The Playground

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THE PLAYGROUND

Welcome to the Playground! Playground rules are posted on page 33, except for the most important one: *Have fun!*

THE SANDBOX

In this section, we highlight problems that anyone can play with, regardless of mathematical background. But just because these problems are easy to approach doesn't mean that they are easy to solve!

Uniform Three-step (P422). New contributor António Guedes de Oliveira (University of Porto, Portugal) posed this classical ruler-and-compass construction challenge. Given an arbitrary triangle ABC, construct points P and Q on rays $\overrightarrow{AC}$ and $\overrightarrow{BC}$, respectively, so that all three segments AP, PQ, and QB are congruent (as in the example in figure 1).

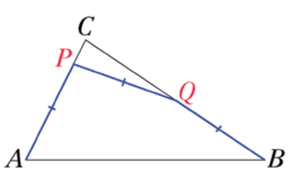

Figure 1. Three equal segments in a triangle.

THE MONKEY BARS

These open-ended problems don't have a previously known exact solution, so we intend for readers to fool around with them. The Playground will publish the best submissions received (proofs encouraged but not required).

Holes Uncovered (P423). Maintenance holes are round supposedly so that covers can't drop into the holes. Of course, even a round hole must have some "lip" (area filled in to reduce the aperture) to prevent the cover from falling in. However, any reduction to the hole diameter suffices, so the infimum of lip areas that will work is zero.

Suppose we only allow lips that leave a single convex aperture. What's the infimum lip area that will prevent the cover of a hole in the shape of an equilateral triangle from falling in? Note that the remaining aperture is *not* required to be a triangle—it need only prevent the equilateral triangular cover from passing through in any orientation (see figure 2 for an example). As always, submit your best shape even if you don't have a full optimization.

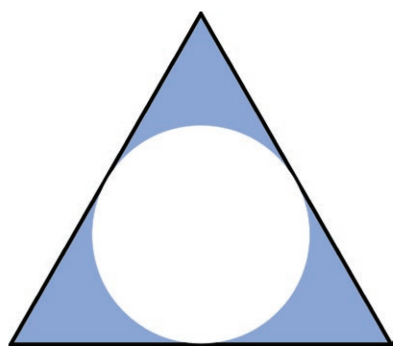

Figure 2. The (nonoptimal) blue shaded lip will prevent the cover of the equilateral triangular hole outlined in black from falling through.

THE ZIP LINE

This section offers problems with connections to articles that appear in the magazine. Not all Zip Line problems require you to read the corresponding article, but doing so can never hurt, of course.

Buffon Squared (P424). Frequent solver Randy K. Schwartz (Schoolcraft College) submitted this problem. It not only connects with his article "Approximating Pi with a Checkerboard" (p. 26), but also continues the *Playground's* recent annual series of riffs on the Buffon experiment.

Suppose a square tile with unit diagonal is tossed randomly onto an infinite checkerboard composed of unit(-side) square cells. What is the probability that the tile lands entirely within one cell?

THE JUNGLE GYM

Any type of problem may appear in the Jungle Gym—climb on!

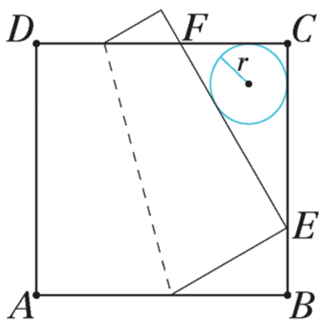

Figure 3. Folding one vertex of a square onto a nonadjacent side.

Folder's Inequality (P425). A traditional sangaku inspired this problem from returning contributor Yagub Aliyev (ADA University, Azerbaijan). (See "Seeking Sangaku" in the November 2016 issue of *Math Horizons* for background on this Japanese mode of mathematics.) Make a single straight-line fold in a unit-square piece of paper $ABCD$ so that vertex A lands on edge BC, say at point E. Then edge AD lands so that it intersects CD; call the point of intersection F (see figure 3).

Let r be the inradius of triangle CEF. Show that $r = 1 - \overline{EF}$ and that $\sqrt{r} \le \sqrt{2} - 1$.

THE CAROUSEL—OLDIES, BUT GOODIES

In this section, we present an old problem that we like so much, we thought it deserved another go-round. Try this, but be careful—old equipment can be dangerous. Answers appear at the end of the column.

Bi-thagorean Theorem? (C34). Art Kalish (SUNY College, Old Westbury) recommended this classic. Label an arbitrary triangle with sides a, b, and c opposite vertices A, B, and C, respectively. Show that $\angle C$ is twice $\angle B$ if and only if $c^2 = ab + b^2$.

