



If the areas of the shaded regions are equal then the area of the region under the given cubic curve from $x = 0$ to $x = a$ is equal to the area of the rectangle with width a and height $c = 8a - 27a^3$. Thus we have

$\displaystyle\int_0^a 8x - 27x^3 = a(8a - 27a^3)$

$\left(4x^2 - \dfrac{27}{4}x^4\right)\Big|_0^a = 8a^2 - 27a^4$

$4a^2 - \dfrac{27}{4}a^4 = 8a^2 - 27a^4$

$\dfrac{81}{4}a^4 = 4a^2$

$81a^4 = 16a^2$

$81a^2 = 16$ (since $a \neq 0$)

$a = \sqrt{\dfrac{16}{81}} = \dfrac{4}{9}$

Then $c = 8a - 27a^3 = \dfrac{32}{9} - \dfrac{27 \cdot 4^3}{9^3} = \dfrac{32}{9} - \dfrac{64}{27} = \dfrac{96-64}{27} = \dfrac{32}{27}$

25 Assume that a, b, and c are all positive.

Case 1. At least 2 of a, b, and c are equal. Without loss of generality we can assume that $a = b$. Then the intersection of the xy-plane and the ellipsoid is a circle whose equation is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, $z = 0$.

Case 2. The numbers a, b, and c are all distinct. Without loss of generality we can assume that $a > b > c$. The intersection of the xy-plane and the ellipsoid is an ellipse whose equation is $\frac{x^2}{a^2}+\frac{y^2}{b^2} = 1$, $z = 0$. The intersection of the zy-plane and the ellipsoid is an ellipse whose equation is $\frac{z^2}{c^2} + \frac{y^2}{b^2} = 1$, $x = 0$. Now rotate the first ellipse until it coincides the second ellipse so that the y-axis is always in its plane. Then one if its axes is always b, and the other one is changing continuously from $a > b$ to $c < b$. Becuase of continuity, at some point it is equal to b.

27 If the function $a_1 \cos x + a_2 \cos(2x) + \ldots + a_{30} \cos(30x)$ takes on only positive values, then its integral over any interval must be positive (because it is the area of the region under the graph of the function). However,

$$\int_0^{2\pi} (a_1 \cos x + a_2 \cos(2x) + \ldots + a_{30} \cos(30x))dx =$$

$$\left(a_1 \sin x + \frac{a_2}{2} \sin(2x) + \ldots + \frac{a_{30}}{30} \sin(30x)\right)\Big|_0^{2\pi} = 0.$$

Therefore, the given function must take on negative values as well.

29 At $x = 0$ we have $\frac{x}{x^2+1} = \arctan x = x$.

Let's compare the derivatives of these functions for $x > 0$. The derivatives are:

$$\left(\frac{x}{x^2+1}\right)' = \frac{1-x^2}{(x^2+1)^2},$$

$$(\arctan x)' = \frac{1}{x^2+1} = \frac{1+x^2}{(x^2+1)^2},$$

$$(x)' = 1 = \frac{x^2+1}{x^2+1}.$$

We see that for all $x > 0$, $\left(\frac{x}{x^2+1}\right)' < (\arctan x)' < (x)'$, therefore the curve $y = \frac{x}{x^2+1}$ lies below the curve $y = \arctan x$ which lies below the line $y = x$.

31 Obviously, the 4-dimensional volume of a 4-dimensional ball is proportional to the fourth power of its radius. Suppose $V = cr^4$ where c is a constant. Then the 3-dimensional volume of the boundary of this ball is $S = V' = 4cr^3$. Therefore $\frac{V}{S} = \frac{cr^4}{4cr^3} = \frac{r}{4}$. If $r = 4$, $\frac{V}{S} = 1$.

33 $\int_0^1 (\sqrt[3]{1-x^7} - \sqrt[7]{1-x^3})dx = \int_0^1 \sqrt[3]{1-x^7}dx - \int_0^1 \sqrt[7]{1-x^3}dx.$

The integral $\int_0^1 \sqrt[3]{1-x^7}dx$ is equal to the area of the region bounded by $y = \sqrt[3]{1-x^7}$, the x-axis, and the y-axis.

The integral $\int_0^1 \sqrt[7]{1-x^3}dx$ is equal to the area of the region bounded by $y = \sqrt[7]{1-x^3}$, the x-axis, and the y-axis.

Equations $y = \sqrt[3]{1-x^7}$ and $y = \sqrt[7]{1-x^3}$ can be rewritten as $x^7 + y^3 = 1$ and $x^3 + y^7 = 1$ respectively. It is easy to see that both curves pass through $(1, 0)$ and through $(0, 1)$, and these two curves are symmetric about the line $y = x$. Thus the areas of the two regions described above are equal, therefore the difference of the integrals $\int_0^1 \sqrt[3]{1-x^7}dx$ and $\int_0^1 \sqrt[7]{1-x^3}dx$ is 0. Thus $\int_0^1 (\sqrt[3]{1-x^7} - \sqrt[7]{1-x^3})dx = 0$.

35 Since $f(0) = 0$ and $\sin(0) = 0$, $|f(x)| < |\sin(x)|$ for all x, and the slope of $y = \sin(x)$ at $(0, 0)$ is 1, we have $|f'(0)| < 1$.

Since $|f'(0)| = |a_1 + 2a_2 + \ldots + na_n|$, the required inequality follows.

37 Let the curve be given by $y = f(x)$. Since it passes through $(3, 2)$, $f(3) = 2$.

At a point $P(a, f(a))$, the tangent line has slope $f'(a)$, and equation $y - f(a) = f'(a)(x - a)$. Its x-intercept is $\left(0, -\frac{f(a)}{f'(a)} + a\right)$. The part of the tangent line that lies in the first quadrant is bisected by P iff $2a = -\frac{f(a)}{f'(a)} + a$. Thus $af'(a) = -f(a)$. Since this must be true for every point on the curve in the first quadrant, we have the differential equation $xf'(x) = -f(x)$. Any function of the form $f(x) = \frac{c}{x}$ is a solution of this equation. Using the condition $f(3) = 2$, we find $c = 6$. So $f(x) = \frac{6}{x}$ satisfies the required condition.