Mathematics Magazine

ISSN: 0025-570X (Print) 1930-0980 (Online) Journal homepage: http://maa.tandfonline.com/loi/umma20

Problems and Solutions

PROBLEMS

Proposals

To be considered for publication, solutions should be received by November 1, 2018.

2046. *Proposed by Ioan Băetu, Botoşani, Romania.*

For integers m, n such that $1 \leq m < n$, let S_n be the group of all permutations of $\{1, 2, \ldots, n\}$, let F be the set of permutations $\sigma \in S_n$ such that $\sigma(m) < \sigma(m+1) < \cdots < \sigma(n)$, and let T be the set of transpositions in F. Prove that there exists a unique subgroup G of S_n such that $T \subset G \subset F$.

2047. *Proposed by George Stoica, Saint John, New Brunswick, Canada.*

Let (a_n) be a sequence of nonzero real numbers such that

$$\lim_{n\to\infty} n\left(\left|\frac{a_n}{a_{n+1}}\right| - 1\right)$$

exists and is strictly positive. Prove or disprove: The sequence (a_n) is necessarily convergent.

2048. *Proposed by Julien Sorel, Piatra Neamt, PNI, Romania.*

Three points A, B, C are chosen uniformly at random in the three-quarter disk

$$\mathcal{Q} = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1, \text{ and either } x \leq 0 \text{ or } y \leq 0\}$$

obtained by removing the first quadrant of the unit disk. What is the probability that the origin $O = (0, 0)$ lies inside $\triangle ABC$?

Math. Mag. **91** (2018) 230–238. doi:10.1080/0025570X.2018.1456271

We invite readers to submit original problems appealing to students and teachers of advanced undergraduate mathematics. Proposals must always be accompanied by a solution and any relevant bibliographical information that will assist the editors and referees. A problem submitted as a Quickie should have an unexpected, succinct solution. Submitted problems should not be under consideration for publication elsewhere.

Proposals and solutions should be written in a style appropriate for this MAGAZINE.

Authors of proposals and solutions should send their contributions using the Magazine's submissions system hosted at http://mathematicsmagazine.submittable.com. *More detailed instructions are available there. We encourage submissions in PDF format, ideally accompanied by LaTeX source. General inquiries to the editors should be sent to* mathmagproblems@maa.org.

2049. *Proposed by Scott Duke Kominers, Harvard University, Cambridge, MA.*

Show that any finite set of squares in the plane (possibly of different sizes and not necessarily disjoint) has a subset consisting of non-overlapping squares that together cover at least 7% of the area covered by the full set.

2050. *Proposed by Sung Soo Kim, Hanyang University, Ansan, Korea.*

Find the number of sequences $a_1, a_2, \ldots, a_9$ in $\{1, 2, 3\}$ such that

(i) $a_1 = a_2 = 1$, and

(ii) the nine pairs (a_1, a_2), (a_2, a_3), $\ldots$, (a_8, a_9), (a_9, a_1) are the same as the nine pairs $(1, 1)$, $(1, 2)$, $\ldots$, $(3, 2)$, $(3, 3)$ in some order.

Quickies

1081. *Proposed by Lokman Gökçe, Adana, Turkey.*

Let $\triangle ABC$ be an obtuse "golden triangle" with angles $\angle BAC = 108^\circ$, $\angle BCA = \angle CBA = 36^\circ$. Let R and r be the circumradius and inradius of $\triangle ABC$, respectively, and let I be its incenter. Show that

$$AI + BI + CI = 3R - 2r, \qquad \text{and}$$
$$AI^2 + BI^2 + CI^2 = 4R^2 - 6Rr.$$

1082. *Proposed by Michel Bataille, Rouen, France.*

Let f be a nonnegative, continuous function on $[0, 1]$. Prove that

$$\frac{1}{6}\int_0^1 f(x)\,dx + \frac{1}{2}\int_0^1 (f(x))^5\,dx \geq \frac{1}{3}\int_0^1 (f(x))^4\,dx + \frac{1}{5}\int_0^1 (f(x))^2\,dx.$$

Solutions

When an operation on primes commutes **June 2017**

2021. *Proposed by Mihai Caragiu, Ohio Northern University, Ada, OH.*

For an integer $n > 1$, let $\Pi(n)$ be the greatest prime factor of n. Consider a binary operation '$*$', on the set $\mathcal{P} = \{2, 3, 5, 7, \ldots\}$ of all primes, defined by $p * q = \Pi(2p + q)$ for all primes p, q. Find all distinct primes p, q such that $p * q = q * p$.

