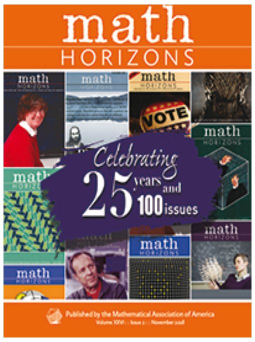


Taylor & Francis
Taylor & Francis Group

Math Horizons

ISSN: 1072-4117 (Print) 1947-6213 (Online) Journal homepage: http://maa.tandfonline.com/loi/umho20

# The Playground

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# THE PLAYGROUND

Welcome to the Playground! Playground rules are posted on page 33, except for the most important one: *Have fun!*

## THE SANDBOX

*In this section, we highlight problems that anyone can play with, regardless of mathematical background. But just because these problems are easy to approach doesn't mean that they are easy to solve!*

**Hexiversary (P378).** This problem is dedicated to Noam Elkies, who submitted the related problem "Rigid Hexagon" to the *third* issue of *Math Horizons*, as well as "Quite a Construction" to the *98th* issue, making him the longest-term contributor on record. A polygon is called *cyclic* if all of its vertices lie on a circle, the radius of which is then known as the *circumradius* of the polygon. What is the circumradius of a cyclic hexagon with three sides of length 25 and three sides of length 100?

**Average Roundup (P379).** Curtis Bennett of Cal State University, Long Beach, contributed this problem. A professor demonstrates parallel processing as follows: Each of the $2^5$ students in the class receives a (independently uniformly distributed) random whole number from 1 to $2^5$, inclusive. The students pair up, take the average of their two numbers (rounding to the nearest whole number, with halves rounding up), and give that number to one of the pair. The students who received the averages repeat the process with their new numbers, and this whole operation continues until only one number remains. What is 100 times the expected difference between this final number and the average of the original $2^5$ numbers assigned to the students?

## THE ZIP-LINE

*This section offers problems with connections to articles that appear in the magazine. Not all Zip-Line problems require you to read the corresponding article, but doing so can never hurt, of course.*

**Splitting Piles (P380).** Tanya Khovanova sent additional examples of games in the spirit of her article "PRIME STEP Plays Games" on page 10. Some of her examples were from the Moscow Olympiad, and inspired by them, we have the following problem: Two players alternate turns in a game that uses two nonempty piles of counters. Each turn consists of discarding one of the two piles completely, and then splitting the other pile into two non-empty piles. The smaller pile can always be discarded; the larger pile can only be discarded if the smaller pile is at least half its size. A player who cannot move (because both piles have just one counter) loses the game. Determine, with proof, which player has a winning strategy when the piles start with 25 and 100 counters.

## THE JUNGLE GYM

*Any type of problem may appear in the Jungle Gym—climb on!*

**Bargain Bernoulli (P381).** David Benko sent us another problem from the University of South Alabama. Used-car salesman Bargain Bernoulli (BB for short) has a hot rod for sale. Its value is \$2,500, but that value depreciates by 1% per month (so its value will be $0.99 \times 2{,}500$ after one month, $0.99^2 \times 2{,}500$ after two months, and so on). However, cognizant of a natural ability to charm the customer, BB is going to set the price of the car at $D > 2{,}500$ dollars. That price will also be marked down 1% per month to keep pace with depreciation. Based on BB's track record, assume that the probability the car is sold in a given month is $V / (100P - 99V)$, where $P$ is the price that month and $V$ is the value that month. What is the maximum expected revenue from selling the car (over all choices BB might make for $D$)?

## THE CAROUSEL—OLDIES, BUT GOODIES

*In this section, we present an old problem that we like so much, we thought it deserved another go-round. Try this, but be careful—old equipment can be dangerous. Answers appear at the end of the column.*

**Why That Fraction? (C23).** In honor of the 25th anniversary of *Math Horizons*, we present the sneakiest problem from the first issue. Let $S$ be a set of 25 distinct real numbers. Prove that there are two elements, $a$ and $b$, in $S$ such that

$$0 < \frac{a-b}{1+ab} < \sqrt{6}-\sqrt{3}+\sqrt{2}-2.$$

## APRIL WRAP-UP

**Frugal Firepower (P370).** David Seppala-Holtzman of St. Joseph's College New York gave us the following problem: A customer orders five identical perfectly spherical cannonballs from Adderley's Cannonball Emporium, and it's your job to pack them for shipping. The Emporium ships only in rectangular boxes but can construct such boxes with any desired dimensions. You have a choice of packing the cannonballs so their centers form a square pyramid or two triangular pyramids, as in figure 1, but you can orient the arrangements however you like inside the box. The shipping cost will be proportional to the sum of the length, width, and height of the box. Determine which arrangement allows you to minimize the shipping cost.

