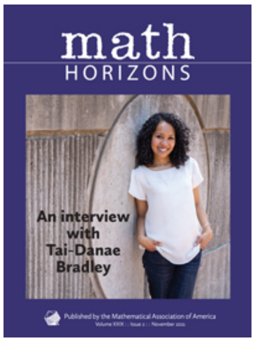

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The Playground

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THE PLAYGROUND

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THE SANDBOX

In this section, we highlight problems that anyone can play with, regardless of mathematical background. But just because these problems are easy to approach doesn't mean that they are easy to solve!

Uncommon Multiples? (P426). Here's a more abstract question from David Seppala-Holtzman (St. Joseph's College), author of the One World Octagon problem from the February issue. You are given natural numbers G and L such that L is a nontrivial multiple of G. How many *unordered* pairs of numbers have greatest common divisor G and least common multiple L?

THE MONKEY BARS

These open-ended problems don't have a previously-known exact solution, so we intend for readers to fool around with them. The Playground will publish the best submissions received (proofs encouraged but not required).

Tile Trays (P427). The "Circus Tent Puzzle" on *MathPickle.com* inspired this problem. You are producing sets of regular polygonal tiles, all of the same unit side length. A k-gon will sell for $3k - 7$ dollars (so an equilateral triangle is \$2, a square is \$5, a regular pentagon \$8, etc.). Each set of tiles comes with a tray that will fit them all without overlap, in an *edge-to-edge* arrangement: any two tiles that share any portion of an edge must share an entire edge.

The longer the perimeter of the tray, the more difficult it is to make. Hence, you wish to minimize the perimeter of the tray; note that the tiles need not completely fill it.

For each n, what is the minimum perimeter of a tray for a set of tiles that will sell for exactly n dollars? For example, an early interesting case is \$13. There are three possible sets of tiles: a hexagon and a triangle, a pentagon and a square, or a square and four triangles. But the latter set allows the least perimeter, as illustrated in figure 1.

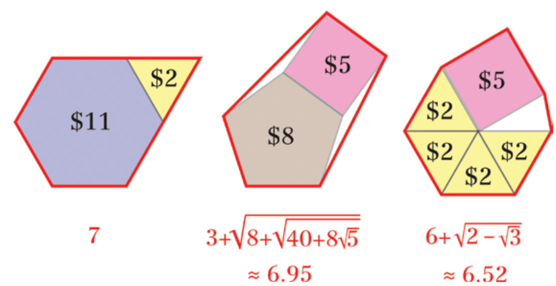

Figure 1. Three sets of tiles that sell for \$13 with their minimal-perimeter trays in red.

THE ZIP LINE

This section offers problems with connections to articles that appear in the magazine. Not all Zip Line problems require you to read the corresponding article, but doing so can never hurt, of course.

Candy Circle (P428). This problem follows up on one in Florentino and Higginbottom's article "Three-Pile Candy Sharing" (p. 12). Three students start with piles of $2j$, $2j \pm 2$, and $2j + 2k$ candies, respectively, for integers $j > 1$ and $k > -j$. In each round, all the students give each of the others half of their pile, and then receive (from an external supply) one additional candy if they happen to end up with an odd number of candies.

Show that after a finite number of rounds, all students have the same number of candies, and determine how many rounds it will take to reach this state. Can you determine the final number of candies each student will have at the end?

THE JUNGLE GYM

Any type of problem may appear in the Jungle Gym—climb on!

Exponential Harmony (P429). From Romania, the Playground received this problem by Dorin Marghidanu (Colegiul National 'A.I. Cuza'). Given positive real numbers a, b, and c, let h be the reciprocal of their sum. Prove that

$$(a^a b^b c^c)^h + (a^b b^c c^a)^h + (a^c b^a c^b)^h \le a + b + c.$$

THE CAROUSEL—OLDIES, BUT GOODIES

In this section, we present an old problem that we like so much, we thought it deserved another go-round. Try this, but be careful—old equipment can be dangerous. Answers appear at the end of the column.

