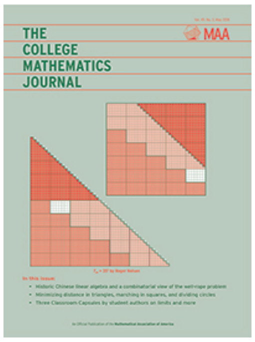

The College Mathematics Journal

ISSN: 0746-8342 (Print) 1931-1346 (Online) Journal homepage: http://maa.tandfonline.com/loi/ucmj20

Problems and Solutions

PROBLEMS AND SOLUTIONS

EDITORS

Jerzy Wojdylo
CMJ Problems
Department of Mathematics
Southeast Missouri State University
One University Plaza
Cape Girardeau, MO 63701
cmjproblems@maa.org

Charles N. Curtis
CMJ Solutions
Mathematics Department
Missouri Southern State University
3950 Newman Road
Joplin, MO 64801
cmjsolutions@maa.org

This section contains problems intended to challenge students and teachers of college mathematics. We urge you to participate actively *both* by submitting solutions and by proposing problems that are new and interesting. To promote variety, the editors welcome problem proposals that span the entire undergraduate curriculum.

Proposed problems should be sent to **Jerzy Wojdylo**, either by email (preferred) as a pdf, TEX, or Word attachment or by mail to the address provided above. Whenever possible, a proposed problem should be accompanied by a solution, appropriate references, and any other material that would be helpful to the editors. Proposers should submit problems only if the proposed problem is not under consideration by another journal.

Solutions to the problems in this issue should be sent to **Chip Curtis**, either by email as a pdf, TEX, or Word attachment (preferred) or by mail to the address provided above, no later than December 15, 2018.

PROBLEMS

1126. *Proposed by George Stoica, Saint John, New Brunswick.*

Prove that $\lim_{x\to\infty} \prod_{n=1}^{\infty}(1 - x^{-n}) = 1.$

1127. *Proposed by D. M. Bătineţu-Giurgiu, Matei Basarab National College, Bucharest, Romania and Neculai Stanciu, George Emil Palade Secondary School, Buzău, Romania.*

For each positive integer n, let $e_n = \left(1 + \frac{1}{n}\right)^n$ and $\lim_{n\to\infty} e_n = e$, Euler's constant, and let

$$s_n = -2\sqrt{n} + \sum_{k=1}^{n} \frac{1}{\sqrt{k}}$$

and $\lim_{n\to\infty} s_n = s$, the Ioachimescu constant. Find $\lim_{n\to\infty} \frac{e - e_n}{(s_n - s)^2}$.

1128. *Proposed by Arthur L. Holshouser, Charlotte, NC, and Benjamin G. Klein, Davidson College, Davidson, NC.*

Let a, b, c, d be positive integers such that a and b are relatively prime, similarly c and d, and $\frac{a}{b} < \frac{c}{d}$. Find necessary and sufficient conditions on a, b, c, d such that if x and y are relatively prime positive integers with $\frac{a}{b} < \frac{x}{y} < \frac{c}{d}$, then $u = -dx + cy$ and $v = bx - ay$ are relatively prime.

doi.org/10.1080/07468342.2018.1445359

1129. *Proposed by David M. Bradley, University of Maine, Orono, ME.*

The natural logarithm satisfies the functional equation $\log(xy) = \log(x) + \log(y)$ for positive real x and y. It also satisfies the inequality $\log(x) \le x - 1$ for positive real x. Show that these two properties characterize the natural logarithm. That is, if the function $f : \mathbb{R}_{>0} \to \mathbb{R}$ satisfies the functional equation $f(xy) = f(x) + f(y)$ and the inequality $f(x) \le x - 1$ for all positive real x and y, then $f(x) = \log(x)$ for all positive real x.

1130. *Proposed by Michael Goldberg, Baltimore Polytechnic Institute, Baltimore, MD, and Mark Kaplan, Towson University, Towson, MD.*

Let T_0 be an arbitrary triangle with vertices A_0, B_0, C_0 and corresponding side lengths a_0, b_0, c_0. Construct triangle T_1 whose vertices A_1, B_1, C_1 are the centers of the squares constructed on the sides A_0B_0, B_0C_0, C_0A_0 outside T_0, respectively; call the corresponding side lengths a_1, b_1, c_1. Continue in this way to build triangles T_2, T_3, etc. Show that there exists $\gamma > 0$ such that there exist finite nonzero limits

$$\lim_{n\to\infty} \frac{a_n}{\gamma^n} = \lim_{n\to\infty} \frac{b_n}{\gamma^n} = \lim_{n\to\infty} \frac{c_n}{\gamma^n},$$

i.e., such that the sequence of triangles $\{T_n \text{ scaled by } 1/\gamma^n\}$ converges to an equilateral triangle.