FEBRUARY WRAP-UP

Editor's note: The Skidmore College Problem Solvers solved P410 Hexsquare but were unfortunately not listed in the April issue.

Dense Dominoes (P403). Another update on the problem of finding small-perimeter "proper" packings of the set of double-n dominoes, in which adjacent squares of different dominoes must have matching numbers:

Figure 4. Double-5 dominoes in a 7×8 rectangle and double-12 dominoes in a 16×16 rectangle.

Erik Richey (Chatham University) found proper packings with smaller perimeter than Ganning's and Praul's previously conjectured minimum (see the *Playground* roundup for November 2020): $n = 5$ in a 7×8 rectangle (figure 4); $n = 8$ in 11×11; $n = 10$ in 13×14; $n = 11$ in 14×16; and $n = 12$ in a 16×16 square (figure 4).

Notably, these include the first packings discovered that achieve fewer than one empty cell per non-double domino. Also, one can show that the 7×8 packing of the double-5 set achieves the minimum possible perimeter (a smaller perimeter only allows seven empty cells, which are not enough to provide necessary "gaps" next to each of the 15 non-doubles).

Based on the numerical results above, Richey conjectures that the minimum rectangle perimeter for a proper packing of a double-n set of dominoes is $P = 4n + 2\lfloor n/3 \rfloor + 8$.

Improved packings or a proof or disproof of this conjecture would be welcome in the *Playground*.

One World Octagon (P414). David Seppala-Holtzman (St. Joseph's College) sent this problem about One World Trade Center (OWTC) in Manhattan. The building rests on a square prism, 200 feet on a side and 185 feet high. The 150 foot-by-150 foot square parapet is centered over the base viewed from above, rotated 45° relative to the base, and is 1,368 feet high. Each corner of the parapet is connected by straight beams to the two nearest corners of the prism base. Is there a height at which the horizontal cross section of OWTC is a regular octagon? If so, what is it?

Yes, OWTC has a regular-octagonal cross section **861** feet above the ground. We received a solution from the Missouri State University problem-solving group, and partial solutions from Randy K. Schwartz and from problem-solving groups at Cal Poly Pomona, Georgia Southern University, and Skidmore College. Figure 5 helps to visualize the solution.

First, the symmetries of the building around its central vertical axis imply that all cross sections of the building are equiangular. We now follow the method from Missouri State. Consider the cross section at a height that is a fraction $0 < \lambda < 1$

of the way from the top of the prism base to the height of the parapet. Because each of the nearly vertical faces of the tower is an isosceles triangle, by similar triangles half of the edges of the cross section have length 150λ and the other half have length 200(1 – λ). In the regular cross section, these are all equal, yielding λ = 4/7. This value means that the regular cross section occurs at a height of 184 + (4/7)(1368 – 185) = 861 feet.

Small Squiral (P415). Let square S_0 have vertices $A_0 = (0,1)$, $B_0 = (-1,0)$, $C_0 = (0,-1)$, and $D_0 = (1,0)$. A *squiral* is a finite sequence of n squares $S_i = (A_i, B_i, C_i, D_i)$ with A_{i+1} on side A_iB_i, B_{i+1} on B_iC_i, and C_{i+1} on C_iD_i, and such that each line B_iA_i has positive slope, except B_nA_n is vertical.

The *size* of a squiral is the side length of the final square $\overline{A_nB_n}$. What is the smallest possible size of any squiral?

The infimum of squiral sizes (not achieved) has the lovely expression $\sqrt{2}\mathbf{e}^{-\tau/8}$ (where $\tau = 2\pi$ is the angle measure of a circle). We received insightful submissions from Lucas and Alex Perry (San Francisco University HS) and Randy K. Schwartz, and, for the first time for a Monkey Bars problem, complete solutions from Jesús Sistos together with Jesús Liceaga and the Eagle Problem Solvers (Georgia State University), from the CNU Problem Solving Seminar, and from the Missouri State University problem-solving group.

