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Problems and Solutions

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PROBLEMS AND SOLUTIONS

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This section contains problems intended to challenge students and teachers of college mathematics. We urge you to participate actively *both* by submitting solutions and by proposing problems that are new and interesting. To promote variety, the editors welcome problem proposals that span the entire undergraduate curriculum.

Proposed problems should be sent to **Greg Oman**, either by email (preferred) as a pdf, TEX, or Word attachment or by mail to the address provided above. Whenever possible, a proposed problem should be accompanied by a solution, appropriate references, and any other material that would be helpful to the editors. Proposers should submit problems only if the proposed problem is not under consideration by another journal.

Solutions to the problems in this issue should be sent to **Chip Curtis**, either by email as a pdf, TEX, or Word attachment (preferred) or by mail to the address provided above, no later than November 15, 2020. Sending both pdf and TEXfiles is ideal.

PROBLEMS

1176. *Proposed by Xiang-Qian Chang, MCPHS University, Boston, MA.*

Let $A_{n\times n}$ be an $n \times n$ positive semidefinite Hermitian matrix. Prove that the following inequality holds for any pair of integers $p \geq 1$ and $q \geq 0$:

$$\frac{\mathrm{Tr}(A^p) + \mathrm{Tr}(A^{p+1}) + \cdots + \mathrm{Tr}(A^{p+q})}{\mathrm{Tr}(A^{p+1}) + \mathrm{Tr}(A^{p+2}) + \cdots + \mathrm{Tr}(A^{p+q+1})} \leq \frac{r_A}{\mathrm{Tr}(A)},$$

where r_A is the rank of A and Tr is the trace function.

1177. *Proposed by Ovidiu Furdui and Alina Sîntămărian, Technical University of Cluj-Napoca, Cluj-Napoca, Romania.*

Find all continuous functions $f : \mathbb{R} \to \mathbb{R}$ which satisfy the following equation for all $x \in \mathbb{R}$:

$$f(-x) = 1 + \int_0^x \sin t f(x-t)\,dt.$$

1178. *Proposed by Cezar Lupu, Texas Tech University, Lubbock, TX, and Vlad Matei, University of California Irvine, Irvine, CA.*

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Consider a triangle ABC. Let $\mathcal{C}$ be the circumcircle of ABC, r the radius of the incircle, and R the radius of $\mathcal{C}$. Let arc(BC) be the arc of $\mathcal{C}$ opposite A, and define arc(CA) and arc(AB) similarly. Let $\mathcal{C}_A$ be the circle tangent internally to the sides AB, AC, and the arc BC not containing A, and let R_A be its radius. Define $\mathcal{C}_B$, $\mathcal{C}_C$, R_B, and R_C similarly. Prove that the following inequality holds:

$$4r \leq R_A + R_B + R_C \leq 2R.$$

1179. *Proposed by Greg Oman, University of Colorado, Colorado Springs, Colorado Springs, CO.*

Let R be a ring, and let I be an ideal of R. Say that I is *small* provided $|I| < |R|$ (i.e., I has smaller cardinality than R). Suppose now that R is an infinite commutative ring with identity which is not a field. Suppose further that R possesses a small maximal ideal M_0. Prove the following:

1. there exists a maximal ideal M_1 of R such that $M_1 \neq M_0$, and
2. M_0 is the *unique* small maximal ideal of R.

1180. *Proposed by Luke Harmon, University of Colorado, Colorado Springs, Colorado Springs, CO.*

In both parts, R denotes a commutative ring with identity. Prove or disprove the following:

1. there exists a ring R with infinitely many ideals with the property that every nonzero ideal of R is a subset of but finitely many ideals of R, and
2. there exists a ring R with infinitely many ideals with the property that every proper ideal contains (as a subset) but finitely many ideals of R.

Highest power of two dividing an entry of a matrix

1151. *Proposed by Gregory Dresden, Washington and Lee University, Lexington, VA.*

Fix an odd integer b in set $M = \begin{pmatrix} 1 & b \\ 4 & 5 \end{pmatrix}$. For a positive integer n, let $e(n)$ denote the exponent of the highest power of 2 that divides an entry of M^n. In other words, $2^{e(n)}$ divides some entry in M^n, but no larger power of 2 divides an entry of M^n. Find $e(n)$ as a function of n.

Solution by Omran Kouba, Higher Institute for Applied Sciences and Technology, Damascus, Syria.