SOLUTIONS

An inequality involving sums

1101. *Proposed by Mehtaab Sawhney (student), University of Pennsylvania, Philadelphia, PA.*

Suppose $x_1, \ldots, x_n$ and $y_1, \ldots, y_m$ are real numbers satisfying

$$\sum_{i=1}^{n} x_i = 0 \quad \text{and} \quad \sum_{j=1}^{m} y_j = 0.$$

Show, for any real numbers $a_1, \ldots, a_n$ and $b_1, \ldots, b_m$, that

$$2\sum_{i=1}^{n}\sum_{j=1}^{m} x_i y_j |a_i - b_j| \;\ge\; \sum_{i=1}^{n}\sum_{j=1}^{n} x_i x_j |a_i - a_j| + \sum_{i=1}^{m}\sum_{j=1}^{m} y_i y_j |b_i - b_j|.$$

Solution by the proposer.

We give a solution to the following more general problem.
Given $\sum_{i=1}^{n} x_i = \sum_{j=1}^{m} y_j = 0$, show that

$$2\sum_{i=1}^{n}\sum_{j=1}^{m} x_i y_j |\vec{a}_i - \vec{b}_j| \ge \sum_{i=1}^{n}\sum_{j=1}^{n} x_i x_j |\vec{a}_i - \vec{a}_j| + \sum_{i=1}^{m}\sum_{j=1}^{m} y_i y_j |\vec{b}_i - \vec{b}_j|$$

with $\vec{a}_i, \vec{b}_j \in \mathbb{R}^\ell$ and $|\cdot|$ being the Euclidean norm.

Note that

$$\begin{aligned}\mathbb{E}_{|\vec{u}|=1}[\vec{u}\cdot\vec{v}] &= \mathbb{E}_{|\vec{u}|=1}\left[\vec{u}\cdot\left(\frac{\vec{v}}{|\vec{v}|}\right)|\vec{v}|\right]\\ &= |\vec{v}|\ \mathbb{E}_{|\vec{u}|=1}[\vec{u}\cdot\langle 1,0,\ldots,0\rangle]\\ &= C|\vec{v}|\end{aligned}$$

with C being some positive constant. Therefore, the result follows if the analogous one-dimensional inequality holds, as

$$\begin{aligned}&\frac{1}{C}\left(2\sum_{i=1}^{n}\sum_{j=1}^{m}x_iy_j|\vec{a}_i-\vec{b}_j| - \sum_{i=1}^{n}\sum_{j=1}^{n}x_ix_j|\vec{a}_i-\vec{a}_j| - \sum_{i=1}^{m}\sum_{j=1}^{m}y_iy_j|\vec{b}_i-\vec{b}_j|\right)\\ &= \mathbb{E}_{|\vec{u}|}\left[2\sum_{i=1}^{n}\sum_{j=1}^{m}x_iy_j|\vec{a}_i\cdot\vec{u}-\vec{b}_j\cdot\vec{u}| - \sum_{i=1}^{n}\sum_{j=1}^{n}x_ix_j|\vec{a}_i\cdot\vec{u}-\vec{a}_j\cdot\vec{u}|\right.\\ &\qquad\left. - \sum_{i=1}^{m}\sum_{j=1}^{m}y_iy_j|\vec{b}_i\cdot\vec{u}-\vec{b}_j\cdot\vec{u}|\right]\\ &\geq \mathbb{E}_{|\vec{u}|=1}[0] = 0.\end{aligned}$$