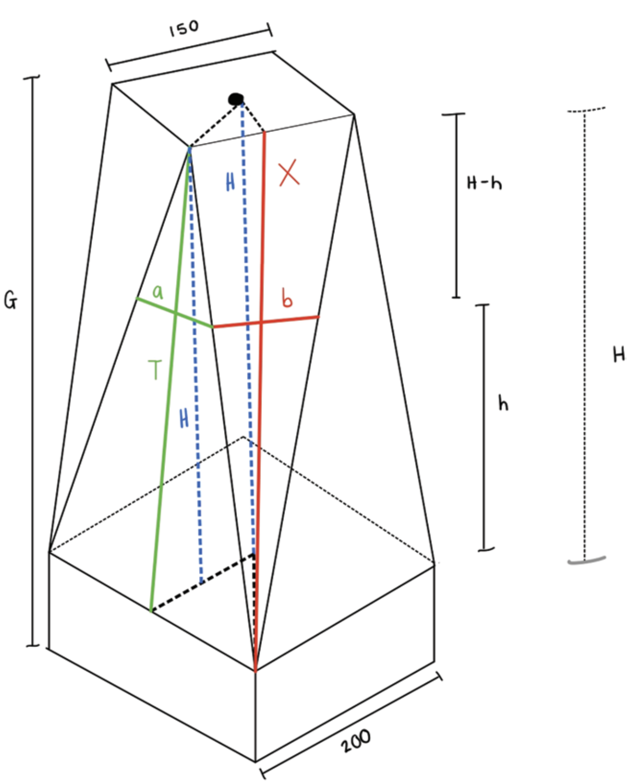

Figure 5. A geometric schematic of One World Trade Center provided by the Cal Poly Pomona group.

All submissions identified the key angle θ as shown in figure 6 (provided by the Perry brothers).

Because s_2, the side of the inner square, is the hypotenuse of a right triangle with angle θ at vertex E, some trigonometry tells us that

$$\frac{s_2}{s_1} = \frac{1}{\sin\theta + \cos\theta}.$$

Thus, to find the smallest squiral, we seek a sequence of angles $\theta_1, \ldots, \theta_n$ that sums to $\alpha = \tau/8$ (the squares overall rotate an eighth of a turn) and that maximizes the product

$$\prod_{i=1}^{n} \sin\theta_i + \cos\theta_i.$$

For a fixed n, this maximum occurs when each θ_i is equal to α/n. Sistos proved this maximum by combining the arithmetic-geometric mean and Jensen's inequalities; the CNU Problem Seminar used Lagrange multipliers; and the Missouri State group found a contradiction presuming the maximum occurred with any θ_i and θ_j distinct.

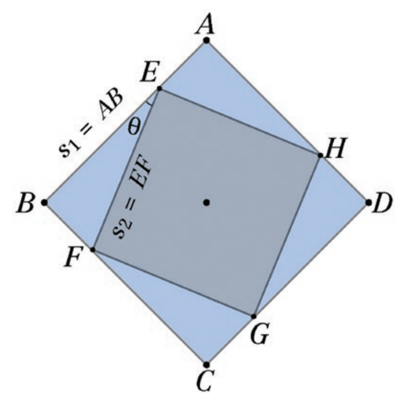

Figure 6. Two generic consecutive squiral squares.

All groups noted that $(\sin(\alpha/n) + \cos(\alpha/n))^n$ is a strictly increasing sequence. Hence, there is no minimum squiral size, but the infimum is $\sqrt{2}$ (the initial square side) divided by the limit of this sequence. It's easier to find the logarithmic limit:

$$\alpha \cdot \lim_{n\to\infty} \frac{\ln(\sin(\alpha/n) + \cos(\alpha/n))}{\alpha/n} = \alpha \cdot \lim_{x\to 0} \frac{\ln(\sin x + \cos x)}{x}.$$

Using L'Hôpital's rule, we see the limit is $\alpha \cdot 1$, for a minimum squiral size of $\sqrt{2}/e^{\alpha} = \sqrt{2}/e^{\tau/8}$.

The Missouri State solvers generalized to a nested spiral of regular k-gons, for which the infimum size has the elegant form $2\sin(\alpha)e^{-\alpha\tan\alpha}$ where now $\alpha = \tau/2k$.

Coarse Squares (P416). Jim Propp recommended this problem connected with his article "The Square Root of Pi." Call a positive whole number *coarse* if it has a prime factor larger than three. Determine the value of the sum

$$\frac{1}{25} + \frac{1}{49} + \frac{1}{100} + \frac{1}{121} + \frac{1}{169} + \cdots,$$

that is, the sum of the reciprocals of the squares of the coarse numbers.

The series sums to **$(\pi^2 - 9)/6$**. Brian Beasley (Presbyterian College), Tom Edgar, Dmitry Fleischman, Christopher Havens (Prison Mathematics Project), and Randy Schwartz submitted solutions, along with problem-solving groups from Georgia Southern University and Missouri State University, and a partial solution from the Skidmore College problem solvers.