Solution by Abhay Goel (student), Kalamazoo College, Kalamazoo, MI.

We have $2 * 5 = \Pi(9) = 3 = \Pi(12) = 5 * 2$ and $2 * 23 = \Pi(27) = 3 = \Pi(48) = 23 * 2$, so we have solutions $\{p, q\} = \{2, 5\}$ or $\{2, 23\}$. We prove that there are no others.

Assume that distinct primes p, q satisfy $p * q = q * p = r$, say. Thus, r is a prime, $r \mid 2p + q$, and $r \mid 2q + p$. Hence, $r \mid 3(p + q) = (2p + q) + (2q + p)$. Since r is prime, either $r = 3$ or $r \mid p + q$. However, if $r \mid p + q$, then $r \mid p = (2p + q) - (p + q)$ and $r \mid q = (2q + p) - (p + q)$; since p, q are different primes, they cannot have a common prime factor r. Thus, we must have $r = 3$. Since r is the largest prime divisor of each

$2p+q$ and $2q+p$, both these numbers must be of the form 2^a3^b for some integer exponents $a \geq 0$, $b \geq 1$. We prove that $p=2$ or $q=2$. Indeed, if we had $p \neq 2$ and $q \neq 2$, then p and q, and hence $2p+q$ and $2q+p$, would both be odd; thus,

$$2p+q=3^b \quad \text{and} \quad 2q+p=3^c, \qquad \text{for some integers } b, c \geq 1.$$

Solving for p, q we obtain,

$$p = 2\cdot 3^{b-1} - 3^{c-1} \quad \text{and} \quad q = 2\cdot 3^{c-1} - 3^{b-1}.$$

If we had $b > 1$ and $c > 1$ then the expressions above show that $3 \mid p$ and $3 \mid q$, contradicting the assumption that p and q are distinct primes; therefore, $b = 1$ or $c = 1$. Now, if $b = 1$, then $p = 2 - 3^{c-1} < 2$; if $c = 1$, then $q = 2 - 3^{b-1} < 2$, a contradiction. Thus, we conclude that $p = 2$ or $q = 2$. Without loss of generality, assume $p = 2$ and $q > 2$. Since $2p + q = 4 + q$ is odd,

$$4+q=3^b \quad \text{and} \quad 2q+2=2^a3^c \qquad \text{for some integers } a, b, c \geq 1.$$

It follows that

$$3(3^{b-1}-1) = 3^b - 3 = q+1 = 2^{-1}(2q+2) = 2^{a-1}3^c.$$

Clearly, $3^c \leq 2^{a-1}3^c < 3^b$, so we must have $b > c \geq 1$. Thus, the left-hand side of the equation above is divisible by 3 but not by 3^2, while the right-hand side is divisible by 3^c. Hence, we must have $c = 1$, so the equation above gives

$$3^{b-1} - 1 = 2^{a-1} \qquad \text{for integers } a, b \geq 1.$$

The solutions to this Diophantine equation are well known, namely $(a-1, b-1) = (1, 1)$ or $(3, 2)$ (a proof is given below for completeness). Thus, $b = 2$ or $b = 3$, giving $q = 3^2 - 4 = 5$ or $q = 3^3 - 4 = 23$ (and $p = 2$), and so proving that the only solutions to the problem are $\{2, 5\}$ and $\{2, 23\}$. Now we prove:

If $2^n + 1 = 3^m$ for nonnegative integers n, m, then either $n = 1$ and $m = 1$, or else $n = 3$ and $m = 2$.

Clearly, these are the all the solutions with $n \leq 3$, so it only remains to show that there are no solutions with $n \geq 4$. If $n \geq 4$, then 2^n is divisible by $2^4 = 16$, so we have

$$3^m = 2^n + 1 \equiv 1 \pmod{16}.$$

The smallest proper power of 3 satisfying the above congruence is $3^4 = 81 \equiv 1 \pmod{16}$, hence $4 \mid m$. However, we also have $3^4 = 81 \equiv 1 \pmod{5}$, hence $3^m \equiv 1 \pmod{5}$, so

$$2^n = 3^m - 1 \equiv 1 - 1 = 0 \pmod{5}.$$

Since no power of 2 is a multiple of 5, this contradiction shows that there are no solutions with $n \geq 4$, completing the proof.