Thus, it suffices to prove the original problem,

$$2\sum_{i=1}^{n}\sum_{j=1}^{m}x_iy_j|a_i-b_j| \geq \sum_{i=1}^{n}\sum_{j=1}^{n}x_ix_j|a_i-a_j| + \sum_{i=1}^{m}\sum_{j=1}^{m}y_iy_j|b_i-b_j|$$

with a_i, b_j being real numbers and $\sum_{i=1}^{n} x_i = \sum_{j=1}^{m} y_j = 0$. Furthermore, shifting $\{a_i\}$ and $\{b_j\}$ by a sufficiently large positive constant, it suffices to prove this inequality with a_i, b_j being positive. The key is to realize that since $\sum_{i=1}^{n} x_i = \sum_{j=1}^{m} y_j = 0$, it follows that the desired inequality is equivalent to

$$\begin{aligned}&2\sum_{i=1}^{n}\sum_{j=1}^{m}x_iy_j\left(\frac{|a_i-b_j|-a_i-b_j}{2}\right)\\ &\geq \sum_{i=1}^{n}\sum_{j=1}^{n}x_ix_j\left(\frac{|a_i-a_j|-a_i-a_j}{2}\right) + \sum_{i=1}^{m}\sum_{j=1}^{m}y_iy_j\left(\frac{|b_i-b_j|-b_i-b_j}{2}\right).\end{aligned}$$

However, using the identity

$$\frac{x+y-|x-y|}{2} = \min(x,y)$$

with $x, y \geq 0$, the rewritten inequality is equivalent upon rearranging to

$$\sum_{i=1}^{n}\sum_{j=1}^{n}x_ix_j\min(a_i,a_j) + \sum_{i=1}^{m}\sum_{j=1}^{m}y_iy_j\min(b_i,b_j) \geq 2\sum_{i=1}^{n}\sum_{j=1}^{m}x_iy_j\min(a_i,b_j).$$

Define

$$\lambda_{[0,z]}(x) = \begin{cases} 1 & x \le z \\ 0 & x > z \end{cases}$$

and let $f(t) = \sum_{i=1}^{n} x_i \lambda_{[0,a_i]}(t)$ and $g(t) = \sum_{j=1}^{m} y_j \lambda_{[0,b_j]}(t)$. The final inequality follows as

$$\begin{aligned}
&\int_0^\infty (f(t) - g(t))^2 \, dt \\
&= \sum_{i=1}^{n}\sum_{j=1}^{n} x_i x_j \min(a_i, a_j) + \sum_{i=1}^{m}\sum_{j=1}^{m} y_i y_j \min(b_i, b_j) - 2\sum_{i=1}^{n}\sum_{j=1}^{m} x_i y_j \min(a_i, b_j) \\
&\ge 0.
\end{aligned}$$

No other solutions were received.

A binomial identity

1102. *Proposed by Mehtaab Sawhney (student), University of Pennsylvania, Philadelphia, PA.*

Prove, for all nonnegative integers n, that

$$\sum_{k=0}^{n} \binom{2k}{k} = \sum_{k=0}^{\lfloor n/3 \rfloor} 3^{n-3k} \binom{n-k}{2k} \binom{2k}{k}.$$

Solution by Radouan Boukharfane, University of Poitiers, France.

We use generating functions. For $|x| < \frac{1}{4}$, the generating function of the left-hand side is

$$\begin{aligned}
f(x) &= \sum_{n=0}^{\infty}\sum_{k=0}^{n} \binom{2k}{k} x^n \\
&= \sum_{\ell=0}^{\infty}\sum_{k=0}^{\infty} \binom{2k}{k} x^{\ell+k} \\
&= \sum_{\ell=0}^{\infty} x^\ell \sum_{k=0}^{\infty} \binom{2k}{k} x^k \\
&= \frac{1}{1-x} . \frac{1}{\sqrt{1-4x}}
\end{aligned}$$

and the generating function of the right-hand side is

$$g(x) = \sum_{n=0}^{\infty} \sum_{k=0}^{\lfloor n/3 \rfloor} 3^{n-3k} \binom{n-k}{2k} \binom{2k}{k} x^n$$

$$
\begin{aligned}
&= \sum_{k=0}^{\infty} \binom{2k}{k} 3^{-3k} \sum_{\ell=0}^{\infty} \binom{\ell+2k}{2k} (3x)^{\ell+3k} \\
&= \sum_{k=0}^{\infty} \binom{2k}{k} x^{3k} (1-3x)^{-2k-1} \\
&= \frac{1}{1-3x} \sum_{k=0}^{\infty} \binom{2k}{k} \left(\frac{x^3}{(1-3x)^2} \right) \\
&= \frac{1}{1-3x} \cdot \frac{1}{\sqrt{1-\frac{4x^3}{(1-3x)^2}}} \\
&= \frac{1}{(1-x)\sqrt{1-4x}}.
\end{aligned}
$$

Also solved by HARRIS KWONG, St. U. New York Fredonia; NORTHWESTERN U. MATH PROBLEM SOLVING GROUP; ÁNGEL PLAZA and FRANCISCO PERDOMO (jointly), U. Las Palmas de Gran Canaria, Spain; ROB PRATT, Washington, DC; and the proposer.