Conaxial Conundrum? (C35). A (right circular) cone can be viewed as a surface of revolution of a line called its *generator*. Classically, the intersection of a right circular cone with a plane that is anywhere between perpendicular to the cone's axis and parallel to its generator yields an ellipse. At what point in its interior does the axis of revolution of the cone pierce such an ellipse? Hint: it is neither the ellipse's centroid nor a focus.

APRIL WRAP-UP

Icosatwins (P418). This reverse-video version of Problem 1099 in *The College Mathematics Journal* came to us from Wenwen Du and Paul Peck of Glenville State College. You are going to color the triangular faces of an icosahedron blue and white so that every white triangle shares exactly one of its edges with another white triangle. What is the maximum number of white triangles that your icosahedron can have?

There can be at most **12** triangles colored white. The Missouri State problem-solving group submitted this solution, and the Georgia Southern University solvers sent a different proof.

Consider the five triangles surrounding any vertex. If four or more are white, then at least one white triangle will have two neighboring triangles shaded in white. Hence, there are at most three white triangles meeting at any vertex. But there are 12 vertices in the icosahedron, and each triangle touches three vertices, so there are at most $12 \times 3 / 3 = 12$ white triangles, as desired. Figure 2 shows a coloring that achieves this upper bound (note the exterior unbounded region represents the twelfth white triangle).

The Missouri State group also notes that the centers of the blue triangles of the coloring in figure 2 comprise the vertices of a cube (of edge length $\phi^2 / 3$). Moreover, the centroids of the six white regions form an octahedron of *unit* edge length—if you find a compact proof of this fact, please submit it to the Playground.

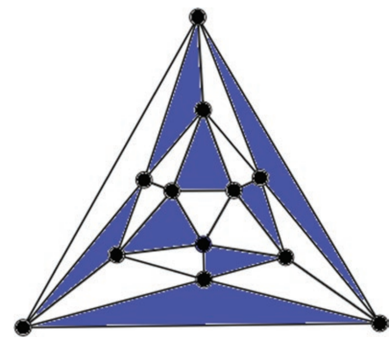

Figure 2. Face-coloring of an icosahedron with 12 white triangles, each adjacent to exactly one other.

OEIS Challenge (P419). You create a triangular diagram somewhat like the arithmetic triangle (also known as Halayudha's or Pascal's triangle), except that the number above and to the right tells you how many numbers up the diagonal to the left to add up, using 1s for any extra terms needed.

2
2 2
2 3 2
2 4 5 2
2 5 9 8 2
2 6 14 20 13 2
⋮

To your surprise, you check and discover that this particular number triangle is not listed in the On-Line Encyclopedia of Integer Sequences (OEIS; which can be found at *oeis.org*).

Your challenge was to submit one interesting fact about this triangle to the Playground. It could be a closed form for some or all of the terms, or a computation of the row sums of the triangle, or a relationship between some entries of this triangle and another well-known sequence, etc. If the Playground reader community collects enough material of interest for this sequence to be accepted into the OEIS, we all win the Challenge!

Carl Libis (Columbia Southern University) sent a number of interesting observations without proof; presumably they can generally be verified by induction. First, to set notation: we write $a(r,p)$ for an entry of the triangle, where r represents the row number from the top down and starting from 0, and p signifies the position within a row, numbered 0 to r from left to right. By convention $a(r,p) = 1$ for $r < 0$, $p < 0$, or $p > r$. The defining recursion for the triangle is then

$$a(r,p) = \sum_{i=1}^{a(r-1,p)} a(r-i, p-i),$$

with initial condition $a(0,0) = 2$.

It's straightforward to see that $a(r,1) = r + 1$ and that $a(r,2)$ is one less than the rth triangular number. For higher fixed values of p, it's convenient to define

$$S_3(r,p) = a(r,p) + a(r,p-1) + a(r,p-2).$$

Carl finds that for $p = 3$, 4, or 5 and $r > p$,

$$a(r+1,p) = S_3(r,p) - a(r,p-4) - 1.$$

However, this formula doesn't directly generalize to larger p:

- For $r > 6$,
  $a(r+1,6) = S_3(r,6) - a(r,2) + 1;$
- For $r > 7$,
  $a(r+1,7) = S_3(r,7) - a(r,3) + 2a(r,1) - 5.$

The similarity of these additional two cases suggests that there might be some formula for an arbitrary entry in terms of just entries in the prior row.