The answer is $e(n) = 2 + \nu_2(n)$ where as usual we will use $\nu_2(k)$ to denote the exponent of the highest power of 2 which divides k. The characteristic polynomial of M is $P(X) = X^2 - 6X + 5 - 4b$. So, from the equality $M^2 = 6M + (4b-5)I_2$ we deduce that $M^n = 6M^{n-1} + (4b-5)M^{n-2}$ for all $n \geq 2$. Thus, if we write $M^n = \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix}$, then all four sequences $(a_n)_{n\geq 0}$, $(b_n)_{n\geq 0}$, $(c_n)_{n\geq 0}$, and $(d_n)_{n\geq 0}$ satisfy the same recurrence relation (namely $x_n = 6x_{n-1} + (4b-5)x_{n-2}$,) but they differ in their initial conditions:

$$(a_0, a_1) = (1, 1), \quad (b_0, b_1) = (0, b), \quad (c_0, c_1) = (0, 4), \quad (b_0, b_1) = (1, 5).$$

- The sequence $(a_n)_{n\geq 0}$ satisfies the recurrence $a_n \equiv a_{n-2} \mod 2$ and because $a_0 = a_1 = 1$ we see that a_n is odd for every n. The same argument shows that d_n is odd for every n.
- Let $(\delta_n)_{n\geq 0}$ the sequence defined recursively by

$$\delta_0 = 0,\ \delta_1 = 1, \quad \text{and} \quad \delta_n = 6\delta_{n-1} + (4b-5)\delta_{n-2} \quad \text{for } n \geq 2.$$

  Then a simple induction shows that $b_n = b\delta_n$ and $c_n = 4\delta_n$ for all n. Since b is odd we see that $\nu_2(b_n) = \nu_2(\delta_n)$ while $\nu_2(c_n) = 2 + \nu_2(\delta_n)$ and $\nu_2(a_n) = \nu_(d_n) = 0$. We conclude that

$$e(n) = 2 + \nu_2(\delta_n).$$

- Let $\ell = (b+1)/2 \in \mathbb{Z}$, and suppose that $\ell \neq 0$. We define $\alpha = 3 + 2\sqrt{2\ell}$ and $\beta = 3 - 2\sqrt{2\ell}$, the two zeros of the second degree trinomial $X^2 - 6X + 5 - 4b = 0$. Then

$$\begin{aligned} \delta_n &= \frac{\alpha^n - \beta^n}{\alpha - \beta} = \frac{1}{4\sqrt{2\ell}} \sum_{k=0}^{n} \binom{n}{k} 3^{n-k} (2\sqrt{2\ell})^k (1 - (-1)^k) \\ &= \sum_{0\leq k<n/2}^{n} \binom{n}{2k+1} 3^{n-2k-1} (8l\ell)^k \equiv n3^{n-1} \mod 8. \end{aligned}$$

  In particular, this proves that if $n = 2m+1$ (i.e., n is odd), then $\nu_2(\delta_n) = 0$. On the other hand,

$$\frac{\alpha^n+\beta^n}{2} = \sum_{k=0}^{n} \binom{n}{k} 3^{n-k} (2\sqrt{2\ell})^k \frac{1+(-1)^k}{2}$$
$$= \sum_{0\le k\le n/2}^{n} \binom{n}{2k} 3^{n-2k} (8\ell)^k = 1 \mod 2.$$

The equality $\delta_{2n} = (\alpha^n+\beta^n)\delta_n$ thus implies that $\nu_2(\delta_{2n}) = 1+\nu_2(\delta_n)$. This shows inductively that $\nu_2(\delta_{2^k(2m+1)}) = k + \nu_2(\delta_{2m+1}) = k$; that is, $\nu_2(\delta_n) = \nu_2(n)$.

- It remains to consider the case $b = -1$. In this case $\delta_n = n3^{n-1}$, and the fact that $\nu_2(\delta_n) = \nu_2(n)$ for all $n \ge 1$ is immediate in this case.

Combining the above results we see that $e(n) = 2 + \nu_2(n)$, as announced.

