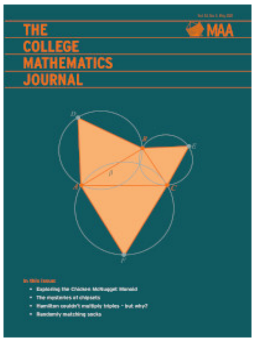

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Problems and Solutions

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PROBLEMS AND SOLUTIONS

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This section contains problems intended to challenge students and teachers of college mathematics. We urge you to participate actively *both* by submitting solutions and by proposing problems that are new and interesting. To promote variety, the editors welcome problem proposals that span the entire undergraduate curriculum.

Proposed problems should be sent to **Greg Oman**, either by email (preferred) as a pdf, TEX, or Word attachment or by mail to the address provided above. Whenever possible, a proposed problem should be accompanied by a solution, appropriate references, and any other material that would be helpful to the editors. Proposers should submit problems only if the proposed problem is not under consideration by another journal.

Solutions to the problems in this issue should be sent to **Chip Curtis**, either by email as a pdf, TEX, or Word attachment (preferred) or by mail to the address provided above, no later than November 15, 2021. Sending both pdf and TEX files is ideal.

CORRECTIONS

1. The problem labeled number 2000 in the March 2021 issue should have been number 1200.
2. The statement of Problem 1191 in the January 2021 issue had an error. Please use the following version instead.

   1191. *Proposed by Herb Bailey, Rose-Hulman Institute of Technology, Terre Haute, IN.*

   An isosceles triangle has incenter I, circumcenter O, side length S, and base length W. Show that there is a unique value of $\frac{S}{W}$ so that there exists a point P on one of the two sides of length S such that triangle IOP is equilateral. Find this value.
3. Problem 1193 of the January 2021 issue of The College Mathematics Journal was correctly attributed to George Stoica; the problem statement, however, is not correct. It was a duplicate of Problem 1188 from the November 2020 issue of the Journal. Please the following version instead. Submissions from the earlier version of 1193 received in time for publication in the November 2021 issue will be included with those for problem 1188.

   1193. *Proposed by George Stoica, Saint John, New Brunswick, Canada.*

   Let $f : \mathbb{Q} \times \mathbb{Q} \to \mathbb{Q}$ be a function such that $y \to f(a, y)$ is a polynomial over $\mathbb{Q}$ for every $a \in \mathbb{Q}$ and $x \to f(x, b)$ is a polynomial over $\mathbb{Q}$ for every $b \in \mathbb{Q}$. Is it true that $f(x, y)$ is a polynomial in $(x, y) \in \mathbb{Q}^2$?

The editors apologize for the errors.

doi.org/10.1080/07468342.2021.1909994

1201. *Proposed by Eugen Ionascu, Columbus State University, Columbus, GA.*

Consider the ellipsoid $\frac{x^2}{4} + y^2 + z^2 = 1$ and the ellipse E which is the intersection of the ellipsoid with the plane $ax + by + cz = 0$, where $P = (a, b, c)$ is a random point on the unit sphere (so $a^2 + b^2 + c^2 = 1$). Now consider the random variable A_E, the area of the ellipse E. If the point P is chosen on the unit sphere with uniform distribution with respect to the area, what is the expected value of A_E?

1202. *Proposed by Cezar Lupu, Texas Tech University, Lubbock, TX.*

Let A and B be $n \times n$ matrices with complex entries such that B is not invertible and $AB^2 - B^2A = \operatorname{adj}(B)$. Prove that $(\operatorname{adj}(B))^2 = O_n$, the $n \times n$ zero matrix. Here, $\operatorname{adj}(B)$ is the adjugate matrix of B.

1203. *Proposed by Greg Oman, University of Colorado at Colorado Springs, Colorado Springs, CO.*

Let $X \subseteq \mathbb{Z}$ and let $k > 1$ be an integer. Say that X is k *sum-free* if the sum of any k elements of X (with repetitions allowed) is *not* in X.

