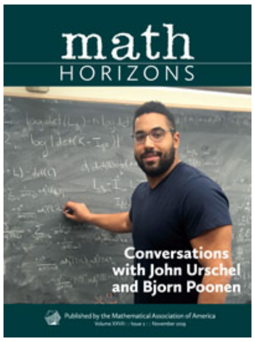

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The Playground

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THE PLAYGROUND

Welcome to the Playground! Playground rules are posted on page 33, except for the most important one: *Have fun!*

THE SANDBOX

In this section, we highlight problems that anyone can play with, regardless of mathematical background. But just because these problems are easy to approach doesn't mean that they are easy to solve!

Additive Roots (P394). Yevgeniy Sokolovsky of Fair Lawn, NJ, asked Playground readers to solve this simple-looking equation. Let A be a real constant. For what values of $x_1,\ldots,x_n$ is

$$\sum_{i=1}^{n}\sqrt{A-x_i}=n+\sqrt{(1-A)n+\sum_{i=1}^{n}x_i}\,?$$

Two Rights Make What? (P395). As shown in figure 1, quadrilateral Q has consecutive interior angles of $\tau/4$, $\tau/12$, and $\tau/4$, where $\tau=2\pi$. The lengths of the two edges forming the $\tau/12$ angle are 7539 and $4353\sqrt{3}$, respectively. What is the length of the other edge of Q that meets the edge of length $4353\sqrt{3}$?

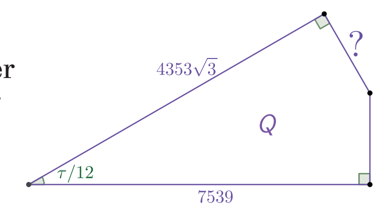

Figure 1. The ASASA property for quadrilaterals?

THE ZIP-LINE

This section offers problems with connections to articles that appear in the magazine. Not all Zip-Line problems require you to read the corresponding article, but doing so can never hurt, of course.

Finding Feebner (P396). James Propp, author of "Who Mourns the Tenth Heegner Number?" (p. 18), shared with us a sequence of questions that will take you on a similar trajectory to the one described in his article. Call a number a "Feebner" number if it is a Fibonacci number and it is between $2\cdot10^{1000}$ and $3\cdot10^{1000}$.

1) Show that there is at most one Feebner number.
2) What is the probability that for a randomly chosen natural number k, there is a Fibonacci number between $2k$ and $3k$?
3) Determine whether there is a Feebner number.
4) Find, with proof, an infinite sequence of numbers l_n such that for every n, there is no Fibonacci number between $2l_n$ and $3l_n$.

THE JUNGLE GYM

Any type of problem may appear in the Jungle Gym—climb on!

Cramped Buffon (P397). George-Louis Leclerc, Comte de Buffon, is well known for having computed the probability p_c that a unit-length "needle" dropped at random onto a plane ruled with parallel lines a unit distance apart will cross one of the lines. However, imagine now that you are performing this experiment not with an infinite plane on which to drop the needle, but rather an infinite strip of paper w units wide, with a *single* line running down its center.

Let $P(w)$ be the probability that a randomly dropped needle crosses this center line, under the condition that it lands entirely on the strip of paper, that is, no portion of the needle extends outside of the strip. For what value of w is $P(w) = p_c$?

THE CAROUSEL—OLDIES, BUT GOODIES

In this section, we present an old problem that we like so much, we thought it deserved another go-round. Try this, but be careful—old equipment can be dangerous. Answers appear at the end of the column.

Threedian Triad (C27). By analogy with "median," call a segment connecting a vertex of a triangle to a *trisection* point of the opposite side a "threedian." In an arbitrary triangle ABC, draw successive corresponding threedians from each vertex, as shown in figure 2. What fraction of the area of ABC is the area of the triangle bounded by these three threedians?

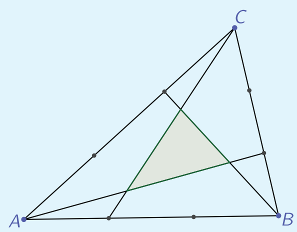

Figure 2. A triangle with three corresponding "threedians."