Also solved by Michel Bataille (France), Brian Beasley, Bruce Burdick, Robert Calcaterra, John Christopher, Joseph DiMuro, Wenwen Du & Paul Peck, James Duemmel, Dmitry Fleischman, Michael Goldenberg & Mark Kaplan, Russell Gordon, Graham Lord, Rick Mabry, Missouri State University Problem Solving Group, Michael Reid, Celia Schacht, Nicholas Singer, Skidmore College Problem Group, John Smith, David Stone & John Hawkins, Joseph Walsh, Edward White & Roberta White, and the proposer. There were two incomplete or incorrect solutions.

Poorly distributed irrational geometric sequences (modulo 1) **June 2017**

2022. *Proposed by Mihàly Bencze, Bucharest, Romania.*

Given an irrational number $\beta > 1$, show that there exists a number $\alpha \in (1, 2)$, such that

$$0 < \{\alpha\beta^n\} < \frac{1}{\beta - 1} \qquad \text{for all } n \in \mathbb{N},$$

where $\{x\}$ denotes the fractional part of x.

Solution by Northwestern University Math Problem Solving Group, Northwestern University, Evanston, IL.

Let $\beta > 1$ be irrational. If $1 < \beta \leq 2$, then $1/(\beta - 1) \geq 1 > \{\alpha\beta^n\}$ for all n and all α, so we only need to choose α ensuring that the inequality $\{\alpha\beta^n\} > 0$ holds, i.e., that $\alpha\beta^n$ is not an integer, for any $n \in \mathbb{N}$. In case β is algebraic, choose α to be any transcendental number in $(1, 2)$; if β is transcendental, let α be any irrational algebraic number in $(1, 2)$. Then, $\alpha\beta^n$ is transcendental (or, when $n = 0$, at least irrational) and hence not an integer for arbitrary $n \in \mathbb{N}$, completing the proof when $\beta \leq 2$.

Henceforth, assume $\beta > 2$. Define a sequence $(d_n)_{n\geq 0}$ of integers recursively by:

$$d_0 = 1, \qquad \text{and} \qquad d_n = \lceil \beta d_{n-1} \rceil \quad \text{for } n \geq 1,$$

where $\lceil x \rceil$ denotes the least integer no less than x. Since d_0 and β are positive, it is clear that $d_n \geq 1$ for all n. Let $\alpha_n = d_n\beta^{-n}$. Since d_n is a positive integer and β is irrational, βd_n is not an integer, so $d_n = \lceil \beta d_{n-1} \rceil > \beta d_{n-1}$ for all $n \geq 1$. Thus, $\alpha_n = d_n\beta^{-n} > d_{n-1}\beta^{-(n-1)} = \alpha_{n-1}$ for all $n \geq 1$, so $(\alpha_n)_{n\geq 0}$ is strictly increasing. Next, we have, for $n \geq 1$:

$$\begin{aligned} 0 < \alpha_n - \alpha_{n-1} &= d_n\beta^{-n} - d_{n-1}\beta^{-(n-1)} = (d_n - \beta d_{n-1})\beta^{-n} \\ &= (\lceil \beta d_{n-1} \rceil - \beta d_{n-1})\beta^{-n} < \beta^{-n}, \end{aligned}$$

since $\lceil x \rceil - x < 1$ for all x. Therefore, the sequence (α_n) is strictly increasing, and we have

$$1 = \alpha_0 < \alpha_n = \alpha_0 + \sum_{i=1}^{n} (\alpha_i - \alpha_{i-1}) < 1 + \sum_{i=1}^{n} \beta^{-n} = \frac{\beta}{\beta - 1}$$

for all $n \geq 1$. By the monotone sequence theorem, the sequence (α_n) has a limit $\alpha = \lim_{n\to\infty} \alpha_n$ such that $1 < \alpha \leq \beta/(\beta - 1) < 2$ (by the assumption $\beta > 2$), hence $\alpha \in (1, 2)$, and moreover $\alpha_n < \alpha$ for all n since (α_n) is strictly increasing; furthermore,

$$0 < \alpha - \alpha_n = \sum_{i=n+1}^{\infty} (\alpha_i - \alpha_{i-1}) < \sum_{i=n+1}^{\infty} \beta^{-i} = \frac{\beta^{-n}}{\beta - 1},$$

hence $0 < \alpha\beta^n - d_n = \beta^n(\alpha - \alpha_n) < 1/(\beta - 1) < 1$ (since $\beta > 2$). It follows that $\{\alpha\beta^n\} = \alpha\beta^n - d_n \in (0, 1/(\beta - 1))$ for all n, concluding the proof.