Ideals of a polynomial ring

1103. *Proposed by Greg Oman, U. of Colorado, Colorado Springs, CO.*

Let $R = \mathbb{Z}[X_i \mid i \in \mathbb{R}]$ be the polynomial ring over $\mathbb{Z}$ in uncountably many variables indexed by the real numbers. Prove or disprove: There exists a countable collection $\{I_n \mid n \in \mathbb{N}\}$ of ideals of R with the following two properties.

(1) The factor ring R/I_n is countable for every $n \in \mathbb{N}$, and

(2) $\bigcap_{n\in\mathbb{N}} I_n = \{0\}$.

Hint: Does there exist a commutative ring S with identity containing R as a subring and a collection $\{I_n \mid n \in \mathbb{N}\}$ of ideals of S which satisfies (1) (with R replaced with S) and (2)?

Solution by Souvik Dey (student), Indian Statistical Institute, Kolkata, India.

Consider the ring $S = \mathbb{Q}^{\mathbb{N}}$ (the ring with respect to pointwise addition and multiplication of all real sequences with all terms rational). Let $B \subseteq \mathbb{R}$ be a set of $2^{\aleph_0}$ (the cardinality of $\mathbb{R}$) algebraically independent real numbers over $\mathbb{Q}$. (This is possible since the transcendence degree of $\mathbb{R}$ over $\mathbb{Q}$ is the cardinality of the continuum, so we can take B to be a transcendence basis.) For each $b \in B$, choose a sequence $x(b) \in S$ of rational numbers converging to b. (This is possible as $\mathbb{Q}$ is dense in $\mathbb{R}$.) These sequences $x(b)$ will then be algebraically independent in S over $\mathbb{Q}$, since any polynomial with rational coefficients in the $x(b)$ converges to the corresponding polynomial in the b, and so in particular cannot be the zero sequence because B is algebraically independent over the rational numbers. Thus, the $x(b)$ generate a subring of S which, since B and $\mathbb{R}$ have the cardinality, is isomorphic to $\mathbb{Z}[X_i \mid i \in \mathbb{R}] = R$. Now let $J_n \subseteq S$ be the ideal of sequences whose nth coordinate is 0. We then have $\cap_{n\in\mathbb{N}} J_n = \{0\}$, and $S/J_n \cong \mathbb{Q}$ is countable for all $n \in \mathbb{N}$ since J_n is the kernel of the nth coordinate projection map $S \to \mathbb{Q}$. Hence, we have found our required ideals.

Also solved by the proposer.

An ordered field minus transitivity

1104. *Proposed by Greg Oman, University of Colorado, Colorado Springs, CO.*
An *ordered field* consists of a field F along with a binary relation $<$ on F which satisfies the following.

(a) (transitivity) For any $a, b, c \in F$, if $a < b$ and $b < c$, then $a < c$.
(b) (trichotomy) For any $a, b \in F$, exactly one of $a < b$, $a = b$, and $b < a$ holds.
(c) For all $a, b, c \in F$, if $a < b$, then $a + c < b + c$.
(d) For all $a, b, c \in F$, if $a < b$ and $0 < c$, then $ac < bc$.

Now consider dropping the transitivity axiom; call an order $<$ on a field F which satisfies (b), (c), and (d) a *pseudo-order*. Let p be a prime. Show that there exists a pseudo-order on the finite field $\mathbb{Z}/\langle p\rangle$ if and only if $p \equiv 3 \pmod 4$.

Solution by Abhay Goel, Kalamazoo College.

First, suppose F is a field with a pseudo-order $<$. We must have $1 > 0$: More generally, we show that $x^2 > 0$ for nonzero $x \in F$. For $x < 0$, adding $-x$ gives $0 < -x$ and multiplying by $-x$ gives $(0)(-x) < (-x)(-x)$, i.e., $0 < x^2$. For $x > 0$, multiplying by x gives $(x)(x) > (0)(x)$, i.e., $x^2 > 0$. So nonzero squares are always positive and, in particular, $1 = 1^2$ is positive.