APRIL WRAP-UP

Picture Perfect (P386). Ioana Mihăilă (Cal Poly Pomona) submitted this problem about how she accidentally caught a railing in a vacation photo she took. She decides to crop the photo to remove the railing but can only crop it in a rectangular shape with sides parallel to the original photo. (See figure 3 for one possible cropping.) Mihăilă has two goals: to maximize the area of the cropped photo and to preserve the 4:3 ratio of the sides of the photo, as in the original. Determine the conditions on lengths a and b, as shown in the figure, that will allow her to satisfy both goals at once.

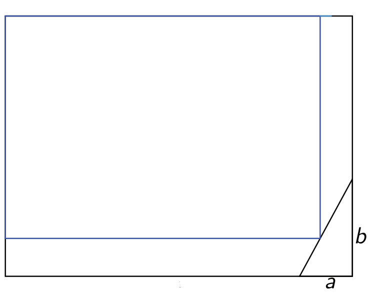

Figure 3. Cropping a photo to remove a linear railing.

We received a solution from Randy K. Schwartz (Schoolcraft College) and a partial solution from Vasile Teodorovici (NSERC Canada). Call the width and height of the cropped portion (dark blue in figures 3 and 4) x and y, respectively. If this is a maximal-area crop, then we know that the area decreases if we make the cropped area w wider and $\frac{b}{a}w$ less tall, so that the railing remains out of the picture. (Figure 4 illustrates shifting the crop in this way.) In other words,

$$xy > (x+w)(y - \tfrac{b}{a}w) = xy + wy - \tfrac{b}{a}wx - \tfrac{b}{a}w^2$$
$$0 > y - \tfrac{b}{a}x - \tfrac{b}{a}w.$$

Because this is true for any w no matter how small, we conclude that $0 \geq y - bx/a$, that is, that $x/y \geq a/b$. Symmetrically, considering sliding the corner to make the cropped region slightly taller but less wide shows that $x/y \leq a/b$. Thus, the maximum crop occurs when $x/y = a/b$; if this also happens when the cropped region has the same 4:3 aspect ratio as the original photo, we conclude that $\mathbf{a/b = 4/3}$.

Benford Again? (P387). In Art Benjamin's article "The Long and Short of Benford's Law," he introduced a continuous Benford variable as a random variable with domain [1,10) and probability density $f(x) = (x \ln 10)^{-1}$.

Call a random variable X *discretely Benford* if for each d from 1 to 9, d is the most significant digit in the decimal expansion of X with probability $\log_{10}(d+1) - \log_{10} d$. Benjamin's article mentions that the product of two continuous Benford variables is discretely Benford. Show that any positive real constant times a continuous Benford variable is also discretely Benford.

We received the following solution from Randy K. Schwartz. First note that unless the most significant digit of k is d, then there is some m such that the most significant digit of kx is d if and only if $kx \in [d10^m, (d+1)10^m)$. If the most significant digit of k is d, then let h be the fractional part of $k/10^m$, where m is chosen so that the quotient is between 1 and 10. (The integer part of this

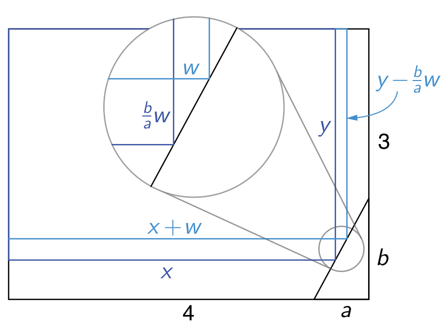

Figure 4. A slight shift to the cropping rectangle.

quotient is d.) In this case, the most significant digit of kx is d if and only if

$$kx \in [(d+h)10^m,(d+1)10^m) \cup [d10^{m+1},(d+h)10^{m+1}).$$

Using the given density of x, we have in the former case that the probability that the most significant digit of kx is d is

$$\begin{aligned} &P\left(kx \in [d10^m,(d+1)10^m]\right) \\ &= P\left(x \in [d10^m/k,(d+1)10^m/k]\right) \\ &= \int_{\frac{d10^m}{k}}^{\frac{(d+1)10^m}{k}} \frac{dx}{x\ln 10} \\ &= \frac{1}{\ln 10}\left(\ln\left(\frac{(d+1)10^m}{k}\right) - \ln\left(\frac{d10^m}{k}\right)\right) \\ &= \frac{1}{\ln 10}\left(\ln(d+1)+\ln(10^m/k)-\ln d-\ln(10^m/k)\right) \\ &= \log(d+1)-\log(d), \end{aligned}$$