We received no correct solutions to this problem. The square pyramid arrangement is better, but only by a small margin. Note this problem isn't about the (unit spherical) cannonballs at all, but only about their centers. Since every sphere is tangent to

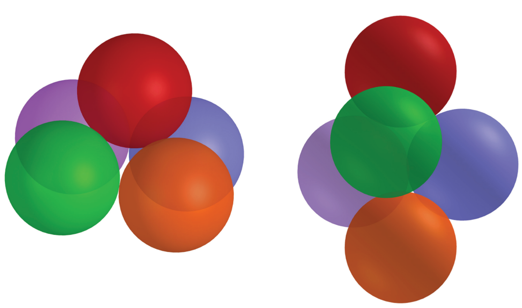

**Figure 1.** Two possible cannonball arrangements.

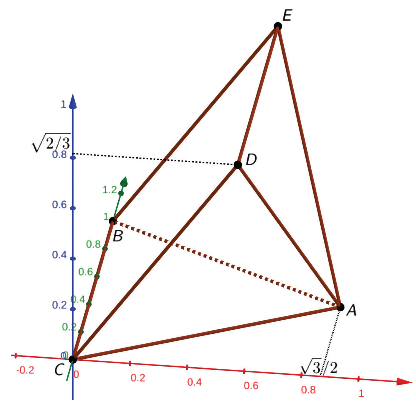


**Figure 2.** Square pyramid in a minimal-boxsize orientation.

the walls it touches, the arrangement of the centers of the cannonballs fits into an $(l,w,h)$-box if and only if the cannonballs fit into an $(l+2,w+2,h+2)$-box. So the question becomes that of finding the smallest box into which a regular square pyramid or a regular triangular bipyramid will fit.

We think of this question as minimizing the sum of the lengths of the projections of the arrangement onto the three coordinate axes—which we call the *boxsize*—as the orientation of the arrangement changes. This optimization is relatively straightforward in the case of the square pyramid. We can assume (by possibly reflecting in a horizontal plane) that the apex $A$ of the square pyramid is not the lowest point. Then by rotating the pyramid about its axis through $A$ and the center of the square face, we can arrange that the two lowest vertices (of the square face) have the same $z$-coordinate, without increasing the boxsize. Now the projection of the arrangement onto the $xy$-plane is a rectangle possibly with one additional point, and we can rotate about the $z$-axis to arrange that the two lowest vertices also have the same $x$-coordinate, without increasing the boxsize. This results in a 1-parameter family of configurations (rotating around the $y$-axis) which is easily optimized; the optimum turns out to correspond to the pyramid lying on one of its triangular sides (see figure 2), with boxsize $l+w+h=\sqrt{3}/2+1+\sqrt{2/3}\approx 2.68$.

The triangular bipyramid is rather more delicate. Without loss of generality, we can assume that the apex $A$ of one pyramid lies at the origin, and the other apex $B$ lies in the first quadrant. It is not hard to see that we can arrange one of the three equatorial vertices, say $C$, to lie on one of the axial planes, say the $xy$-plane, without increasing the boxsize.

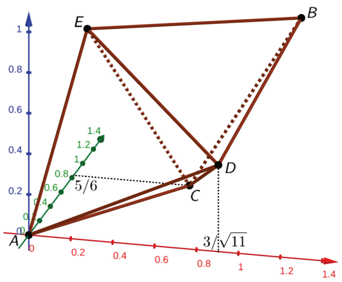


**Figure 3.** Triangular bipyramid in a minimal-boxsize orientation.

That yields a two-parameter family of configurations, which may be optimized graphically/numerically, or through a lengthy analysis of which points are highest/farthest left/farthest back in various portions of the parameter space. The optimum (depicted in figure 3) occurs when another equatorial vertex $D$ lies on the $xz$-plane and edge $BE$ is parallel to the $xy$-plane, with boxsize

$$l + w + h = 32/9\sqrt{11} + 8/9 + \sqrt{8/11} \approx 2.81.$$

**Juggling Numbers (P371).** From the University of South Alabama, David Benko sent a problem about a juggler spinning plates. A *plate of type* $l$ starts a juggling trick spinning at $l$ rotations per second. Every plate slows down by one rotation per second each second. When needed, the juggler can "boost" the spin rate of a plate of type $l$ by $l$ rotations per second. If a plate is not boosted before or at the instant its spin rate reaches zero, it falls off its support, ending the trick. The juggler can perform at most one boost to one plate at the end of each second during the trick. Determine the unordered triples $(l,m,n)$ of positive integers such that the juggler can keep three plates, of types $l$, $m$, and $n$, spinning indefinitely.