1. Let $n > 1$ be an integer. Prove that there is an infinite $X \subseteq \mathbb{Z}$ which is k sum-free for $1 < k \leq n$.
2. Is there an infinite $X \subseteq \mathbb{Z}$ which is k sum-free for every integer $k > 1$?

1204. *Proposed by George Stoica, Saint John, New Brunswick, Canada.*

Let $(x + iy)^n := P_n(x, y) + iQ_n(x, y)$ for $n = 1, 2, \ldots$, and let $f(x, y)$ be a continuous function along a (simple, closed, smooth) curve C in $\mathbb{R}^2$ such that the following equation holds:

$$\int_C f(x, y)dP_n(x, y) = \int_C f(x, y)dQ_n(x, y) = 0, n = 1, 2, \ldots.$$

Prove that $f(x, y)$ is constant along C.

1205. *Proposed by George Stoica, Saint John, New Brunswick, Canada.*

Let R be a (associative) prime ring, and suppose that $f: R \to R$ is a nontrivial surjective ring endomorphism. If $[a, f(a)] = 0$ for all $a \in R$, prove that R is commutative (recall that the *commutator* $[xy] := xy - yx$ for $x, y \in R$, and R is *prime* if R is nontrivial and for all $a, b \in R$: if $arb = 0$ for all $r \in R$, then either $a = 0$ or $b = 0$).

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An inequality involving the trace

1176. *Proposed by Xiang-Qian Chang, MCPHS University, Boston, MA.*

Let $A_{n\times n}$ be an $n \times n$ positive semidefinite Hermitian matrix. Prove that the following inequality holds for any pair of integers $p \geq 1$ and $q \geq 0$:

$$\frac{\mathrm{Tr}(A^p) + \mathrm{Tr}(A^{p+1}) + \cdots + \mathrm{Tr}(A^{p+q})}{\mathrm{Tr}(A^{p+1}) + \mathrm{Tr}(A^{p+2}) + \cdots + \mathrm{Tr}(A^{p+q+1})} \leq \frac{r_A}{\mathrm{Tr}(A)},$$

where r_A is the rank of A and Tr is the trace function.

Solution by Michel Bataille, Rouen, France.

We assume that A is a non-zero matrix.

The matrix A is similar to a diagonal matrix $D = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_k, 0, \ldots, 0)$ where $\lambda_1, \lambda_2, \ldots, \lambda_k$ are the positive eigenvalues of A. Since similar matrices have the same rank and the same trace, we have $k = r_A$ and $\mathrm{Tr}(A) = \lambda_1 + \lambda_2 + \cdots + \lambda_k$. Also, for any positive integer m, A^m is similar to D^m, hence $\mathrm{Tr}(A^m) = \lambda_1^m + \lambda_2^m + \cdots + \lambda_k^m$. Without loss of generality, we suppose that $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_k$. Then, from Chebychev's inequality, we have

$$(\lambda_1^m + \lambda_2^m + \cdots + \lambda_k^m)(\lambda_1 + \lambda_2 + \cdots + \lambda_k) \leq k(\lambda_1^{m+1} + \lambda_2^{m+1} + \cdots + \lambda_k^{m+1})$$

so that

$$\mathrm{Tr}(A^m) \leq \frac{k}{\mathrm{Tr}(A)}\mathrm{Tr}(A^{m+1}).$$

It is immediately deduced that

$$\mathrm{Tr}(A^p) + \mathrm{Tr}(A^{p+1}) + \cdots + \mathrm{Tr}(A^{p+q}) \leq \frac{k}{\mathrm{Tr}(A)}(\mathrm{Tr}(A^{p+1}) + \mathrm{Tr}(A^{p+2}) + \cdots + \mathrm{Tr}(A^{p+q+1})),$$

and the required result follows (since $k = r_A$).

