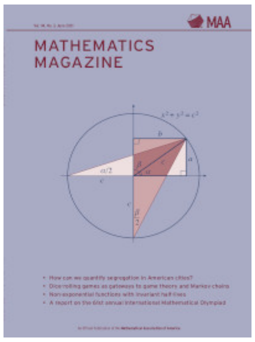


Mathematics Magazine

ISSN: (Print) (Online) Journal homepage: https://maa.tandfonline.com/loi/umma20

Problems and Solutions

To cite this article: (2021) Problems and Solutions, Mathematics Magazine, 94:3, 228-238, DOI: 10.1080/0025570X.2021.1909344

To link to this article: https://doi.org/10.1080/0025570X.2021.1909344

Published online: 23 Jun 2021.

PROBLEMS

Proposals

To be considered for publication, solutions should be received by November 1, 2021.

2121. *Proposed by Seán M. Stewart, Bomaderry, Australia.*

Evaluate

$$\int_0^{\frac{1}{2}} \frac{\arctan(x)}{x^2 - x - 1}\, dx.$$

2122. *Proposed by Ahmad Sabihi, Isfahan, Iran.*

Let

$$G(m, k) = \max\{\gcd((n+1)^m + k, n^m + k) | n \in \mathbb{N}\}.$$

Compute $G(2, k)$ and $G(3, k)$.

2123. *Proposed by Albert Natian, Los Angeles Valley College, Valley Glen, CA.*

An urn contains n balls. Each ball is labeled with exactly one number from the set

$$\{a_1, a_2, \cdots, a_n\}, \qquad a_1 > a_2 > \cdots > a_n,$$

(so no two balls have the same number). Balls are randomly selected from the urn and discarded. At each turn, if the number on the ball drawn was the largest number remaining in the urn, you win the dollar amount of that ball. Otherwise, you win nothing. Find the expected value of your total winnings after n draws.

Math. Mag. **94** (2021) 228–238. doi:10.1080/0025570X.2021.1909344 © Mathematical Association of America

We invite readers to submit original problems appealing to students and teachers of advanced undergraduate mathematics. Proposals must always be accompanied by a solution and any relevant bibliographical information that will assist the editors and referees. A problem submitted as a Quickie should have an unexpected, succinct solution. Submitted problems should not be under consideration for publication elsewhere.

Proposals and solutions should be written in a style appropriate for this Magazine.

Authors of proposals and solutions should send their contributions using the Magazine's submissions system hosted at `http://mathematicsmagazine.submittable.com`*. More detailed instructions are available there. We encourage submissions in PDF format, ideally accompanied by LaTeX source. General inquiries to the editors should be sent to* `mathmagproblems@maa.org`.

2124. *Proposed by Mircea Merca, University of Craiova, Craiova, Romania.*

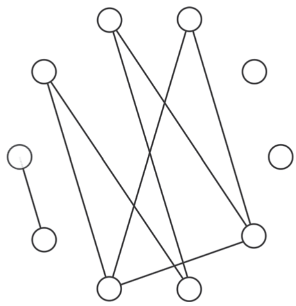
For a positive integer n, prove that

$$\sum_{\substack{\lambda_1+\lambda_2+\cdots+\lambda_k=n\\ \lambda_1\geqslant\lambda_2\geqslant\ldots\geqslant\lambda_k>0}} (-1)^{n-\lambda_1}\frac{\binom{\lambda_1}{\lambda_2}\binom{\lambda_2}{\lambda_3}\cdots\binom{\lambda_k}{0}}{1^{\lambda_1}2^{\lambda_2}\cdots k^{\lambda_k}}=\frac{1}{n!},$$

where the sum runs over all the partitions of n.

2125. *Proposed by Freddy Barrera, Colombia Aprendiendo, and Bernardo Recamán, Universidad Sergio Arboleda, Bogotá, Colombia.*

Given a collection of positive integers, not necessarily distinct, a graph is formed as follows. The vertices are these integers and two vertices are connected if and only if they have a common divisor greater than 1. Find an assignment of ten positive integers totaling 100 that results in the graph shown below.



Quickies

1111. *Proposed by George Stoica, Saint John, NB, Canada.*

Prove that for $A, B \in M_2(\mathbb{C})$, the following conditions are equivalent:

(i) $|\det(A+\lambda B)| = |\det(A-\lambda B)|$ for all $\lambda \in \mathbb{C}$

(ii) $\mathrm{tr}(AB) = \mathrm{tr}(A)\cdot\mathrm{tr}(B)$ or $\det A = \det B = 0$.