In addition, Carl investigates the right-descending columns $c_q(r) = a(r,r-q)$, noting:

- $c_1(r) = H_{r-1}$, where H gives the sum recurrence (Hemachandra or Fibonacci numbers);
- $c_2(r) = 2c_2(r-1) + c_1(r-3) - 1$; and
- $c_3(r) = 2c_3(r-1) + c_2(r-2) + c_1(r-4) - 2$.

In fact, it appears numerically that in general, for $q > 1$, $c_q(r)$ is given by

$$c_q(r-1) + c_1(r-q-1) - q + 1 + \sum_{i=1}^{q-1} c_{q-i+1}(r-i).$$

Whether these findings warrant inclusion in the OEIS remains to be seen, so if you make any further discoveries or find generalizations of the formulas for an entry in terms of the previous row, do submit them.

SET Another Problem (P420). This problem came from Hsu, Ostroff, and Van Meter's article "Set with a Twist." Given a group G with identity e, a 3SET in G is an ordered triple $\langle g_1, g_2, g_3 \rangle$ of three distinct elements of G such that $g_1g_2g_3 = e$.

How many 3SETs are there in a) S_4, b) S_5, or c) S_6? Here S_n is the usual symmetric group, or group of permutations, on n elements.

Dmitry Fleischman and the Missouri State solvers sent solutions, along with partial answers from Wayne Nelson (Lovelock Correctional) and the Georgia Southern solvers. There are **522**, **14082**, and **516402** 3SETs in S_4, S_5, and S_6, respectively.

Note that any ordered pair $\langle g,h \rangle$ from G participates in at most one 3SET, namely $\langle g,h,(gh)^{-1} \rangle$. So when $|G| = N$, we begin with the N^2 ordered pairs and count 3SETs by inclusion-exclusion.

There are three ways this triple could fail to be a 3SET: $g = h$, $g = (gh)^{-1}$, or $h = (gh)^{-1}$. There are N pairs $\langle g,h \rangle$ that satisfy any one of these three equations. Any two or all three of them holding simultaneously is equivalent to $g^3 = e$. Hence, if we let N_3 be the number of elements whose cube is the identity, the number of 3SETs is

$$N^2 - 3N + 3N_3 - N_3 = N(N-3) + 2N_3.$$

For $G = S_n$, $N = n!$. For $n < 6$, the only elements of order 1 or 3 are the identity and the three cycles, yielding $1 + n(n-1)(n-2)/3$ such elements. However, S_6 contains additional elements of order 3: the products of two disjoint three-cycles, of which there are (in S_n) $n!/((n-6)! \cdot 3^2 \cdot 2)$. The numerical values above follow.

Fearsome Cyclocts (P421). This problem was from Arsalan Wares (Valdosta State University). The eight points A through H all lie on the unit circle. Vertical segment AB lies in the right half-plane with B above A, and has the same length as horizontal segment CD in the upper half-plane with C to the right of D. Similarly, vertical segment EF with E above F lies in the left half-plane, and has the same length as horizontal GH in the lower half-plane with G to the left of H. Any eight points satisfying the above conditions constitute the vertices of a special sort of possibly self-intersecting octagon $ABCDEFGH$ that we will call a *cyclocts*. The interior of a cyclocts is defined by the "odd-crossing rule," as shown in the example in figure 3. (Note that depending on the locations of the four segments, there may be as many as four points of self-intersection.) Find the minimum-area and maximum-area cyclocts.

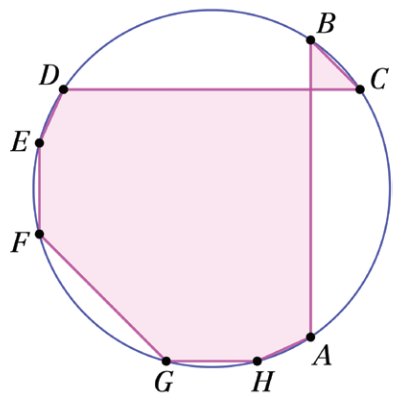

Figure 3. An example cyclocts. A point is in the interior if you must cross its edges an odd number of times to reach the circle.