Also solved by James Duemmel, Bellingham, WA; Dmitry Fleischman, Santa Monica, CA; Jim Hartman, The College of Wooster; Justin Haverlick, Keene Valley, NY; Eugene Herman, Grinnell C.; Koopa Koo, Hong Kong STEAM Academy; Omran Kouba, Damascus, Syria; Elias Lampakis, Kiparissia, Greece; Pi'ilani Noguchi; Northwestern U. Math Problem Solving Group; Sunghee Park, Seoul, Korea; Michael Vowe, Therwil, Switzerland; and the proposer.

An integral equation

1177. *Proposed by Ovidiu Furdui and Alina Sîntămărian, Technical University of Cluj-Napoca, Cluj-Napoca, Romania.*

Find all continuous functions $f : \mathbb{R} \to \mathbb{R}$ which satisfy the following equation for all $x \in \mathbb{R}$:

$$f(-x) = 1 + \int_0^x \sin t f(x-t)dt.$$

Editor's note: This problem already appeared as problem 1163, with solution presented in the November 2020 issue. We apologize for the duplication.

Circles internally tangent to two sides of a triangle and the circumcircle

1178. *Proposed by Cezar Lupu, Texas Tech University, Lubbock, TX, and Vlad Matei, University of California Irvine, Irvine, CA.*

Consider a triangle ABC. Let $\mathcal{C}$ be the circumcirlce of ABC, r the radius of the incircle, and R the radius of $\mathcal{C}$. Let arc(BC) be the arc of $\mathcal{C}$ opposite A, and define arc(CA) and arc(AB) similarly. Let $\mathcal{C}_A$ be the circle tangent internally to the sides AB, AC, and the arc BC no containing A, and let R_A be its radius. Define $\mathcal{C}_B$, $\mathcal{C}_C$, R_B, and R_C similarly. Prove that the following inequality holds:

$$4r \leq R_A + R_B + R_C \leq 2R.$$

Solution by Michael Vowe, Therwil, Switzerland.

The solution of problem **11386** in the American Mathematical Monthly, May 2010, pp. 463-464, shows that

$$R_A = \frac{r}{\cos^2\left(\frac{}{2}\right)}, R_B = \frac{r}{\cos^2\left(\frac{B}{2}\right)}, R_C = \frac{r}{\cos^2\left(\frac{C}{2}\right)}.$$

Since

$$\sec^2\frac{A}{2} + \sec^2\frac{B}{2} + \sec^2\frac{C}{2} = 1 = \left(\frac{4R+r}{s}\right)^2$$

(see [1], p. 58 (63)), we have only to show that

$$3 \leq \left(\frac{4R+r}{s}\right)^2 \leq \frac{2R-r}{r}.$$

But this inequality can be obtained by Mitrinovic et al. [1], p. 166, 3.5. This chain of inequalities yields many sharper versions.

Equality holds if and only if the triangle is equilateral.

[1] D. S. Mitrinović, J. E. Pečarić, V. Volenec (1989), *Recent Advances in Geometric Inequalities*. Dordrecht: Kluwer.

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Also solved by Michel Bataille, *Rouen, France;* Radouan Boukharfane, *Extreme Computing Research Center, Kaust, Thuwal, KSA;* Habib Far, *Lone Star College-Montgomery;* Giuseppe Fera, *Vicenza, Italy;* Subhankar Gayen, *Mahavidyalaya, India;* Koopa Koo, Hong Kong STEAM Academy*;* Sushanth Sathish Kumar, *Portola H.S.;* Elias Lampakis, *Kiparissia, Greece;* Martin Lukarevski, *University Goce Delcev - Stip, North Macedonia;* Volkhard Schindler, *Berlin, Germany;* Daniel Vacaru, *Economical C. "Maria Teiuleanu", Piteşti, Romania; and the proposer.*

Small maximal ideals

1179. *Proposed by Greg Oman, University of Colorado, Colorado Springs, Colorado Springs, CO.*

Let R be a ring, and let I be an ideal of R. Say that I is *small* provided $|I| < |R|$ (i.e., I has a smaller cardinality than R). Suppose now that R is an infinite commutative ring with identity that is not a field. Suppose further that R possesses a small maximal ideal M_0. Prove the following:

1. there exists a maximal ideal M_1 of R such that $M_1 \neq M_0$, and
2. M_0 is the *unique* small maximal ideal of R.