1112. *Proposed by Lokman Gökçe, Istanbul, Turkey.*

Let $\triangle ABC$ be any triangle with $m\angle BAC = 150^\circ$. Let $\triangle ABE$ and $\triangle ACD$ be equilateral triangles whose interiors lie in the exterior of $\triangle ABC$. Denote the intersection of the segments $\overline{BD}$ and $\overline{CE}$ by F. Prove that $FA + FE + FD = BC$.

Solutions

The number of isosceles triangles in various polytopes **June 2020**

2096. *Proposed by H. A. ShahAli, Tehran, Iran.*

Any three distinct vertices of a polytope P form a triangle. How many of these triangles are isosceles if P is (a) a regular n-gon? (b) one of the Platonic solids? (c) an n-dimensional cube?

Solution by Robert Calcaterra, University of Wisconsin-Platteville, Platteville, WI.
Let m denote the number of vertices of P. For a fixed vertex A of P, let $F(P)$ denote the number of unordered triplets of distinct vertices A, B, and C of P for which $AB = AC$, $G(P)$ is the number of such triplets for which $AB = AC = BC$, and $I(P)$ the number of isosceles triangles that can be formed using the vertices of P. Note that since all of the polytopes under consideration are uniform, $F(P)$ and $G(P)$ do not depend on A. Since each equilateral triangle is counted in $F(P)$ for three different choices of A,

$$I(P) = m\,(F(P) - G(P)) + \frac{m}{3}G(P) = mF(P) - \frac{2}{3}mG(P).$$

(a) If P is a regular n-gon, then $F(P) = \lfloor (n-1)/2 \rfloor$. Moreover, $G(P) = 1$ if n is a multiple of 3 and $G(P) = 0$ if not. Therefore,

$$I(P) = \begin{cases} n\left\lfloor \frac{n-1}{2} \right\rfloor & \text{if } 3 \nmid n \\ n\left\lfloor \frac{n-1}{2} \right\rfloor - \frac{2n}{3} & \text{if } 3 | n \end{cases}$$

(b) Let P be a Platonic solid. If A and B are vertices of P, the minimum number of edges of the solid that must be traversed to get from A to B will be called the span from A to B. For the Platonic solids, the spans for two pairs of vertices are the same if and only if the Euclidean distances are the same.

- If P is a tetrahedron, every triplet of distinct vertices forms an isosceles (in fact, equilateral) triangle. Therefore $I(P) = \binom{4}{3} = 4$.
- If P is a cube, then the numbers of vertices with spans 1, 2, and 3 from the fixed vertex A are 3, 3, and 1, respectively. Therefore, $F(P) = \binom{3}{2} + \binom{3}{2} = 6$. Moreover, 0 pairs of the vertices with span 1 from A have span 1 from each other, and 3 pairs with span 2 from A have span 2 from each other. Thus $G(P) = 3$ and $I(P) = 8 \cdot 6 - \frac{2}{3} \cdot 8 \cdot 3 = 32$. (This also follows from part (c) below).
- If P is an octahedron, every triplet of distinct vertices forms an isosceles triangle. Therefore $I(P) = \binom{6}{3} = 20$.
- If P is an icosahedron, then the numbers of vertices with spans 1, 2, and 3 from the fixed vertex A are 5, 5, and 1, respectively. Therefore, $F(P) = \binom{5}{2} + \binom{5}{2} = 20$. Moreover, 5 pairs of the vertices with span 1 from A have span 1 from each other, and 5 pairs with span 2 from A have span 2 from each other; thus $G(P) = 10$ and $I(P) = 12 \cdot 20 - \frac{2}{3} \cdot 12 \cdot 10 = 160$.
- If P is a dodecahedron, then the numbers of vertices with spans 1, 2, 3, 4, and 5 from A are 3, 6, 6, 3, and 1, respectively. So, $F(P) = \binom{3}{2} + \binom{6}{2} + \binom{6}{2} + \binom{3}{2} = 36$. Moreover, 0 pairs of vertices with span 1 from A have span 1 from each other, 3 pairs with span 2 from A have span 2 from each other, 6 pairs

with span 3 from A have span 3 from each other, and 0 pairs with span 4 from A have span 4 from each other; thus, $G(P) = 9$ and $I(P) = 20 \cdot 36 - \frac{2}{3} \cdot 20 \cdot 9 = 600$.