Call a number "smooth" if it is not coarse, that is, if it is of the form $2^a 3^b$ for natural numbers a and b. The requested sum S is the difference of the sum of reciprocals of all squares and the sum of reciprocals of smooth squares. The former was established by Euler to equal $\pi^2/6$. The latter sum factors as the product of the sum of the negative even powers of two and the sum of the negative even powers of three:

$$\left(\sum_{k=0}^{\infty} 2^{-2k}\right)\left(\sum_{k=0}^{\infty} 3^{-2k}\right) = \left(\frac{1}{1-2^{-2}}\right)\left(\frac{1}{1-3^{-2}}\right) = 9/6,$$

so $S = (\pi^2 - 9)/6$.

Taxicab for Squares (P417). In honor of Ramanujan's birthday, Christopher Havens asked for the smallest positive integer n such that $1729n^2 + 1$ is a perfect square.

The requested number is slightly larger than a trillion. Computations of it arrived from Randy Schwartz and problem-solving groups at Georgia Southern University, Missouri State University, and Skidmore College.

Note that if n is as requested, then for some m, $m^2 - 1729n^2 = 1$. Further, note that

$$\frac{m}{n} - \sqrt{1729} = \frac{m - n\sqrt{1729}}{n} = \frac{1}{n(m + n\sqrt{1729})} < \frac{1}{2n^2}.$$

By a well-known theorem on the accuracy of continued fraction convergents, m/n must be a convergent of the continued fraction for $\sqrt{1729}$, which is

$$[41;1,1,2,1,1,2,1,2,1,8,1,1,27,5,6,$$
$$5,27,1,1,8,1,2,1,2,1,1,2,1,1,82,\ldots]$$

where the portion after the semicolon repeats infinitely thereafter. (Can you show that it is no accident that $82 = 2 \cdot 41$ and that the remainder of the period is a palindrome?)

With this information, one can test that the first suitable convergent is p_{29}/q_{29}. (Further general theory allows you to determine the index 29 from the repeating period length without needing to check all of the intermediate convergents.) Hence, the smallest positive integer n such that $1729n^2 + 1$ is a perfect square is $q_{29} =$ **1,072,885,712,316**. A remarkable fact about such so-called Pell equations is that their least (positive) integer solution can be quite large, even when their coefficient (here 1729) is relatively small.

CAROUSEL SOLUTION

Erect an isosceles triangle BCD on side a of the given triangle and extending side b, as shown in figure 7. Then the following successive statements are equivalent:

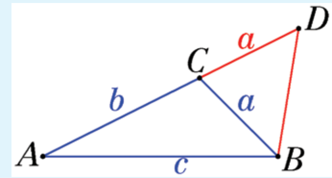

Figure 7. An arbitrary triangle ABC with isosceles triangle erected on side *a* and extending side *b*.

- $\angle C = 2\angle B$
- $\angle BCD = (\tau/2) - 2\angle B$
- $\angle CBD = \angle BDA = \angle B$
- Triangle ADB is similar to triangle ABC
- $\overline{AD}/\overline{AB} = \overline{AB}/\overline{AC}$
- $(a+b)/c = c/b$
- $c^2 = ab + b^2$

Thus, we have demonstrated that the given property of the angles of a triangle ($\angle C = 2\angle B$) is equivalent to the condition that a certain quadratic form ($ab + b^2 - c^2$) on the side lengths, with integer coefficients, takes on a zero value. Note that the Pythagorean theorem has a perfectly analogous statement, with angle condition $\angle A + \angle B = \tau/4$ and quadratic form $a^2 + b^2 - c^2$.

The Law of Cosines yields two more such facts:

$$\angle C = \tau/6 \Leftrightarrow a^2 + b^2 - ab - c^2 = 0$$
$$\angle C = \tau/3 \Leftrightarrow a^2 + b^2 + ab - c^2 = 0.$$

If any Playground reader knows of or can discover any other theorems of this form, the editor would be very interested to hear.

SUBMISSION & CONTACT INFORMATION

The Playground features problems for students at the undergraduate and (challenging) high school levels. Problems and solutions should be submitted to MHproblems@maa.org and MHsolutions@maa.org, respectively (PDF format preferred). Paper submissions can be sent to Glen Whitney, UCLA Math Dept., 520 Portola Plaza MS 6363, Los Angeles, CA 90095. Please include your name, email address, and affiliation, and indicate if you are a student. If a problem has multiple parts, solutions for individual parts will be accepted. Unless otherwise stated, problems have been solved by their proposers.

The deadline for submitting solutions to problems in this issue is October 31, 2021.

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