Editor's Note. Celia Schacht and George Stoica independently communicated that the property stated in the problem was proved by R. Tijdeman in 1972. (R. Tijdeman, Note on Mahler's 3/2-problem, *Det Kongelige Norske Videnskabers Selskabs Skrifter* **16** (1972) 1–4.) Celia Schacht further pointed out that A. Dubickas has improved upon Tijdeman's result in recent years, and also remarked that the numbers α satisfying the stated property for a fixed irrational $\beta > 2$ form a zero-measure set of exceptions to Weyl's equidistribution property (H. Weyl, Über die Gleichverteilung von Zahlen modulo Eins, *Mathematische Annalen* **77** (1916) 313–352): For any irrational $\beta > 1$ and almost every $\alpha \in \mathbb{R}$, the fractional parts $\{\alpha\beta^n\}$ $(n = 0, 1, \ldots)$ are uniformly distributed in $[0, 1)$ (in particular, they are dense in $[0, 1)$).

Also solved by Robert Calcaterra, Souvik Dey (India), Soo Young Kim (South Korea), Reiner Martin (Germany), Michael Reid, Celia Schacht, Nicholas C. Singer, George Stoica (Canada), Enrique Treviño, and the proposer. There was one incomplete or incorrect solution.

Expressing natural numbers in the form $a + a + b + c$ **June 2017**

2023. *Proposed by Mircea Merca, Craiova, Romania.*

For every natural number n, let $f(n)$ be the number of representations of n in the form

$$n = a + a + b + c$$

where a, b, c are distinct positive integers such that $b < c$. Show that there are infinitely many values of n such that $f(n + 1) < f(n)$.

Solution by Michael Reid, University of Central Florida, Orlando, FL.
For a positive integer m, let $g(m)$ denote the number of expressions $m = b + c$ with b, c positive integers and $b < c$. For each integer b in the range $1 \le b < m/2$ there is exactly one such expression, so $g(m) = \lfloor (m - 1)/2 \rfloor$. Next, let $\widetilde{f}(n)$ be the number of expressions $n = a + a + b + c$ with a, b, c positive integers and $b < c$. For each a in the range $1 \le a < n/2$ there are exactly $g(n - 2a)$ such expressions, hence

$$\begin{aligned}\widetilde{f}(n) &= \sum_{a=1}^{\lfloor \frac{n-1}{2} \rfloor} g(n-2a) = \sum_{a=1}^{\lfloor \frac{n-1}{2} \rfloor} \left\lfloor \frac{n-2a-1}{2} \right\rfloor = \sum_{a=1}^{\lfloor \frac{n-1}{2} \rfloor} \left(\left\lfloor \frac{n-1}{2} \right\rfloor - a \right) \\ &= \frac{1}{2} \left\lfloor \frac{n-1}{2} \right\rfloor^2 - \frac{1}{2} \left\lfloor \frac{n-1}{2} \right\rfloor = \frac{1}{2} \left\lfloor \frac{n-1}{2} \right\rfloor \left\lfloor \frac{n-3}{2} \right\rfloor .\end{aligned}$$

Finally, to compute $f(n)$, we subtract from $\widetilde{f}(n)$ the number of expressions $n = a + a + b + c$ where either $a = b$ or $a = c$. For each $a \ge 1$ with $n - 2a > a$, there is exactly one such expression to exclude (namely $n = a + a + a + (n - 3a)$ if $a < n - 3a$, $n = a + a + (n - 3a) + a$ if $n - 3a < a$), except when $n - 3a = a$, in which case there are none to exclude. Thus, $f(n) = \widetilde{f}(n) - \lfloor (n - 1)/3 \rfloor + \delta(n)$, where $\delta(n) = 1$ if 4 divides n; otherwise, $\delta(n) = 0$. This gives the explicit formula

$$f(n) = \frac{1}{2} \left\lfloor \frac{n-1}{2} \right\rfloor \left\lfloor \frac{n-3}{2} \right\rfloor - \left\lfloor \frac{n-1}{3} \right\rfloor + \delta(n).$$

For any positive integer n of the form $12k + 9$ (k a nonnegative integer), we have:

$$\begin{aligned} f(n) &= f(12k+9) = \tfrac{1}{2}(6k+4)(6k+3) - (4k+2) + 0 = 18k^2 + 17k + 4, \\ f(n+1) &= f(12k+10) = \tfrac{1}{2}(6k+4)(6k+3) - (4k+3) + 0 = 18k^2 + 17k + 3. \end{aligned}$$

This provides infinitely many n satisfying $f(n + 1) = f(n) - 1 < f(n)$.