(c) Let P be a cube in $\mathbb{R}^n$. We may view the vertices of P as binary n-tuples, so that the distance between two vertices is the square root of the number of components at which they differ. The number of vertices of P at distance $\sqrt{k}$ from A is $\binom{n}{k}$ for $k = 0, 1, \dots, n$. Recall that

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n \text{ and } \sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}.$$

Therefore,

$$\begin{aligned} F(P) &= \sum_{k=1}^{n-1} \frac{1}{2} \binom{n}{k} \left(\binom{n}{k} - 1 \right) = \frac{1}{2} \left(\sum_{k=1}^{n-1} \binom{n}{k}^2 - \sum_{k=1}^{n-1} \binom{n}{k} \right) \\ &= \frac{1}{2} \left(\left(\binom{2n}{n} - 2 \right) - (2^n - 2) \right) \\ &= \frac{1}{2} \left(\binom{2n}{n} - 2^n \right) \end{aligned}$$

For the vertices A, B, and C to form an equilateral triangle with sides of length $\sqrt{k}$, three disjoint subsets, say X, Y, and Z, must be chosen from $\{1, 2 \dots, n\}$ in such a way that the components of A differ from those of B at precisely the positions in $X \cup Y$, the components of A differ from those of C at precisely the positions in $X \cup Z$, and the components of B differ from those of C at precisely the positions in $Y \cup Z$. This forces $|X \cup Y| = |X \cup Z| = |Y \cup Z| = k$, which yields $|X| = |Y| = |Z| = \ell$ and $k = 2\ell$. There will be $n - 3\ell$ positions at which the components of A, B, and C all agree (the positions in the complement of $X \cup Y \cup Z$). Note that each equilateral triangle will be generated twice using this procedure because interchanging Y and Z will reverse the roles of B and C. Therefore (using multinomial coefficients), we have

$$G(P) = \frac{1}{2} \sum_{\ell=1}^{\lfloor n/3 \rfloor} \binom{n}{n - 3\ell, \ell, \ell, \ell} \text{ and}$$

$$I(P) = 2^{n-1} \left(\binom{2n}{n} - 2^n \right) - \frac{2^n}{3} \sum_{\ell=1}^{\lfloor n/3 \rfloor} \binom{n}{n - 3\ell, \ell, \ell, \ell}$$

Also solved by Allen J. Schwenk, Albert Stadler (Switzerland), and the proposer. There were two incomplete or incorrect solutions.

A series involving the floor, ceiling, and round functions **June 2020**

2097. *Proposed by Omran Kouba, Higher Institute for Applied Sciences and Technology, Damascus, Syria.*

For a real number $x \notin \frac{1}{2} + \mathbb{Z}$, denote the nearest integer to x by $\langle x \rangle$. For any real number x, denote the largest integer smaller than or equal to x and the smallest integer larger than or equal to x by $\lfloor x \rfloor$ and $\lceil x \rceil$, respectively. For a positive integer n let

$$a_n = \frac{2}{\langle \sqrt{n} \rangle} - \frac{1}{\lfloor \sqrt{n} \rfloor} - \frac{1}{\lceil \sqrt{n} \rceil}.$$

(a) Prove that the series $\sum_{n=1}^{\infty} a_n$ is convergent and find its sum L.

(b) Prove that the set

$$\left\{ \sqrt{n}\left(\sum_{k=1}^{n} a_k - L\right) : n \geq 1 \right\}$$

is dense in [0, 1].

Solution by Hongwei Chen, Christopher Newport University, Newport News, VA.

(a) We show that the sum converges to zero. To see this, first, we can easily check the following facts:

$$\begin{aligned} \langle \sqrt{n} \rangle &= k, \text{for } n \in [k(k-1)+1, k(k+1)], \\ \lfloor \sqrt{n} \rfloor &= k, \text{for } n \in [k^2, (k+1)^2), \\ \lceil \sqrt{n} \rceil &= k+1, \text{for } n \in (k^2, (k+1)^2]. \end{aligned}$$

These imply that $a_{k^2} = 0$ and

$$a_n = \frac{2}{k} - \frac{1}{k} - \frac{1}{k+1} = \frac{1}{k(k+1)}, \text{ for } n \in (k^2, k(k+1)],$$

$$a_n = \frac{2}{k+1} - \frac{1}{k} - \frac{1}{k+1} = -\frac{1}{k(k+1)}, \text{ for } n \in (k(k+1), (k+1)^2).$$