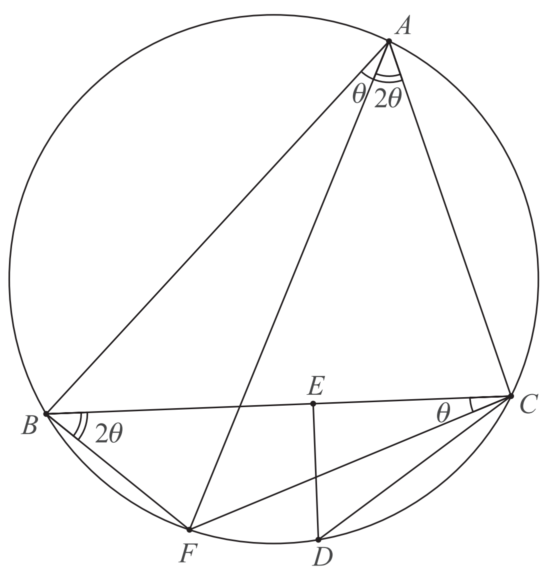
Also solved by Armstrong Problem Solvers; Levent Batakci, Case Western Reserve U.; Ali Deeb and Hafez Al-Assad (jointly), Higher Inst. for Applied Sciences and Technology, Syria; Brendan Dosch (student), North Central C.; James Duemmel, Bellingham, WA; Florida Atlantic U. Problem Solving Group; Neville Fogarty and Chris Kennedy (jointly), Christopher Newport U.; George Washington U. Problems Group; Eugene Herman, Grinnell C.; John Kieffer, U. of Minnesota Twin Cities; Koopa Tak Lun Koo, Chinese STEAM Academy, Hong Kong; A. Bathi Kasturiarachi, Kent St. U. at Stark; Carl Libis, Columbia Southern U.; Albert Natian, Los Angeles Valley C.; Northwestern U. Math Problem Solving Group; Éric Pité, Paris, France; Francisco Perdomo and Ángel Plaza (jointly), Universidad de Las Palmas de Gran Canaria, Spain; Arthur Rosenthal, Salem St. U.; Ioannis Sfikas, Athens, Greece; Jacob Siehler, Gustavus Adolphus C.; Enrique Treviño, Lake Forest C.; Edward White, Frostburg, MD; and the proposer.

An inequality for the area of a triangle

1152. *Proposed by Yagoub Aliev, ADA University, Baku, Azerbaijan.*

Let R be the radius of the circumscribed circle of triangle ABC. Let D be a point on the arc BC which does not contain A, and drop perpendicular DE to BC. Now take point F on the same arc such that $\angle CAF = 2\angle BAF$. Prove that $8R \cdot \text{Area}(CDE) \le CF^3$.

Solution by Michel Bataille, Rouen, France.



Let $\theta = \angle BAF$. Then we have $\angle CAF = 2\theta$, $\angle BCF = \theta$, $\angle CBF = 2\theta$ so that $CF = 2R\sin 2\theta$.

Let $t = \angle BCD$. We note that $t < 90°$ and $t = \angle BAD < \angle BAC = 3\theta$, hence $t < \min(90°, 3\theta)$ and that $\angle CBD = 180° - (180° - 3\theta) - t = 3\theta - t$. From the latter, we obtain $CD = 2R\sin(3\theta - t)$. Since $CE = CD\cos t$, we deduce that

$$\text{Area}(CDE) = \frac{1}{2}CE \cdot CD \sin t = \frac{1}{2}CD^2 \sin t \cos t = R^2 \sin 2t \sin^2(3\theta - t).$$

Now, let $f(t) = \sin 2t \sin^2(3\theta - t)$ for $0 < t < \min(90°, 3\theta)$. Then, the derivative of the function f satisfies

$$\begin{aligned} f'(t) &= 2(\cos 2t)\sin^2(3\theta - t) + (\sin 2t)(-2\sin(3\theta - t)\cos(3\theta - t)) \\ &= 2\sin(3\theta - t)\sin(3\theta - 3t). \end{aligned}$$

Since $\sin(3\theta - t) > 0$, $f'(t)$ has the same sign as $\sin(3\theta - 3t)$. It readily follows that $f(t)$ is maximal when $t = \theta$, that is, $f(t) \le \sin 2\theta \sin^2 2\theta = \sin^3 2\theta$. We conclude that

$$8R \cdot \text{Area}(CDE) = 8R^3 f(t) \le 8R^3 \sin^3 2\theta = CF^3.$$

Also solved by Hafez Al-Assad, Higher Inst. for Applied Sciences and Technology, Syria; George Washington U. Problems Group; Koopa Tak Lun Koo, Chinese STEAM Academy, Hong Kong; Kee-Wai Lau, Hong Kong, China; and the proposer.

Probability of equally spaced points

1153. *Proposed by Eugen Ionascu, Columbus State University, Columbus, GA.*

Two points B and C are chosen independently at random with uniform distribution on segment $\overline{AD}$. Show that the probability that A, B, C, and D are equally spaced from some point P not on $\overleftrightarrow{AB}$ is equal to $\mathcal{P} = (15 - 16\ln 2)/9$. (By "equally spaced," we mean that the angles $\angle APB$, $\angle BPC$, and $\angle CPD$ have the same measure.)

Solution by the George Washington University Problems Group.

We first identify the locus of points P from which A, B, and C are equally spaced.