Solution by Anthony Bevelacqua, University of North Dakota, Grand Forks, ND.

We will need the following basic result about cardinality: If A or B is infinite then $|A \times B| = \max(|A|, |B|)$.

Since R is not a field there exists a non-zero non-unit $a \in R$. Let $Ra = \{ra \mid r \in R\}$ and $R[a] = \{r \in R \mid ra = 0\}$. It's clear that both Ra and $R[a]$ are ideals of R. Since a is a non-unit we have $1 \notin Ra$, and since a is not zero we have $1 \notin R[a]$. Thus both Ra and $R[a]$ are proper ideals of R. The map $R \to Ra$ given by $r \mapsto ra$ is a ring epimorphism with kernel $R[a]$ so, by the first isomorphism theorem, we have $Ra \cong R/R[a]$. Hence $|R| = |Ra \times R[a]| = \max(|Ra|, |R[a]|)$. Thus R possesses a proper ideal I of cardinality $|R|$. Let M_1 be a maximal ideal of R containing I. Then $|I| \leq |M_1| \leq |R|$ so M_1 has cardinality $|R|$. Since $|M_0| < |R|$ we have $M_1 \neq M_0$. Thus we've shown 1.

Now assume M_0 and N are distinct small maximal ideals of R. Then, since they are distinct maximal ideals, we have $R = M_0 + N$. Since $M_0 + N = \{x + y \mid (x, y) \in M_0 \times N\}$ and R is infinite we have M_0 or N is infinite. Now

$$|R| \leq |M_0 \times N| = \max(|M_0|, |N|) < |R|,$$

a contradiction. This establishes 2.

Also solved by Paul Budney, Sunderland, MA; Eagle Problem Solvers, Georgia Southern U.; Elias Lampakis, Kiparissia, Greece; and the proposer.

Ideals in ideals

1180. *Proposed by Luke Harmon, University of Colorado, Colorado Springs, Colorado Springs, CO.*

In both parts, R denotes a commutative ring with identity. Prove or disprove the following:

1. there exists a ring R with infinitely many ideals with the property that every nonzero ideal of R is a subset of but finitely many ideals of R, and
2. there exists a ring R with infinitely many ideals with the property that every proper ideal of R contains (as a subset) but finitely many ideals of R.

Solution by Bill Dunn, Lone Star College Montgomery, Conroe, TX.

For 1, let R be the ring of integers. Every ideal I of R is principal, $I = (n)$, for some positive integer n. Suppose I is nonzero, $n \neq 0$. Then I is a subset of any other ideal $J = (m)$ if and only if m divides n. Because there are only finitely many positive integer divisors of n, there are only finitely many ideals of R that contain I.

For 2, suppose such a ring R existed. Because R has infinitely many ideals, it must have infinitely many proper ideals. Also, R must be Artinian because, by hypothesis on every proper ideal containing but finitely many ideals of R, any decreasing sequence of ideal must terminate.

However, an Artinian ring has only finitely many maximal ideals. Because every proper ideal is contained in some maximal ideal, one of these maximal ideals must contain infinitely many ideals of R, contradicting the hypothesis.

Therefore, such a ring R does not exist.

Also solved by Anthony Bevelacqua, U. of N. Dakota; Paul Budney, Sunderland, MA; Eagle Problem Solvers, Georgia Southern U.; and the proposer.

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