Therefore, for $k^2 \leq n \leq (k+1)^2$, we have $\sum_{m=1}^{k^2} a_m = 0$ and

$$0 \leq \sum_{m=1}^{n} a_m \leq \frac{1}{k(k+1)} \cdot [k(k+1) - k^2] = \frac{1}{k+1}$$

As $n \to \infty$, we have $k \to \infty$ and so

$$\sum_{n=1}^{\infty} a_n = \lim_{n \to \infty} \sum_{m=1}^{n} a_m = 0.$$

(b) Let $x \in [0, 1]$. We show that there exists a subsequence from the set $\{\sqrt{n} \sum_{m=1}^{n} a_m\}$, which converges to x. Notice that there exist two integer sequences p_k and q_k with $0 \leq p_k \leq q_k$ such that $p_k/q_k \to x$, as $k \to \infty$. Let $n_k = q_k^2 + p_k$. Then

$$q_k^2 \leq n_k \leq q_k^2 + q_k < \left(q_k + \frac{1}{2}\right)^2.$$

This implies that

$$\langle \sqrt{n_k} \rangle = q_k, \ \lfloor \sqrt{n_k} \rfloor = q_k, \ \lceil \sqrt{n_k} \rceil = q_k + 1.$$

Therefore, as $k \to \infty$, we have

$$\sqrt{n_k} \sum_{m=1}^{n_k} a_m = \sqrt{n_k} \cdot \frac{n_k - q_k^2}{q_k(q_k+1)} = \frac{p_k}{q_k} \cdot \frac{\sqrt{n_k}}{q_k+1} \to x.$$

This proves that the set $\{\sqrt{n} \sum_{m=1}^{n} a_m\}$ is dense in [0, 1].

Also solved by Elton Bojaxhiu (Germany) & Enkel Hysnelaj (Australia), Brian Bradie, Robert Calcaterra, Dmitry Fleischman, Maxim Galushka (UK), GWstat Problem Solving Group, Eugene A. Herman, Walter Janous (Austria), Donald E. Knuth, Sushanth Sathish Kumar, Elias Lampakis (Greece), Shing Hin Jimmy Pa (Canada), Allen Schwenk, Albert Stadler (Switzerland), and the proposer. There was one incorrect or incomplete solution.

A zigzag sequence of random variables **June 2020**

2098. *Proposed by Albert Natian, Los Angeles Valley College, Valley Glen, CA.*

Let $Z_0 = 0$, $Z_1 = 1$, and recursively define random variables $Z_2, Z_3, \ldots,$ taking values in [0, 1] as follows: For each positive integer k, Z_{2k} is chosen uniformly in $[Z_{2k-2}, Z_{2k-1}]$, and Z_{2k+1} is chosen uniformly in $[Z_{2k}, Z_{2k-1}]$.

Prove that, with probability 1, the limit $Z^* = \lim_{n\to\infty} Z_n$ exists and find its distribution.

Solution by Northwestern University Math Problem Solving Group, Northwestern University, Evanston, IL.
We will prove:

1. The limit Z^* exists.
2. The limit Z^* has probability density $f(x) = 2x$ on [0, 1].

Proof of 1. We have that $[Z_0, Z_1] \supseteq [Z_2, Z_1] \supseteq [Z_2, Z_3] \supseteq [Z_4, Z_3] \supseteq \ldots$ is a sequence of nested closed intervals. By the nested interval theorem, their intersection will be non-empty, and will consist of a unique point precisely if the sequence of lengths of the nested intervals tends to zero. We prove that this happens with probability 1.

Let I_n $(n = 0, 1, 2, , \ldots)$ be the nth interval in the sequence, and L_n = length of I_n, i.e., $L_{2k} = Z_{2k+1} - Z_{2k}$ and $L_{2k+1} = Z_{2k+1} - Z_{2k+2}$. Pick $\delta > 0$. We will prove by induction that the probability of $L_n > \delta$ is $P(L_n > \delta) \leq (1-\delta)^n$. Since $P(L_n > 1) = 0$ the result is trivially true for $\delta \geq 1$, so we may assume $1 > \delta > 0$.