Lemma: Let B be a point on the segment $\overline{AC}$ with $r = AB$, $s = BC$, and $r < s$. Let F be the point on $\overrightarrow{BA}$ such that $BF = 2rs/(s - r)$. The locus of points P not on $\overleftrightarrow{AB}$ from which A, B, and C are equally spaced is the circle of diameter BF with the points B and F removed.

Proof: Let A be at the origin, let $B = (r, 0)$, and let $C = (r + s, 0)$. Now pick any point P in the plane with coordinates (a, b). The line AP has equation $bx - ay = 0$, and the distance from B to this line is $|br|/\sqrt{a^2 + b^2}$. The line CP has equation $bx - (a - r - s)y - b(r + s) = 0$, and the distance from B to this line is $|bs|/\sqrt{(a - r - s)^2 + b^2}$. The points A, B, and C are equally spaced from P if and only if these distances are equal, in other words when

$$\frac{r}{\sqrt{a^2 + b^2}} = \frac{s}{\sqrt{(a - r - s)^2 + b^2}}.$$

Squaring both sides and rearranging this leads to the equivalent equation

$$\left(a+\frac{r^2}{s-r}\right)^2+b^2=\frac{r^2s^2}{(s-r)^2},$$

which is the circle passing through B with center on the negative x-axis and with diameter $rs/(s-r)$. The lemma follows.

Assume without loss of generality that $AD=1$, and let $x=AB$ and $y=AC$. The point (x, y) is in the unit square $[0, 1]^2$, and the probability we seek is the area of the set of points (x, y) such that the resulting B and C together with A and D are equally spaced. This requires that the locus of points from which A, B, and C are equally spaced intersects the locus of points from which B, C, and D are equally spaced. We examine the case $y>x$ for convenience.

When $y-x>x$ and $y-x>1-y$, then BC is larger than both AB and CD, and the two loci are nonintersecting circles, one through B and containing A, the other through C and containing D.

When $y-x<x$ and $y-x<1-y$, then BC is smaller than both AB and CD, and the two loci are intersecting circles, one through B containing C, the other through C containing B.

The delicate case is when BC is larger than just one of AB and CD. Assume that $x<y-x<1-y$. The two loci are a circle through B and a circle through C, both containing A. According to the lemma, the circle through B will lie entirely inside the circle through C if and only if

$$x-\frac{2x(y-x)}{(y-x)-x}>y-\frac{2(y-x)(1-y)}{(1-y)-(y-x)}.$$

Hence the two circles intersect at a point not on the line AD if and only if

$$x-\frac{2x(y-x)}{(y-x)-x}<y-\frac{2(y-x)(1-y)}{(1-y)-(y-x)},$$

which is equivalent to $3xy+y<4x$, or $y<4x/(3x+1)$. The case $x>y$ is exactly the same, and the points are equally spaced if and only if $x<4y/(3y+1)$, or $y>x/(4-3x)$.

Hence (x, y) leads to equally spaced points if and only if

$$\frac{x}{4-3x}<y<\frac{4x}{3x+1}.$$

The area of this region is

$$\int_0^1\left(\frac{4x}{3x+1}-\frac{x}{4-3x}\right)dx=\int_0^1\left(\frac{4}{3}-\frac{4/3}{3x+1}+\frac{1}{3}-\frac{4/3}{4-3x}\right)dx=\frac{5}{3}-\frac{8}{9}\ln 4,$$

which equals the probability as claimed.

Also solved by Randy Schwartz, Schoolcraft C.; Andrew Simoson, King U.; Lawrence Weill (emeritus), California St. U., Fullerton; and the proposer. One incomplete solution was received.

The derivative of a product of sines

1154. *Proposed by Ovidiu Furdui and Alina Sintamarian, Technical University of Cluj-Napoca, Romania.*

For a positive integer n, define $f_n : \mathbb{R} \to \mathbb{R}$ by $f_n(x) = \sin x \sin 2x \sin 3x \cdots \sin nx$. Calculate $f_n^{(n)}(0)$.

Solution by Henry Ricardo, Westchester Area Math Circle.

We will show that $f_n^{(n)}(0) = (n!)^2$.

The crucial observation is that $f_n^{(n)}(0)$ is $n!$ times the coefficient of x^n in the Taylor expansion of $f_n(x)$ about the origin. We have $\sin(kx) = kx + O(x^3)$, which gives

$$f_n(x) \;=\; \prod_{k=1}^{n}(kx + O(x^3)) \;=\; n!\,x^n + O(x^{n+2}).$$

Thus $f_n^{(n)}(0) = n! \cdot n! = (n!)^2$.