Also solved by Dawson Bolus, Robert Calcaterra, Dmitry Fleischman, Graham Lord, Missouri State University Problem Solving Group, Celia Schacht, Nicholas Singer, David Stone, Brendan Sullivan, Enrique Treviño, Edward White, and the proposer.

Multiple binary expansions of 1 to a fractional base **June 2017**

2024. *Proposed by George Stoica, Saint John, New Brunswick, Canada.*

A *binary expansion* is an expression of the form

$$0.d_1d_2d_3\dots d_i\dots$$

where each numeral (digit) d_i is either 0 or 1 ($i = 1, 2, 3, \dots$). Given a real number $\beta > 1$ (called the *base*) the *base-β value* of the binary expansion above is

$$(0.d_1d_2d_3\dots)_\beta = \sum_{i=1}^{\infty} \frac{d_i}{\beta^i}.$$

(i) If $1 < \beta < 2$, show that some binary expansion has base-β value equal to 1; in fact, if $\beta \le \phi$, where $\phi = (1 + \sqrt{5})/2$ is the golden ratio, then there are infinitely many such expansions.

(ii) Find all binary expansions with value 1 when $\beta = \phi$.

Editor's Note. Due to an editorial error, part (*i*) of the original published problem (*Math. Mag.* **90** (2017) 231–238) asked for a proof that infinitely many binary expansions have base-β value equal to 1 for any $\beta \in (1, 2)$. This, however, is not true: The Komornik–Loreti constant $\gamma \doteq 1.7872356\dots$ has the property that the value 1 corresponds to a unique base-γ binary expansion, but to multiple binary expansions to any base $\beta \in (1, \gamma)$. (V. Komornik and P. Loreti, Unique Developments in Non-Integer Bases, *Amer. Math. Monthly* **105** (1998) 636–639.) The critical base γ is the unique positive number such that

$$1 = (0.d_1d_2\dots)_\gamma = (0.1101001100101101\dots)_\gamma$$

where $d_1, d_2, \dots$ is the Thue–Morse sequence recursively defined by $d_0 = 0$, $d_{2n} = d_n$, and $d_{2n+1} = 1 - d_n$ for $n = 0, 1, 2, \dots$. The solution below only proves that infinitely many expansions with value 1 exist for $\beta \le \phi \doteq 1.618\dots$. We apologize for this mistake.

Solution by the editors.
(*i*) Fix $\beta \in (1, 2)$. As customary, we identify a binary expression $\mathbf{d} = 0.d_1d_2\dots$ with its value $(\mathbf{d}) = (0.d_1d_2\dots)_\beta$. A terminating binary expansion $0.d_1d_2\dots d_n$ means $0.d_1\dots d_n000\dots$, while $0.d_1d_2\dots d_k\overline{1}$ means $0.d_1d_2\dots d_k111\dots$ (an expansion with a tail of unit digits). By $\mathbf{d}_{\le n}$ (resp., $\mathbf{d}_{<n}$) we mean the expansion $0.d_1\dots d_n$ (resp., $0.d_1\dots d_{n-1}$). When we speak about the digits $d_1, d_2, \dots$ of an expansion, we ignore the leading "0."; in particular, $\mathbf{d}_{\le 0} = \mathbf{d}_{<1} = 0.$ is a valid expansion regarded as having no digits and value zero. Let $B = 1/(\beta - 1) = 0.\overline{1}$. Note that $B > 1$ since $1 < \beta < 2$. We prove that any value $\upsilon \in [0, B]$ has a binary expansion $\mathbf{d}$. Recursively define the digits d_i ($i = 1, 2, \dots$) by letting

$$d_i = \begin{cases} 0, & \text{if } (\mathbf{d}_{<i}) + \beta^{-i} > \upsilon, \\ 1, & \text{if } (\mathbf{d}_{<i}) + \beta^{-i} \le \upsilon. \end{cases} \tag{1}$$

Straightforward induction shows that $0 \le \upsilon - (\mathbf{d}_{\le k}) \le \beta^{-k}B$ for $k = 0, 1, 2, \dots$; moreover, if $d_k = 0$, the stronger bound $\upsilon - (\mathbf{d}_{\le k}) \le \beta^{-k}$ holds (in particular, $d_k = 0$

cannot be followed by an infinite sequence of unit digits). Clearly, $\beta^{-k} \to 0$ as $k \to \infty$ (since $\beta > 1$), so the equality $\upsilon = 0.d_1 d_2 \ldots$ follows, proving the existence of an expansion with given value $\upsilon \in [0, B]$.