Now, note that if $p \equiv 1 \bmod 4$ or if $p = 2$, then -1 is a quadratic residue modulo p. That is, we have some element $x \in \mathbb{F}_p$ with $x^2 = -1$. But this contradicts the above, since $x^2 = -1$ must be positive, and 1 must be positive, so adding the relations $-1 > 0$ and $1 > 0$ gives $0 > 0$. This shows the necessity of $p \equiv 3 \bmod 4$.

Finally, we show sufficiency. Given $p \equiv 3 \bmod 4$, we define $<$ on $\mathbb{F}_p$ as follows. For $x \neq y$, $x < y$ if and only if $y - x$ is a quadratic residue modulo p.

First, note that this satisfies trichotomy. Indeed, if $x \neq y$, then $y - x$ is nonzero, so the Legendre symbol $(\frac{y-x}{p})$ is nonzero. Furthermore,

$$\left(\frac{x-y}{p}\right) = \left(\frac{(-1)(y-x)}{p}\right) = \left(\frac{-1}{p}\right)\left(\frac{y-x}{p}\right) = -\left(\frac{y-x}{p}\right)$$

by choice of p. So, exactly one of $x - y$ and $y - x$ is a quadratic residue, so that exactly one of $x < y$ or $y < x$ holds.

Second, this is clearly additive. If $x < y$ and $c \in F$ is arbitrary, then $x + c < y + c$ since $(y + c) - (x + c) = y - x$ is a quadratic residue by assumption.

Finally, if $c > 0$, then $(\frac{c}{p}) = 1$, so

$$\left(\frac{yc - xc}{p}\right) = \left(\frac{c}{p}\right)\left(\frac{y-x}{p}\right) = 1$$

shows that $yc > xc$ as well. Therefore, it is also invariant under multiplication by positive elements, so it is indeed a pseudo-order as desired.

Also solved by Souvik Dey (student), Indian Stat. Inst., India; Robert Doucette, McNeese St. U.; Eugene Herman, Grinnell C.; Luke Kiernan (student), U. Michigan; Missouri St. U. Problem Solving Group; Kangrae Park, Seoul Science H. S., South Korea; and the proposer. Three incomplete solutions were received.

A three-variable inequality

1105. *Proposed by Ángel Plaza, University of Las Palmas de Gran Canaria, Spain.*
Let x, y, z be positive real numbers and k a nonnegative integer. Prove that

$$\sum_{\text{cyclic}} \frac{x^{2k+2}+y^{2k+2}}{z^{2k+1}} \ \geq\ (xyz)^{k+1} \sum_{\text{cyclic}} \frac{1}{x^{3k+2}} + 3\sqrt[3]{xyz}.$$

Solution by Henry Ricardo, Westchester Area Math Circle.

Two applications of the AM-GM inequality yield

$$\begin{aligned}\sum_{\text{cyclic}} \frac{x^{2k+2}+y^{2k+2}}{z^{2k+1}} &\geq 2\sum_{\text{cyclic}} \frac{(xy)^{k+1}}{z^{2k+1}} \\ &= (xyz)^{k+1}\sum_{\text{cyclic}} \frac{1}{x^{3k+2}} + \sum_{\text{cyclic}} \frac{(xy)^{k+1}}{z^{2k+1}} \\ &\geq (xyz)^{k+1}\sum_{\text{cyclic}} \frac{1}{x^{3k+2}} + 3\sqrt[3]{\frac{(xy)^{k+1}(yz)^{k+1}(zx)^{k+1}}{(xyz)^{2k+1}}} \\ &= (xyz)^{k+1}\sum_{\text{cyclic}} \frac{1}{x^{3k+2}} + 3\sqrt[3]{xyz}.\end{aligned}$$

Also solved by Michel Bataille, Rouen, France; Radouan Boukharfane, U. Poitiers, France; Saumya Dubey, Rutgers U.; James Duemmel, Bellingham, WA; Habib Far, Lone Star C. Montgomery; Dmitry Fleischman, Santa Monica, CA; Eugene Herman, Grinnell C.; Younghun Jo, Seoul Science H. S., South Korea; Harris Kwong, St. U. New York Fredonia; Wei-Kai Lai and John Risher (student), U. S. Carolina Salkehatchie; Jun Sung Oak, Yonsei U., South Korea; Seung Won Park, Yonsei U., South Korea; Paolo Perfetti, U. Roma Tor Vergata, Italy; Digby Smith, Mount Royal U.; Neculai Stanciu, Buzău, Romania and Titu Zvonaru, Comăneşti, Romania; Michael Vowe, Therwil, Switzerland; and the proposer.