Editor's note: Many of the other submitted solutions were essentially identical to the featured solution.

Also solved by Ulrich Abel, Technische Hochschule, Mittelhessen, Germany; Pedro Acosta De Leon (student), Mass. Inst. of Tech.; Hafez Al-Assad, Higher Inst. for Applied Sciences and Technology, Syria; Armstrong Problem Solvers; Farrukh Rakhimjanovich Ataev, Westminster International U., Uzbekistan; Ethan Bamberger (student), N. Central C.; Michel Bataille, Rouen, France; Brian Bradie, Christopher Newport U.; Paul Budney, Sunderland, MA; Bruce Davis, St. Louis Comm. C. at Florissant Valley; Ali Deeb (student), Higher Inst. for Applied Sciences and Technology, Syria; James Duemmel, Bellingham, WA; Bill Dunn, Montgomery C.; Adam Bradie (student) and Neville Fogarty (jointly), Christopher Newport U.; Subhankar Gayen, V. M. Mahavidyalaya; Russ Gordon, Whitman C.; Raymond Greenwell, Hofstra U.; GWstat Problem Solving Group; George Washington U. Problems Group; Allyson Hahn (student), N Central C. Eugene Herman, Grinnell C.; Tom Jager, Calvin C.; Spencer Davey, Mark Norman (students), and Kelly Jahns, Spokane Comm. C.; A. Bathi Kasturiarachi, Kent St. U. at Stark; Dan Kemp, S. Dakota St. U.; Koopa Tak Lun Koo, Chinese STEAM Academy, Hong Kong; Kee-Wai Lau, Hong Kong, China; Carl Libis, Columbia Southern U.; Robert McCaffery, The Citadel; Missouri State U. Problem Solving Group; Albert Natian, Los Angeles Valley C.; José Heber Nieto, Universidad del Zulia, Maracaibo, Venezuela; Northwestern U. Math Problem Solving Group; Sung Hee Park, Seoul Science High S.; Maxwell Pirtle, St. Anthony's High S.; Éric Pité, Paris, France; Francisco Perdomo and Ángel Plaza (jointly), Universidad de Las Palmas de Gran Canaria, Spain; Gary Raduns, Roberts Wesleyan C.; Arthur Rosenthal, Salem St. U.; Seán Stewart, Bomaderry, NSW, Australia; Nora Thornber, Raritan Valley Comm. C.; Enrique Treviño, Lake Forest C.; Stan Wagon, Macalester C.; Lawrence Weill (emeritus), California St. U., Fullerton; Xinyi Zhang (student), Vanderbilt U.; and the proposer. One incorrect solution was received.

There is no field algebraic over the rationals isomorphic to a proper subfield.

1155. *Proposed by Greg Oman, University of Colorado at Colorado Springs, Colorado Springs, CO.*

Let F be a field and suppose that K is a field extension of F. If $\alpha \in K$ is transcendental over F, then it is well-known that $F(\alpha) \cong F(x)$. Thus, for example, $\mathbb{Q}(\pi)$ is isomorphic to its proper subfield $\mathbb{Q}(\pi^2)$. Does there exist a field K, which is algebraic over $\mathbb{Q}$ and a proper subfield F of K such that $K \cong F$?

Solution by Anthony Bevelacqua, University of North Dakota.

Let K be a field algebraic over $\mathbb{Q}$. Suppose F is a subfield of K such that $\phi : K \to F$ is an isomorphism of fields. Note that since $\phi(1) = 1$ we have $\phi(c) = c$ for each $c \in \mathbb{Q}$.

Let $\alpha \in K$ and let $g(X)$ be the minimal polynomial of α over $\mathbb{Q}$. Let $R = \{\alpha_1, \ldots, \alpha_r\}$ be the set of roots of $g(X)$ in K. Since $g(X) \in \mathbb{Q}[X]$ we have for each i

$$0 = \phi(g(\alpha_i)) = g(\phi(\alpha_i))$$

and so $\phi(\alpha_i) = \alpha_j$ for some j. Thus $\phi(R) \subseteq R$. Since ϕ is injective and R is finite we have $\phi(R) = R$. Now $\alpha \in R \subset \operatorname{Im} \phi = F$. Thus $K \subseteq F$, and so $K = F$.

Therefore K is not isomorphic to any proper subfield of K.

Also solved by Paul Budney, Sunderland, MA; Tom Jager, Calvin C.; Missouri State U. Problem Solving Group; Francisco Perdomo and Ángel Plaza (jointly), Universidad de Las Palmas de Gran Canaria, Spain; and the proposer. One incorrect solution was received.