We call an expansion $\mathbf{d}$ chosen according to (1) *greedy* since at every step the digit 1 is chosen if at all possible. Obviously, any expansion (whether greedy or not) with value $\upsilon \in (0, B)$ must have both zero and unit digits.

Next, assume $\beta \leq \phi = (1 + \sqrt{5})/2$. We have $(1 + \beta^{-1})(\beta - 1) = \beta - \beta^{-1} \leq \phi - \phi^{-1} = 1$, hence $1 + \beta^{-1} \leq 1/(\beta - 1) = B$. Consider the base-$\beta$ greedy expansion $\mathbf{d} = 0.d_1 d_2 \ldots$ with value $\upsilon = 1$. Since $0 < 1 < B$, the expansion $\mathbf{d}$ has both zero and unit digits. Since $\mathbf{d}$ is greedy, it does not end with a tail of unit digits, so there is a position k such that $d_k = 1$ and $d_{k+1} = 0$. By construction of the greedy expansion, we have $(\mathbf{d}_{\leq k}) = (\mathbf{d}_{<k}) + \beta^{-k}$ and $(\mathbf{d}_{<k}) + \beta^{-k} \leq 1 < (\mathbf{d}_{<k}) + \beta^{-k} + \beta^{-(k+1)}$. Let $\upsilon' = \beta^k[1 - (\mathbf{d}_{<k})]$; thus, we have $1 \leq \upsilon' < 1 + \beta^{-1} \leq B$. (Note the strict inequality.) The value υ' has a greedy expansion $\mathbf{e} = 0.e_1 e_2 \ldots$ with $e_1 = 1$ (since $\upsilon' \geq 1 > \beta^{-1}$), giving an expansion

$$1 = (\mathbf{d}') = 0.d_1' d_2' \ldots = 0.d_1 d_2 \ldots d_{k-1} 01 e_2 e_3 \ldots$$

different from the greedy expansion $\mathbf{d}$. This construction may be iterated: the greedy expansion $\mathbf{e}$ of $\upsilon' \in [1, B)$ has both zero and unit digits, hence for some position k' we have $e_{k'} = 1$ and $e_{k'+1} = 0$, from which we obtain different expansion $\mathbf{e}' = 0.e_1 e_2 \ldots e_{k'-1} 01 f_2 f_3 \ldots$ with value υ', and hence a new expansion

$$1 = (\mathbf{d}'') = 0.d_1 \ldots d_{k-1} 01 e_2 \ldots e_{k'-1} 01 f_2 f_3 \ldots.$$

Successively repeating this procedure, infinitely many different expansions $\mathbf{d}, \mathbf{d}', \mathbf{d}'', \ldots$ with value 1 are obtained, provided $\beta \leq \phi$.

(*ii*) For the base $\beta = \phi$, we have $1 = \phi^{-1} + \phi^{-2}$, hence $\phi^{-i} = \phi^{-(i+1)} + \phi^{-(i+2)}$. Hence, from any (nonzero) terminating expansion one can obtain another by replacing the trailing digit 1 by the digits 011 (i.e., "100" becomes "011"). Starting with the greedy expansion

$$\mathbf{a} = 0.11 = \phi^{-1} + \phi^{-2} = 1,$$

we obtain the following expansions with value 1:

$$\mathbf{a}' = 0.1011, \quad \mathbf{a}'' = 0.101011, \quad \ldots, \quad \mathbf{a}^{(n)} = 0.1010\ldots1011, \quad \ldots,$$

where in $\mathbf{a}^{(n)}$ there are n pairs "10" before the trailing "11" (thus, $\mathbf{a}$ above is $\mathbf{a}^{(0)}$). (These are precisely the expansions obtained by application of the iterative procedure in the solution to part (*i*), starting with $\mathbf{a} = 0.11$.) We also have

$$\mathbf{b} = 0.0\overline{1} = 0.01111\ldots = \phi^{-2} + \phi^{-3} + \cdots = \frac{\phi^{-2}}{1 - \phi^{-1}} = 1.$$

Performing the substitution "100" for "011" as above, we obtain the following expansions with value 1:

$$\mathbf{b}' = 0.100\overline{1}, \quad \mathbf{b}'' = 0.10100\overline{1}, \quad \ldots, \quad \mathbf{b}^{(n)} = 0.1010\ldots100\overline{1}, \quad \ldots,$$

where in $\mathbf{b}^{(n)}$ there are n pairs "10" before the trailing "$0\overline{1}$" (thus, $\mathbf{b}^{(0)}$ is $\mathbf{b}$). Also,

$$\mathbf{c} = 0.\overline{10} = 0.101010101010\ldots = \sum_{j=0}^{\infty} \phi^{-(2j+1)} = \frac{\phi^{-1}}{1 - \phi^{-2}} = 1.$$