as desired. Similarly, in the latter case,

$$\begin{aligned} &P\left(x \in \left[(d+h)\frac{10^m}{k},(d+1)\frac{10^m}{k}\right) \cup \left[d\frac{10^{m+1}}{k},(d+h)\frac{10^{m+1}}{k}\right]\right) \\ &= \int_{\frac{(d+h)10^m}{k}}^{\frac{(d+1)10^m}{k}} \frac{dx}{x\ln 10} + \int_{\frac{d10^{m+1}}{k}}^{\frac{(d+h)10^{m+1}}{k}} \frac{dx}{x\ln 10} \\ &= \log(d+1)-\log(d+h)+\log(d+h)-\log d \\ &= \log(d+1)-\log d, \end{aligned}$$

again as desired. Hence, kx is discretely Benford.

Clocks Work (P388). Daniel Heath provided this problem extending the setup described in his article "Clockwork Mathematics." In figure 5, the blue cog (with 21 teeth) drives the small red cog (with 9), which is fixed to the large red cog (with 25); they share the same axle. The large red cog drives the green cog (with 23). The coupled red cogs provide another way of controlling the cogs' rotation speeds.

1) How many full turns of the blue cog will be required to return all of the cogs to their illustrated positions?
2) Suppose that the main mechanism of a clock turns a cog clockwise one full turn per minute; this cog is attached the second hand. Design a gear system driven by this cog so that another cog turns clockwise once per hour (for the minute hand) and a third cog turns clockwise twice per day (for the hour hand).

We received a solution from Randy K. Schwartz and partial solutions from Abrar Sheikh (Poolesville HS) and a team from Taylor University (Becca Griggs, Alexander McFarland, Chris Netzley). In part 1, it takes **69** turns of the blue cog to return all gears to their starting positions.

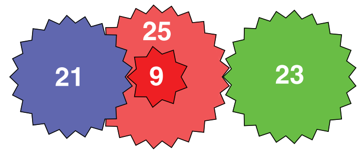

Figure 5. A gear train with coupled cogs.

The gear train for part 2 pictured in figure 6 was submitted by Randy.

As 23 and 25 have no common factor, it takes 23 full turns of the red gear to return the green gear to its original position. Because the greatest common divisor of 21 and 9 is 3, it takes three full turns of the blue gear to return the red gear to its initial position, at which point the red gear has turned seven times. As 7 and 23 have no common factor, this process must be repeated 23 times to return the green gear to its original position, for $3 \times 23 = 69$ turns in all.

The gear train in figure 6 could work for the hands of a clock for the following reasons. (a) There are an even number of gears from the gray "drive" gear to each "hand" (orange for minute, blue for hour), so the hands are rotating in the same clockwise direction. (b) The gear ratios make the orange third gear rotate once for every 60 rotations of the drive gear. So, it will rotate once an hour, and they make the blue fifth gear rotate once for every 12 rotations of the third gear. Thus, it will rotate once every 12 hours. There are some practical difficulties with this gear train, such as the axle of the blue gear running through the orange gear and the fact that the three "hands" are in different locations. For a different kind of challenge, you could think about how to overcome these issues.

Spherical Ruler (P389). You are given a solid sphere of unknown radius R, on which you can make marks, but which you cannot alter in any other way. You have calipers that can exactly measure the distance between any two points in space less than $1.95R$ units apart. And you have a rusty compass that can still draw circles but is stuck at some unknown

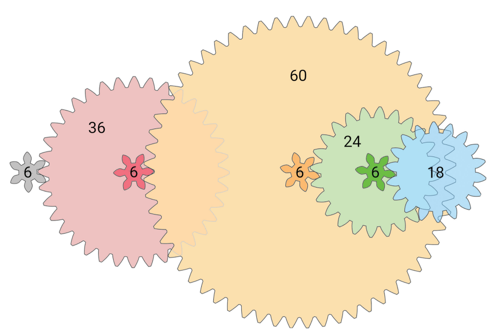

Image courtesy of *geargenerator.com*

Figure 6. A possible gear train for a clock.