Base case: For $n = 0$ the inequality $P(L_0 > \delta) \leq (1-\delta)^0$ obviously holds because $L_0 = 1$, hence $P(L_0 > \delta) = P(1 > \delta) = 1$ and $(1-\delta)^0 = 1$.

Induction step: Assume $P(L_n > \delta) \leq (1-\delta)^n$. Then

$$P(L_{n+1} > \delta) = P(L_n \leq \delta) \cdot P(L_{n+1} > \delta \mid L_n \leq \delta) + P(L_n > \delta) \cdot P(L_{n+1} > \delta \mid L_n > \delta).$$

Note that the first term is zero because if $L_n \leq \delta$ then $L_{n+1} > \delta$ is impossible. On the other hand, if $L_n > \delta$ then we only have $L_{n+1} > \delta$ if the next endpoint Z_{n+2} is selected at a distance less than $L_n - \delta$ from the right or left (depending on the parity of n) endpoint of I_n. The probability is

$$P(L_{n+1} > \delta \mid L_n > \delta) = \frac{L_n - \delta}{L_n} = 1 - \frac{\delta}{L_n} \leq 1 - \delta.$$

Hence

$$P(L_{n+1} > \delta) \leq (1-\delta)^n (1-\delta) = (1-\delta)^{n+1},$$

and this completes the induction.

From here we get $\lim_{n\to\infty} P(L_{n+1} > \delta) = 0$ for every $\delta > 0$, hence $L_n \to 0$ as $n \to \infty$ with probability 1.

Proof of 2. For each $n \geq 0$ define the new random variable U_n, chosen between Z_{2n} and Z_{2n+1} with probability density

$$f_{U_n|Z_{2n}=z_n, Z_{2n+1}=z_{2n+1}}(x) = \frac{2(x - z_{2n})}{(z_{2n+1} - z_{2n})^2}$$

on $[z_{2n}, z_{2n+1}]$, where "$U_n|Z_{2n} = z_{2n}, Z_{2n+1} = z_{2n+1}$" means the random variable U_n given $Z_{2n} = z_{2n}$ and $Z_{2n+1} = z_{2n+1}$ (we ignore the case $z_{2n+1} = z_{2n}$ because its probability is zero).

Since U_n is between Z_{2n} and Z_{2n+1}, its limit U^* will coincide with Z^*.

Next, we will prove by induction that for every $n \geq 0$, the probability density of U_n is always the same, namely $f_{U_n}(x) = 2x$ on $[0, 1]$.

Base case: For $n = 0$ we have $Z_0 = 0$, $Z_1 = 1$, hence $f_{U_0}(x) = \dfrac{2(x-0)}{(1-0)^2} = 2x$ on $[0, 1]$.

Induction step: Assume $f_{U_n}(x) = 2x$. Next, note that U_{n+1} is defined like U_n but with starting points Z_2 and Z_3 in place of Z_0 and Z_1. So, U_{n+1} given $Z_2 = z_2$ and $Z_3 = z_3$ is just U_n mapped from $[0, 1]$ to $[z_2, z_3]$ with the transformation $(z_3 - z_2)U_n + z_2$. By induction hypothesis we have $f_{U_n}(x) = 2x$, and its transformation to $[z_2, z_3]$ will have probability density

$$f_{U_{n+1}|Z_2=z_2, Z_3=z_3}(x) = \frac{2(x - z_2)}{(z_3 - z_2)^2}$$

on $[z_2, z_3]$.

The cumulative distribution function of U_{n+1} is $F_{U_{n+1}}(x) = P(U_{n+1} \leq x)$. By definition U_{n+1} must be in the interval $[Z_2, Z_3]$, while x may be in any of two different intervals, namely $[U_{n+1}, Z_3)$ or $[Z_3, 1]$. So, the event $U_{n+1} \leq x$ can be expressed as the union of $Z_2 \leq Z_3 \leq x$ and $Z_2 \leq U_{n+1} \leq x < Z_3$. Since they are disjoint we have

$$P(U_{n+1} \leq x) = P(Z_2 \leq Z_3 \leq x) + P(Z_2 \leq U_{n+1} \leq x < Z_3)\,.$$

We have that X_2 is random uniform on $[0, 1]$, and X_3 is random uniform on $[Z_2, 1]$, so

$$f_{Z_3|Z_2=z_2}(x) = \frac{1}{1 - z_2},$$

hence

$$P(Z_2 \leq Z_3 \leq x) = \int_0^x \frac{x - z_2}{1 - z_2}\, dz_2 = x + (1 - x)\log(1 - x)\,.$$