We prove that there are no other base-ϕ expansions with value 1. Indeed, let $\mathbf{d} = 0.d_1d_2\ldots$ be any base-ϕ expansion with value 1. If $\mathbf{d}$ differs from $\mathbf{c}$, let the first difference occur at position k. We prove that $\mathbf{d}$ is $\mathbf{a}^{(l-1)}$ if $k = 2l$ is even, $\mathbf{d}$ is $\mathbf{b}^{(l-1)}$ if $k = 2l - 1$ is odd ($l = 1, 2, \ldots$).

If $k = 1$, then $d_1 \neq c_1 = 1$, so $d_1 = 0$; hence, $1 = (\mathbf{d}) \leq 0.0\overline{1} = 1$. Since equality holds, $\mathbf{d}$ must be the expansion $\mathbf{b}$. If $k = 2l - 1$ for some $l > 1$, then $c_k = 1$ and $d_k = 0$, so

$$0.d_kd_{k+1}\ldots = \phi^{k-1}[(\mathbf{d}) - (\mathbf{d}_{<k})] = \phi^{k-1}[(\mathbf{c}) - (\mathbf{c}_{<k})] = 0.\overline{10} = (\mathbf{c}) = 1.$$

As shown above, $0.d_kd_{k+1}\ldots$ is the expansion $\mathbf{b}^{(0)}$, so $\mathbf{d}$ is the expansion $\mathbf{b}^{(l-1)}$. If $k = 2l$ for some $l \geq 1$, then $c_k = 0$ and $d_k = 1$, so

$$0.d_kd_{k+1}\ldots = \phi^{k-1}[(\mathbf{d}) - (\mathbf{d}_{<k})] = \phi^{k-1}[(\mathbf{c}) - (\mathbf{c}_{<k})] = 0.\overline{01} = \phi^{-1} = 0.1.$$

Since $d_k = 1$, we must have $0 = d_{k+1} = d_{k+2} = \ldots$, so $\mathbf{d}$ is the expansion $\mathbf{a}^{(l-1)}$.

Also solved by Ram Dubey, Dmitry Fleischman, Enrique Treviño, and the proposer.

Sign fluctuations of weighted partial sums of a sequence **June 2017**

2025. *Proposed by Valerian Nita, Sterling Heights, MI.*

Let n be a positive integer and let $x_1, x_2, \ldots, x_n$ and $a_1, a_2, \ldots, a_n$ be real numbers such that $\sum_{k=1}^{n} x_k = 0$ and $0 < a_1 < a_2 < \ldots < a_n$. Define $s_1, s_2, \ldots, s_n$ by $s_k = \sum_{j=1}^{k} a_jx_j$ for $k = 1, 2, 3, \ldots, n$. If there is at least one nonzero number among $x_1, x_2, \ldots, x_n$, prove that there is at least one positive and at least one negative number among $s_1, s_2, \ldots, s_n$.

Solution by Michel Bataille, Rouen, France.

There is nothing to prove if $n = 1$, because in this case $x_1 = \sum_{k=1}^{1} x_k = 0$ by hypothesis, so it is not possible for the number x_1 to be nonzero. Henceforth assume $n \geq 2$. Without loss of generality we may assume x_1 is nonzero, because the truth of the statement for $x_m, x_{m+1}, \ldots, x_n$ and $a_m, \ldots, a_n$ implies its truth for $0, \ldots, 0, x_m, \ldots, x_n$ and $a_1, a_2, \ldots, a_n$, and moreover at least one of $x_1, x_2, \ldots, x_n$ is nonzero by assumption. By changing the sign of all numbers $x_1, x_2, \ldots, x_n$ if necessary, we may further assume $x_1 > 0$. Thus, $s_1 = a_1x_1$ is positive, so it remains only to show that one of the numbers $s_2, \ldots, s_n$ is negative. For $k = 1, 2, \ldots, n$, let $X_k = \sum_{j=1}^{k} x_j$. We have, for $2 \leq k \leq n$:

$$\begin{aligned} s_k &= a_1X_1 + a_2(X_2 - X_1) + a_3(X_3 - X_2) + \cdots + a_k(X_k - X_{k-1}) \\ &= a_kX_k + \sum_{j=1}^{k-1}(a_j - a_{j+1})X_j. \end{aligned}$$

