Math 145 **Fall 2009**

Practice Test 2 - Solutions

1. Six integer numbers, a_1, a_2, a_3, a_4, a_5, and a_6 are chosen randomly. Prove that $\prod_{1 \le i < j \le 6} (a_i - a_j)$ is divisible by 10.

   There are two possible remainders (0 and 1) upon division by 2. Since there are more numbers than possible remainders, by Dirichlet's box principle at least two numbers have the same remainder. Then their difference is divisible by 2. Therefore the product of all differences is divisible by 2.
   There are five possible remainders (0, 1, 2, 3, 4) upon division by 5. Since there are more numbers than possible remainders, by Dirichlet's box principle at least two numbers have the same remainder. Then their difference is divisible by 5. Therefore the product of all differences is divisible by 5.
   Since the product is divisble by both 2 and 5 and these are distinct primes, the product is divisible by 10.

   Note. We used Corollary 6.10 here.

2. Solve for x: $|x+1| + 5 - x^2 \ge 0$

   Case I. $x + 1 \ge 0$
   $|x+1| = x+1$, *so the inequality becomes*
   $x + 1 + 5 - x^2 \ge 0$
   $x + 6 - x^2 \ge 0$
   $x^2 - x - 6 \le 0$
   $(x-3)(x+2) \le 0$
   $-2 \le x \le 3$
   The condition $x + 1 \ge 0$ *implies* $x \ge -1$, *so the solution set in this case is* $[-1, 3]$.
   Case II. $x + 1 < 0$
   $|x+1| = -(x+1)$, *so the inequality becomes*
   $-(x+1) + 5 - x^2 \ge 0$
   $-x + 4 - x^2 \ge 0$
   $x^2 + x - 4 \le 0$
   $\left(x - \frac{-1+\sqrt{17}}{2}\right)\left(x - \frac{-1-\sqrt{17}}{2}\right) \le 0$
   $\frac{-1-\sqrt{17}}{2} \le x \le \frac{-1+\sqrt{17}}{2}$
   The condition $x + 1 < 0$ *implies* $x < -1$, *so the solution set in this case is*
   $\left[\frac{-1-\sqrt{17}}{2}, -1\right)$.
   Answer: $\left[\frac{-1-\sqrt{17}}{2}, 3\right]$.

3. Let $F_0 = 0$, $F_1 = 1$, $F_2 = 1$, ..., F_{99} be the first 100 Fibonacci numbers (recall that $F_n = F_{n-1} + F_{n-2}$ for $n \ge 2$). How many of them are even?

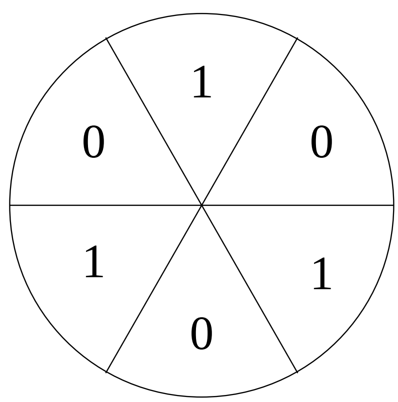
   We compute the first few Fibonacci numbers: 0, 1, 1, 2, 3, 5, 8, 13, 21, *and notice that* *every third of them is even. More precisely, F_n is even if and only if $n \equiv 0 \pmod 3$. Therefore exactly third of $F_1, F_2, \ldots, F_{99}$ is even which gives 33 numbers, and F_0 is even, thus we have 34 even numbers total.*

Proof of the pattern (by Strong Mathematical Induction):

Basis step. If $n = 0$, $F_0 = 0$ is even.

Inductive step. Suppose the statement "F_n is even if and only if $n \equiv 0 \pmod 3$" holds for $0 \leq n \leq k$. We will prove that the statement holds for $n = k + 1$.
Case I. $k + 1 = 1$. Then $k + 1 \not\equiv 0 \pmod 3$, and F_1 is odd.
Case II. $k + 1 = 2$. Then $k + 1 \not\equiv 0 \pmod 3$, and F_2 is odd.
Case III. $k + 1 \geq 3$. Then we consider all possible cases of $k + 1$ modulo 3.
Case IIIA. $k + 1 \equiv 0 \pmod 3$. Then by the inductive hypothesis F_k is odd and F_{k-1} is odd (since $k \equiv 2 \pmod 3$ and $k - 1 \equiv 1 \pmod 3$), so $F_{k+1} = F_k + F_{k-1}$ is even.
Case IIIB. $k + 1 \equiv 1 \pmod 3$. Then by the inductive hypothesis F_k is even and F_{k-1} is odd (since $k \equiv 0 \pmod 3$ and $k - 1 \equiv 2 \pmod 3$), so $F_{k+1} = F_k + F_{k-1}$ is odd.
Case IIIC. $k + 1 \equiv 2 \pmod 3$. Then by the inductive hypothesis F_k is odd and F_{k-1} is even (since $k \equiv 1 \pmod 3$ and $k - 1 \equiv 0 \pmod 3$), so $F_{k+1} = F_k + F_{k-1}$ is odd.

4. A circle is divided into six sectors. Then the numbers 1, 0, 1, 0, 1, 0 are written into the sectors as shown below. We may increase any two neighboring numbers by 1. We may repeat this step as many times as we want. Is it possible to equalize all the numbers?



Solution 1. The parity of the sum of all numbers is an invariant since when we increase two numbers by 1, the sum changes by 2. Initially the sum is odd. If all numbers became equal, the sum would be even. This is impossible.

Solution 2. Let's denote the numbers a_1, a_2, a_3, a_4, a_5, and a_6, so that initially $a_1 = 1$, $a_2 = 0$, $a_3 = 1$, $a_4 = 0$, $a_5 = 1$, and $a_6 = 0$. Then $(a_1 + a_3 + a_5) - (a_2 + a_4 + a_6)$ is an invariant since when we increase two neighboring numbers by 1, both $(a_1 + a_3 + a_5)$ and $(a_2 + a_4 + a_6)$ increase by 1, so their difference does not change. Initially $(a_1 + a_3 + a_5) - (a_2 + a_4 + a_6) = 3$. If all numbers became equal, $(a_1 + a_3 + a_5) - (a_2 + a_4 + a_6)$ would become 0. This is impossible.