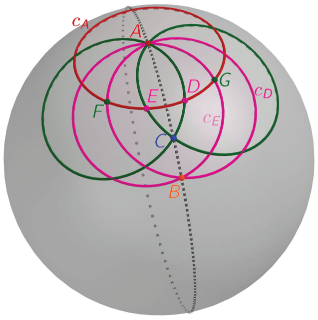

Figure 7. Constructing three points on the same great circle.

radius between $0.5R$ and $1.1R$. Describe a method for finding the exact value of R.

All submitted methods that solve this problem, including ones from Randy Schwartz, Abrar Sheikh, and the proposer, were different. We present Abrar's for its conceptual simplicity, although Randy's was shorter to execute. First, we need to construct points A, B, and C that lie on the same great circle. This can be done as depicted in figure 7. Choose point A on the sphere arbitrarily, and draw the circle c_A (of the fixed radius of the rusty compass) centered on A. Choose two points D and E on circle c_A arbitrarily, and draw the circles c_D and c_E centered on D and E. The new intersection point of c_D and c_E is point B. Let the intersection point of c_D and c_A on the opposite side of E be called F and the intersection of c_E and c_A on the opposite side of D be called G. Finally, point C is the new intersection point of the circles centered on F and G. By symmetry, points A, B, and C lie on the same great circle.

Now measure distances $d = AB$, $e = BC$, and $f = CA$ with the calipers. These are the three sides of a triangle inscribed in a circle of radius R. So, by the circumradius formula,

$$R = \frac{def}{\sqrt{(d+e+f)(d+e-f)(d-e+f)(-d+e+f)}}.$$

CAROUSEL SOLUTION

Label the vertices of the inner triangle closest to A, B, and C as D, E, and F, respectively, as shown in figure 8. Add lines through D parallel to EF, through E parallel to FD, and through F parallel to DE.

We need to determine where these new lines intersect the respective sides of the outer triangle, so let p be the fraction of CB cut off by the line through F, as shown, and q and r be the respective fractions of sides AC and BA, respectively. Then, by using two pairs of similar triangles involving segments BE and EF, we have that

$$\frac{\frac{2}{3}-p}{\frac{1}{3}} = \frac{\frac{2}{3}-r}{r}.$$

Analogously, we also know that

$$\frac{\frac{2}{3}-r}{\frac{1}{3}} = \frac{\frac{2}{3}-q}{q} \text{ and } \frac{\frac{2}{3}-q}{\frac{1}{3}} = \frac{\frac{2}{3}-p}{p}.$$

These equations simplify to $p = 1 - \frac{2}{9r}$, $r = 1 - \frac{2}{9q}$, and $q = 1 - \frac{2}{9p}$, which, after successive substitutions, yield $9p^2 - 9p + 2 = 0$. Discarding the solution $p = 2/3$, which would correspond to F lying on DE, we obtain $p = q = r = 1/3$.

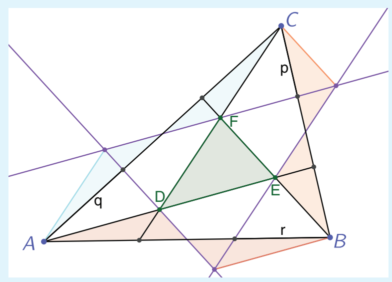

Figure 8. The triangle with lines parallel to the threedians added.

In other words, the newly added lines go through the other trisection points. Once we know that, it is straightforward to deduce that they also bisect the adjacent sides. We can then see that like-shaded triangles in the diagram are congruent, so the area of ABC outside triangle DEF is equal to the area of the three parallelograms adjoining DEF. Moreover, each of these parallelograms consists of two congruent copies of DEF, so the area of DEF is exactly **one-seventh** the area of ABC.

SUBMISSION & CONTACT INFORMATION

The Playground features problems for students at the undergraduate and (challenging) high school levels. Problems and solutions should be submitted to *MHproblems@maa.org* and *MHsolutions@maa.org*, respectively (PDF format preferred). Paper submissions can be sent to Glen Whitney, ICERM, 121 South Main Street, Box E, 11th Floor, Providence, RI 02903. Please include your name, email address, and school affiliation, and indicate if you are a student. If a problem has multiple parts, solutions for individual parts will be accepted. Unless otherwise stated, problems have been solved by their proposers.
The deadline for submitting solutions to problems in this issue is January 11, 2020.

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