Apparently, this cyclocts has so far been too fearsome, as the Playground has not received any submissions. We will hold this problem open until the due date for the problems proposed in this issue.

SUBMISSION & CONTACT INFORMATION

The Playground features problems for students at the undergraduate and (challenging) high school levels. Problems and solutions should be submitted to MHproblems@maa.org and MHsolutions@maa.org, respectively (PDF format preferred). Paper submissions can be sent to Glen Whitney, UCLA Math Dept., 520 Portola Plaza MS 6363, Los Angeles, CA 90095. Please include your name, email address, and affiliation, and indicate if you are a student. If a problem has multiple parts, solutions for individual parts will be accepted. Unless otherwise stated, problems have been solved by their proposers.

The deadline for submitting solutions to problems in this issue is January 15, 2022.

SOLUTION TO NINES CLOCK PUZZLE

Challenge I has solutions for 2, 3, 4 and 9. Challenge II has extra solutions of 8, 9^n, 16^n, and 64^n. We have checked that there are no further solutions up to 144. Sample solutions can be found at *maa.org/mathhorizons/supplemental.htm*. If you find a solution to either challenge not listed here, email Brian Shelburne at (*bshelburne@wittenberg.edu*).

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CAROUSEL SOLUTION

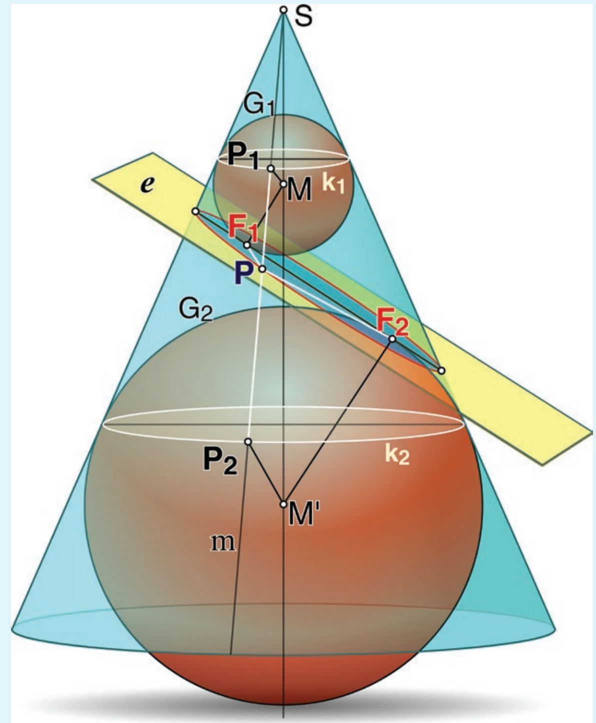

Figure 4. Dandelin spheres of an ellipse, courtesy of Perhelion on Wikimedia.

There are exactly two spheres tangent to both the cone (in one of its circular cross-sections) and to the plane whose intersection with the cone produces the ellipse, one on each side of the plane. These are the so-called *Dandelin spheres*, shown in figure 4. As can also be seen from the figure by considering similar triangles, the intersection of the cone's axis with the interior of the ellipse is the point that divides the segment between the ellipse's foci in the same proportion as the ratio between the radii of the Dandelin spheres.

This is a bit of a trick question, however, in that this intersection point is not uniquely determined by the ellipse itself. Just as with a circle, the same ellipse can be generated by the intersection of its plane with some right circular cone of any apex angle α between 0 and τ / 2. For large α, the "upper" Dandelin sphere has small radius and the lower one has very large radius. On the other hand, as α tends toward 0 and the cone tends toward a cylinder, the radii of the two Dandelin spheres converge to the same value. Hence, the intersection of the cone axis and the ellipse can take on any point between a focus and the centroid of the ellipse, as the cone used to construct the ellipse varies.