The second term can be computed as follows:

$$\begin{aligned}
P(Z_2 \leq U_{n+1} \leq x < Z_3) &= \int_0^x \int_x^1 \int_{z_2}^x f_{U_{n+1}|Z_2=z_2, Z_3=z_3}(t) f_{Z_3|Z_2=z_2}(x)\, dt\, dz_3\, dz_2 \\
&= \int_0^x \int_x^1 \int_{z_2}^x \frac{2(t - z_2)}{(z_3 - z_2)^2} \frac{1}{1 - z_2}\, dt\, dz_3\, dz_2 \\
&= (x - 1)(x + \log(1 - x))\,,
\end{aligned}$$

hence

$$F_{U_{2n+1}}(x) = x + (1-x)\log(1-x) + (x-1)(x+\log(1-x)) = x^2 .$$

Differentiating we get $f_{U_{2n+1}}(x) = 2x$ on [0, 1], and this completes the induction.

Since the distribution of U_n is the same for every n we have that the limit U^* will have the same distribution too. And since $U^* = Z^*$, the same will hold for Z^*, hence $f_{Z^*}(x) = 2x$.

Also solved by Robert A. Agnew, Elton Bojaxhiu (Germany) & Enkel Hysnelaj (Australia), Robert Calcaterra, Shuyang Gao, John C. Kieffer, Omran Kouba (Syria), Kenneth Schilling, and the proposer.

An almost linear functional equation **June 2020**

2099. *Proposed by Russ Gordon, Whitman College, Walla Walla, WA and George Stoica, Saint John, NB, Canada.*

Let r and s be distinct nonzero rational numbers. Find all functions $f : \mathbb{R} \to \mathbb{R}$ that satisfy

$$f\left(\frac{x+y}{r}\right) = \frac{f(x)+f(y)}{s}$$

for all real numbers x and y.

Solution by Eugene A. Herman, Grinnell College, Grinnell, IA.

Clearly the zero function is always a solution and, when $s = 2$, all constant functions are solutions. We show that there are no others. First assume $s \neq 2$. Substituting 0 for both x and y yields $f(0) = 0$. Substituting $y = 0$ and $y = -x$ yield these two identities:

$$f\left(\frac{x}{r}\right) = \frac{f(x)}{s}, \quad f(-x) = -f(x) \quad \text{for all } x \in \mathbb{R}.$$

Given any $x \in \mathbb{R}$, we use induction to show that $f(nx) = nf(x)$ for all $n \in \mathbb{N}$. The base case is a tautology. If $f(nx) = nf(x)$ for some $n \in \mathbb{N}$, then

$$\frac{f((n+1)x)}{s} = f\left(\frac{(n+1)x}{r}\right) = f\left(\frac{nx+x}{r}\right) = \frac{f(nx)+f(x)}{s} = \frac{(n+1)f(x)}{s}$$

and so $f((n+1)x) = (n+1)f(x)$. It follows that $f(x/n) = f(x)/n$ for all $n \in \mathbb{N}$ and hence that $f((m/n)x) = (m/n)f(x)$ for all $m, n \in \mathbb{N}$. Since $f(-x) = -f(x)$, this last statement is also true for m negative. Choose m, n so that $r = n/m$. Therefore

$$\frac{f(x)}{s} = f\left(\frac{x}{r}\right) = \frac{f(x)}{r}$$

and so $f(x) = 0$.

Now assume $s = 2$, and let $t = 2/r$. Thus $t \neq 1$ and

$$f\left(\frac{t}{2}(x+y)\right) = \frac{f(x)+f(y)}{2}, \quad \text{for all } x, y \in \mathbb{R}.$$

Substituting $y = x$ and $y = -x$ yield

$$f(tx) = f(x), \quad \frac{f(x)+f(-x)}{2} = f(0) \quad \text{for all } x \in \mathbb{R}.$$

Thus $f(-x/t) = f(-x)$, and so

$$f\left(\frac{t-1}{2}x\right) = f\left(\frac{t}{2}(x - x/t)\right) = \frac{f(x) + f(-x/t)}{2} = \frac{f(x) + f(-x)}{2} = f(0).$$

Therefore f is a constant function.

