Math 145 **Fall 2005**

Practice Test 2 - Solutions

- Is it true or false that an integer n is divisible by 12 if and only if it is divisible by both 2 and 6?

  False. For example, 6 *is divisible by both* 2 *and* 6 *but is not divisible by* 12*.*

1. Show that $2^{457} + 3^{457}$ is divisible by 5.

   Computing the first few powers of 2 *and* 3 *modulo 5 gives*
   $2^1 = 2$, $2^2 = 4$, $2^3 = 8 \equiv 3 \ (mod\ 5)$, $2^4 = 16 \equiv 1 \ (mod\ 5)$;
   $3^1 = 3$, $3^2 = 9 \equiv 4 \ (mod\ 5)$, $3^3 = 27 \equiv 2 \ (mod\ 5)$, $3^4 = 81 \equiv 1 \ (mod\ 5)$.
   Therefore $2^{457} + 3^{457} \equiv 2^{456+1} + 3^{456+1} \equiv 2^{456} \cdot 2 + 3^{456} \cdot 3 \equiv 2^{4 \cdot 114} \cdot 2 + 3^{4 \cdot 114} \cdot 3 \equiv (2^4)^{114} \cdot 2 + (3^4)^{114} \cdot 3 \equiv 1 \cdot 2 + 1 \cdot 3 \equiv 0 \ (mod\ 5)$.
   Note. There are many other ways to do this problem.

2. Solve for x: $|x+1| + 5 - x^2 \geq 0$

   Case I. $x + 1 \geq 0$
   $|x+1| = x+1$, *so the inequality becomes*
   $x + 1 + 5 - x^2 \geq 0$
   $x + 6 - x^2 \geq 0$
   $x^2 - x - 6 \leq 0$
   $(x-3)(x+2) \leq 0$
   $-2 \leq x \leq 3$
   The condition $x + 1 \geq 0$ *implies* $x \geq -1$*, so the solution set in this case is* $[-1, 3]$.
   Case II. $x + 1 < 0$
   $|x+1| = -(x+1)$, *so the inequality becomes*
   $-(x+1) + 5 - x^2 \geq 0$
   $-x + 4 - x^2 \geq 0$
   $x^2 + x - 4 \leq 0$
   $$\left(x - \frac{-1+\sqrt{17}}{2}\right)\left(x - \frac{-1-\sqrt{17}}{2}\right) \leq 0$$
   $$\frac{-1-\sqrt{17}}{2} \leq x \leq \frac{-1+\sqrt{17}}{2}$$
   The condition $x + 1 < 0$ *implies* $x < -1$*, so the solution set in this case is*
   $\left[\frac{-1-\sqrt{17}}{2}, -1\right]$.
   Answer: $\left[\frac{-1-\sqrt{17}}{2}, 3\right]$.

3. Let $F_0 = 0$, $F_1 = 1$, $F_2 = 1$, $\ldots$, F_{99} be the first 100 Fibonacci numbers (recall that $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$). How many of them are even?

   We compute the first few Fibonacci numbers: 0, 1, 1, 2, 3, 5, 8, 13, 21, *and notice that every third of them is even. More precisely,* F_n *is even if and only if* $n \equiv 0 \ (mod\ 3)$. *Therefore exactly third of* $F_1, F_2, \ldots, F_{99}$ *is even which gives* 33 *numbers, and* F_0 *is even, thus we have* 34 *even numbers total.*

Note. *The pattern described above can be proved by Strong Mathematical Induction.*

Basis step. If $n = 0$, $F_0 = 0$ *is even.*

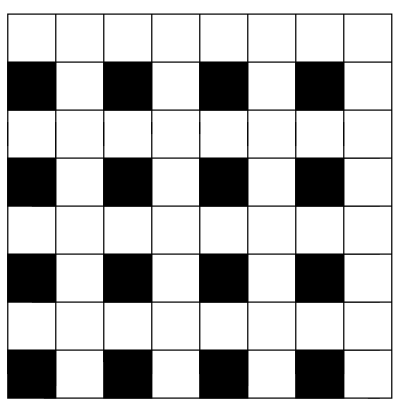
Inductive step. Suppose the statement "F_n *is even if and only if* $n \equiv 0 \pmod 3$*" holds for* $0 \le n \le k$. *We will prove that the statement holds for* $n = k + 1$.
Case I. $k + 1 = 1$. *Then* $k + 1 \not\equiv 0 \pmod 3$, *and* F_1 *is odd.*
Case II. $k + 1 = 2$. *Then* $k + 1 \not\equiv 0 \pmod 3$, *and* F_2 *is odd.*
Case III. $k + 1 \ge 3$. *Then we consider all possible cases of* $k + 1$ *modulo 3.*
Case IIIA. $k + 1 \equiv 0 \pmod 3$. *Then by the inductive hypothesis* F_k *is odd and* F_{k-1} *is odd (since* $k \equiv 2 \pmod 3$ *and* $k - 1 \equiv 1 \pmod 3$*), so* $F_{k+1} = F_k + F_{k-1}$ *is even.*
Case IIIB. $k + 1 \equiv 1 \pmod 3$. *Then by the inductive hypothesis* F_k *is even and* F_{k-1} *is odd (since* $k \equiv 0 \pmod 3$ *and* $k - 1 \equiv 2 \pmod 3$*), so* $F_{k+1} = F_k + F_{k-1}$ *is odd.*
Case IIIC. $k + 1 \equiv 2 \pmod 3$. *Then by the inductive hypothesis* F_k *is odd and* F_{k-1} *is even (since* $k \equiv 1 \pmod 3$ *and* $k - 1 \equiv 0 \pmod 3$*), so* $F_{k+1} = F_k + F_{k-1}$ *is odd.*

4. There are seven 1's and eight -1's on a blackboard. In each step, you may erase any two numbers, say, a and b, and write $-ab$ instead. Show that no matter in what order we erase the numbers, 1 will remain in the end.

   After experimenting with a couple of examples, we notice that the parity of the number of 1*'s is always the same. Here is a proof.*
   If we replace two 1*'s by* -1 *then the number of* 1*'s decreases by* 2*, so its parity is the same as before.*
   If we replace two -1*'s by* -1 *then the number of* 1*'s does not change, so its parity is the same as before.*
   If we replace 1 *and* -1 *by* 1 *then the number of* 1*'s does not change, so its parity is the same as before.*
   We started with an odd number (seven) of 1*'s, therefore an odd number of* 1*'s should remain at the end. Thus* 1 *will remain.*

- Show that if $4 \times 1 \times 1$ bricks and $2 \times 2 \times 2$ cubes fill (without overlap) an $8 \times 8 \times 8$ cube, then the number of $2 \times 2 \times 2$ cubes is even.

  "Color" the small (i.e. $1 \times 1 \times 1$*) cubes of the* $8 \times 8 \times 8$ *cube as follows. Color each other row as in the picture below, and each other row all white.*



  Then the $8 \times 8 \times 8$ *cube contains 64 (which is even) black cubes. Each* $4 \times 1 \times 1$ *brick fills either 0 or 2, so, an even number of black cubes. Thus the* $2 \times 2 \times 2$ *cubes must cover an even number of remaining black cubes. Each* $2 \times 2 \times 2$ *cube fills exactly one black cube, therefore the number of* $2 \times 2 \times 2$ *cubes must be even.*