Note that the coefficients $a_j - a_{j+1}$ above are all negative by hypothesis. We have $X_1 = x_1 > 0$ and $X_n = \sum_{k=1}^{n} x_k = 0$, so there exists $m \in \{2, \ldots, n\}$ such that $X_1, \ldots, X_{m-1}$ are positive and X_m is nonpositive. Thus, we have

$$s_m = a_mX_m + \sum_{j=1}^{m-1}(a_j - a_{j+1})X_j < 0$$

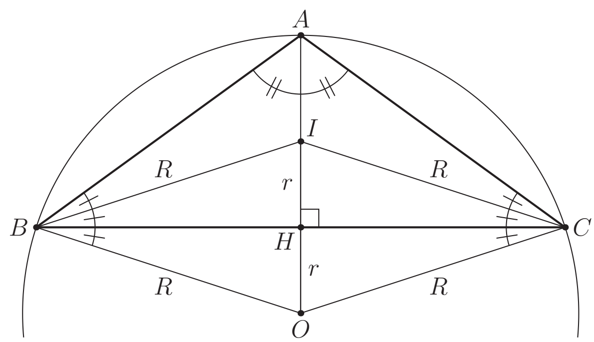
since the term $a_m X_m$ is nonpositive and the terms $(a_j - a_{j+1})X_j$ are negative (there is at least one of the latter, namely $(a_1 - a_2)X_1 < 0$), concluding the proof.

Also solved by Paul Budney, Robert Calcaterra, Richard Daquila, James Duemmel, Dmitry Fleischman, Eugene A. Herman, Dain Kim (South Korea), Miguel Lerma, ONU-SOLVE Group, Edward White, and the proposer.

Answers

Solutions to the Quickies from page 230.

A1081.



Let O be the circumcenter of $\triangle ABC$. Since $OA = OB = OC = R$ and $\triangle ABC$ is isosceles with $AB = AC$, the segments $\overline{AO}$ and $\overline{BC}$ are perpendicular and intersect at the midpoint H of $\overline{BC}$; moreover, $\overline{AH}$ bisects $\angle BAC$, so I lies on $\overline{OA}$ and $IH = r$. It follows that $\angle OBA = \angle OAB = \frac{1}{2}\angle BAC = 54^\circ$, hence $\angle OBC = \angle OBA - \angle CBA = 54^\circ - 36^\circ = 18^\circ$. Since $\overline{BI}$ bisects $\angle CBA$ we also have $\angle CBI = \angle IBA = \frac{1}{2}\angle CBA = 18^\circ = \angle OBC$, and similarly $\angle BCI = \angle OCB = 18^\circ$, so $OBIC$ must be a rhombus, with sides of length $OB = R$ and with $OH = IH = r$. It follows that $AI = OA - OI = R - 2r$, so $AI + BI + CI = (R - 2r) + R + R = 3R - 2r$.

Next, we have $OI^2 = R^2 - 2Rr$ by Euler's theorem. Since $OI = 2r$:

$$\begin{aligned} AI^2 + BI^2 + CI^2 &= (R-2r)^2 + R^2 + R^2 = 3R^2 - 4Rr + (2r)^2 \\ &= 3R^2 - 4Rr + (R^2 - 2Rr) = 4R^2 - 6Rr. \end{aligned}$$

A1082. The left-hand side of the stated inequality is equal to

$$\begin{aligned} L &= \left(\int_0^1 y^5\,dy\right)\left(\int_0^1 f(x)\,dx\right) + \left(\int_0^1 y\,dy\right)\left(\int_0^1 (f(x))^5\,dx\right) \\ &= \int_0^1\int_0^1 \left(y^5 f(x) + y(f(x))^5\right) dx\,dy. \end{aligned}$$

Now, for $a, b \geq 0$, the inequality $a^5 b + ab^5 \geq a^2b^4 + a^4b^2$ holds (since $a^5b + ab^5 - a^2b^4 - a^4b^2 = ab(a-b)^2(a^2 + ab + b^2) \geq 0$). Since f is nonnegative, it follows by integration that $L \geq R$, where

$$\begin{aligned} R &= \int_0^1\int_0^1 \left(y^2(f(x))^4 + y^4(f(x))^2\right) dx\,dy \\ &= \left(\int_0^1 y^2\,dy\right)\left(\int_0^1 (f(x))^4\,dx\right) + \left(\int_0^1 y^4\,dy\right)\left(\int_0^1 (f(x))^2\,dx\right) \end{aligned}$$

is equal to the right-hand side of the stated inequality, completing the proof.