This solution was submitted by the Taylor University team of Zachery Huse, Zachary Saltzgaber, Timothy Hotchkiss, and Jordan Crawford. Another Taylor University team (Drew Anderson, Josh Roth, and Benjamin Ryker) and Nicholas Zelinsky of Seton Hall University submitted partial solutions.

Because the triples are unordered, we can consider $(l,m,n)$ such that $l \le m \le n$. With this convention, the juggler can keep any triple of plates $(l,m,n)$ which is coordinate-wise greater than or equal to at least one of (3,3,3), (2,4,4), or (2,3,6) spinning indefinitely. First note that if the juggler can handle $(l,m,n)$, and $(l',m',n') \ge (l,m,n)$ coordinate-wise, then the same schedule of spinning that keeps the former aloft will also work for the latter. So we need only seek the minimal sets of plates that the juggler can succeed with.

Next note that a plate of type $k$ must be spun at least $1/k$ of the seconds to stay aloft. Thus, if

$$\frac{1}{l} + \frac{1}{m} + \frac{1}{n} > 1,$$

the juggler cannot handle the collection of plates. So we must find the minimal solutions of

$$\frac{1}{l} + \frac{1}{m} + \frac{1}{n} \le 1.$$

Any triple with $l \ge 4$ works, so we need only search $l = 3$ and $l = 2$. If $l = 3$, we see that $m \ge 3$ and $n \ge 3$, yielding the first minimal triple. If $l = 2$, then any triple with $m \ge 5$ works, so we need only seek triples with $2 \le m \le 4$. If $m = 4$, then $n \ge 4$, producing the second minimal triple, and if $m = 3$, then $n \ge 6$, producing the third. There are no solutions with $l = 2$ and $m = 2$.

Finally, we must exhibit spinning schedules that keep each of these triples aloft. *lmnlmn*… works for (3,3,3), *llmnllmn*… for (2,4,4), and *llmmlnllmmln*… for (2,3,6).

**Square Squares Sum Square (P372).** Matthew McMullen offered this problem in the spirit of his article *Playing with Blocks,* which focused on triangular numbers: Find the smallest square number (greater than one) of consecutive nonzero squares that sum to a square.

We present the solution of the Northwestern University problem solving group. Other solutions were submitted by Brian Beasley (Presbyterian College), Dmitry Fleischman, Kellianne Yhip (North Central College), the Missouri State University problem solving group, and a team from Taylor University (Anderson, Roth, and Ryker).

The following 49 consecutive squares sum to a square: $25^2 + 26^2 + \cdots + 73^2 = 127{,}449 = 357^2$.

To see this is minimal, use the standard formula for the sum of the first $k$ squares to show that the sum of the $\ell^2$ consecutive squares starting from $(n+1)^2$ is

$$S(n,\ell) = \ell^2 \, \frac{6n^2 + 6\ell^2 n + 6n + 2\ell^4 + 3\ell^2 + 1}{6}.$$

Therefore, $S(n,\ell)$ is a perfect square if and only if

$$\frac{36S(n,\ell)}{\ell^2} = 36n^2 + 36\ell^2 n + 36n + 12\ell^4 + 18\ell^2 + 6$$

is a perfect square. So $\ell$ cannot be even, since if so, this expression is congruent to 2 mod 4, which no square is. Nor can $\ell$ be three, because if so, this expression is congruent to 6 mod 9, and the only squares modulo 9 are 0, 1, 4, and 7.

Eliminating $\ell = 5$ requires a slightly different approach. Note $S(n,5) = 25(n^2 + 26n + 221)$. If this were a perfect square, then $n^2 + 26n + 221 = m^2$ for some $m$, hence

$m^2 - (n+13)^2 = 52$, which factors as $(m+n+13)(m-n-13) = 2^2 \cdot 13$. Because of the factorization, this equation has no solutions in nonnegative integers $m$ and $n$.

**Quite a Construction (P373).** Noam Elkies of Harvard University provided this problem that he discovered while responding to a seemingly unrelated question online. Suppose $BC$ is a diameter of circle $c$ with center $O$, and $ABC$ is a triangle with a right angle at $B$. Moreover, suppose the bisector of angle $A$ meets $BC$ at $A'$ and $AA' = OC$ (see figure 4). It turns out that the length of $OA'$ is the same as the side of a regular $n$-gon inscribed in $c$. Determine $n$.