Also solved by Michel Bataille (France), Elton Bojaxhiu (Germany) & Enkel Hysnelaj (Australia), Paul Budney, Robert Calcaterra, Walther Janous (Austria), Sushanth Sathish Kumar, Omran Kouba (Syria), Elias Lampakis (Greece), Albert Natian, Kangrae Park (South Korea), Kenneth Schilling, Jacob Siehler, Albert Stadler (Switzerland), Michael Vowe (Switzerland), and the proposers.

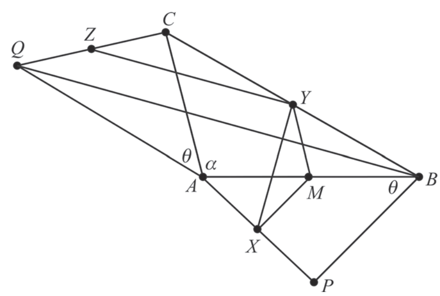
Two congruent triangles on the sides of an arbitrary triangle June 2020

2100. *Proposed by Yevgenya Movshovich and John E. Wetzel, University of Illinois, Urbana, IL.*

Given $\triangle ABC$ and an angle θ, two congruent triangles $\triangle ABP$ and $\triangle QAC$ are constructed as follows: $AQ = AB$, $BP = AC$, $m\angle ABP = m\angle CAQ = \theta$, B and Q are on opposite sides of $\overleftrightarrow{AC}$, and C and P are on opposite sides of $\overleftrightarrow{AB}$, as shown in the figure. Let X, Y, and Z be the midpoints of segments AP, BC, and CQ, respectively.

Show that $\angle XYZ$ is a right angle.

Solution by Sushanth Sathish Kumar (student), Portola High School, Irvine, CA.



Let M be the midpoint of segment AB. Note that $\overline{YZ}$ is a midline of triangle CBQ, and so $\overleftrightarrow{BQ}$ is parallel to $\overleftrightarrow{YZ}$. Thus, it suffices to show that $\overleftrightarrow{XY}$ is perpendicular to $\overleftrightarrow{BQ}$.

Since $\overline{MX}$ and $\overline{MY}$ are midlines of triangles APB and ABC, we have that $MX = BP/2 = AC/2 = MY$. Hence, triangle MXY is isosceles. Moreover, since $\overleftrightarrow{MX} || \overleftrightarrow{BP}$ and $\overleftrightarrow{MY} || \overleftrightarrow{AC}$, we have

$$m\angle XMY = m\angle XMA + m\angle AMY = \theta + 180^\circ - \alpha,$$

where we set $\alpha = m\angle BAC$. It follows that $m\angle MXY = m\angle XYM = (\alpha - \theta)/2$.

We wish to calculate $m\angle(\overleftrightarrow{XM}, \overleftrightarrow{BQ})$, where $m\angle(\ell_1, \ell_2)$ denotes the measure of the non-obtuse angle between ℓ_1 and ℓ_2. Note that

$$m\angle(\overleftrightarrow{XM}, \overleftrightarrow{BQ}) = m\angle PBQ = m\angle PBA + m\angle ABQ.$$

Since $AB = AQ$ and $m\angle BAQ = \alpha + \theta$, we find that $m\angle ABQ = 90^\circ - (\alpha + \theta)/2$. Thus, $m\angle(\overleftrightarrow{XM}, \overleftrightarrow{BQ}) = 90^\circ - (\alpha - \theta)/2$. But since $m\angle(\overleftrightarrow{MX}, \overleftrightarrow{XY}) = (\alpha - \theta)/2$, we find that $m\angle(\overleftrightarrow{BQ}, \overleftrightarrow{XY}) = 90^\circ$, and we are done.

Also solved by Michel Bataille (France), Elton Bojaxhiu (Germany) & Enkel Hysnelaj (Australia), Robert Calcaterra, Prithwijit De (India), J. Chris Fisher, Dmitry Fleischman, Fresno State Problem Solving Group, Marty Getz & Dixon Jones, Eugene Herman, Walther Janous (Austria), Elias Lampakis (Greece), Kee-Wai Lau (Hong Kong), Graham Lord, Elizabeth Mika, Albert Natian, Celia Schacht, Volkhard Schindler (Germany), Albert Stadler (Switzerland), Michael Vowe (Switzerland) and the proposers.

Answers

Solutions to the Quickies from page 229.