The first half of the solution below is by the Northwestern University problem solving group, and the second half is from Dmitry Fleischman. Other solutions were received from Karl Hendela (Seton Hall University), Randy Schwartz (Schoolcraft College), and Vasile Teodorovici (Ottawa, Canada), along with partial solutions from Fred Wang (University of Delaware) and a team from Taylor University (Anderson, Roth, and Ryker).

Segment $OA'$ is the side of a regular 14-gon. Without loss of generality, let the radius of the circle be 1. We shall show that both $OA'$ and $2\sin(\tau/28)$ (the side of a regular 14-gon) are roots of $x^3 - x^2 - 2x + 1$, which can easily be seen to have a unique positive root less than 1.

Let $\alpha$ be the common angle $BAA'$ and $A'AC$. Then $AB = \cos\alpha = 2\cot 2\alpha$, whence $\sin^3\alpha - 2\sin^2\alpha - \sin\alpha + 1 = 0$. On the other hand, if $u = OA'$, then $BA' = 1 - u = \sin\alpha$. Substituting $1-u$ for $\sin\alpha$ in the previous expression yields $u^3 - u^2 - 2u + 1 = 0$, as desired.

For the second half, let $\beta = \tau/28$. Evidently, $3\beta$ and $4\beta$ are complementary angles, so $\sin 4\beta = \cos 3\beta$. Expanding with angle sum formulas, $4\sin\beta\cos\beta\cos 2\beta = \cos\beta(4\cos^2\beta - 3)$. Cancelling $\cos\beta$ and applying standard identities again yields $4\sin\beta(1-2\sin^2\beta) = 1 - 4\sin^2\beta$. Rearranging, this is $(2\sin\beta)^3 - (2\sin\beta)^2 - 2(2\sin\beta) + 1 = 0$, again as desired.

Constructing a regular heptagon is one of the classically insoluble geometry problems (along with trisecting an angle, doubling a cube, and squaring a circle). The reader may pleasantly ponder what additional equipment beyond a ruler and compass would be required to carry out the construction of a heptagon (by way of a 14-gon) implicit in this problem.

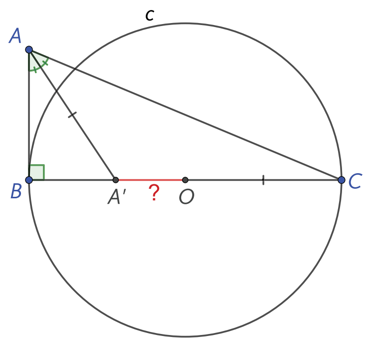


**Figure 4.** What happens when the angle bisector and radius are equal?

Because of a delay in shipping the September issue, we have extended the deadline for the September Playground to December 28.

## CAROUSEL SOLUTION

This problem is dedicated to its original solver, Aleksandr Khazanov. Then a student at Stuyvesant High School in New York, he submitted a perfect paper in the 1994 International Math Olympiad, but has sadly been missing since 2001. Let $T = \{\arctan s : s \in S\}$. Since $T \subset (-\pi/2, \pi/2)$, there are two distinct points in $T$, $\arctan a = \alpha > \beta = \arctan b$, such that $\alpha - \beta < \pi/24$. But note that

$$\tan(\alpha-\beta) = \frac{\tan\alpha - \tan\beta}{1+\tan\alpha\tan\beta} = \frac{a-b}{1+ab}.$$

Therefore,

$$0 = \tan 0 < \frac{a-b}{1+ab} < \tan\frac{\pi}{24} = \sqrt{6} - \sqrt{3} + \sqrt{2} - 2.$$

## SUBMISSION & CONTACT INFORMATION

The Playground features problems for students at the undergraduate and (challenging) high school levels. Problems and solutions should be submitted to *MHproblems@maa.org* and *MHsolutions@maa.org*, respectively (PDF format preferred). Paper submissions can be sent to Glen Whitney, Harvard University Dept. of Mathematics, One Oxford Street, Cambridge, MA 02138. Please include your name, email address, and school affiliation, and indicate if you are a student. If a problem has multiple parts, solutions for individual parts will be accepted. Unless otherwise stated, problems have been solved by their proposers.

The deadline for submitting solutions to problems in this issue is January 11, 2019.

10.1080/10724117.2018.1522883