A1111. The implication (ii)⇒(i) follows immediately from

$$\det(A+\lambda B)=\lambda^2\det B+\lambda(\operatorname{tr}A\operatorname{tr}B-\operatorname{tr}(AB))+\det A,\ \lambda\in\mathbb{C}.$$

Conversely, suppose that (i) holds. Let

$$a_\lambda=\lambda^2\det B+\det A \ \text{ and } \ b=\operatorname{tr}A\operatorname{tr}B-\operatorname{tr}(AB).$$

Then (i) can be rewritten as $|a_\lambda+\lambda b|^2=|a_\lambda-\lambda b|^2,\ \lambda\in\mathbb{C}$, which is equivalent to

$$\operatorname{Re}(\lambda b\overline{a_\lambda})=0,\ \lambda\in\mathbb{C}. \tag{1}$$

If we put $\lambda=n$, $n\in\mathbb{N}$, in (1), we get

$$\operatorname{Re}(nb(n^2\,\overline{\det B}+\overline{\det A}))=0,\ n\in\mathbb{N},$$

that is,

$$\operatorname{Re}(b\,\overline{\det A})=-n^2\operatorname{Re}(b\,\overline{\det B}),\ n\in\mathbb{N},$$

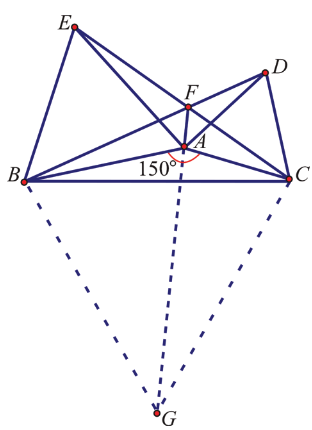
from which it follows that $\operatorname{Re}(b\,\overline{\det B})=0$, and hence $\operatorname{Re}(b\,\overline{\det A})=0$.

If we put $\lambda=in$ $n\in\mathbb{N}$, in (1), we get $\operatorname{Re}(inb(-n^2\,\overline{\det B}+\overline{\det A}))=0,\ n\in\mathbb{N}$, that is,

$$\operatorname{Im}(b\,\overline{\det A})=n^2\operatorname{Im}(b\,\overline{\det B}),\ n\in\mathbb{N},$$

from which it follows that $\operatorname{Im}(b\,\overline{\det B})=0$, and hence $\operatorname{Im}(b\,\overline{\det A})=0$.

Consequently, $b\,\overline{\det A}=b\,\overline{\det B}=0$, and therefore $b=0$ or $\det A=\det B=0$, so condition (ii) holds.



A1112. Construct an equilateral triangle BCG whose interior lies in the exterior of $\triangle ABC$. It is well known that $\overleftrightarrow{AG}$, $\overleftrightarrow{BD}$, and $\overleftrightarrow{CE}$ are concurrent at F and the measure of the angle between any two adjacent lines is $60°$.

Since $m\angle AFE = 120°$, $m\angle ABE = 60°$, $m\angle AFD = 120°$, $m\angle ACD = 60°$, $m\angle BFC = 120°$, and $m\angle BGC = 60°$, we see that $ABEF$, $ACDF$, and $BFCG$ are cyclic quadrilaterals. By Ptolemy's theorem,

$$FE + FA = FB, FD + FA = FC, \quad \text{and} \quad FB + FC = FG = FA + AG.$$

Therefore

$$2FA + FE + FD = FB + FC = FA + AG.$$

Since $2m\angle BAC + m\angle BGC = 360°$, G is the circumcenter of $\triangle ABC$. Hence $AG = GB = BC$. Thus,

$$2FA + FE + FD = FA + BC$$

and we have $FA + FE + FD = BC$ as desired.

Erratum

In the December 2020 issue of this MAGAZINE, we ran a note entitled "When Two Wrongs Make a Right," by Leonard Van Wyk. It has since become clear that the main content of this note had appeared previously in the following publications:

- Brannen, N. S., Ford, B. (2004). Logarithmic Differentiation: Two Wrongs Make a Right. *College Math. J.* 35(5): 388–390.
- Bilodeau, G. E. (1993). An Exponential Rule. *College Math. J.* 24(4): 350–351.

Both Dr. Van Wyk and the Editor offer their sincere apologies to these authors for having been unaware of their work. We thank Gerry Bilodeau for bringing these predecessors to our attention.

Math. Mag. **94** (2021) 238. doi:10.1080/0025570X.2021.1911210