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PROBLEMS AND SOLUTIONS

Edited by Daniel H. Ullman, Daniel J. Velleman & Douglas B. West

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PROBLEMS AND SOLUTIONS

Edited by **Daniel H. Ullman, Daniel J. Velleman**, and **Douglas B. West**

with the collaboration of Paul Bracken, Ezra A. Brown, Zachary Franco, George Gilbert, László Lipták, Rick Luttmann, Hosam Mahmoud, Frank B. Miles, Lenhard Ng, Rajesh Pereira, Kenneth Stolarsky, Richard Stong, Stan Wagon, Lawrence Washington, and Li Zhou.

Proposed problems should be submitted online at
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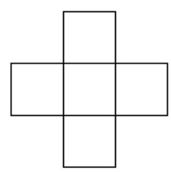
Proposed solutions to the problems below should be submitted by December 31, 2021, via the same link. More detailed instructions are available online. Proposed problems must not be under consideration concurrently at any other journal nor be posted to the internet before the deadline date for solutions. An asterisk (*) after the number of a problem or a part of a problem indicates that no solution is currently available.

PROBLEMS

12265. *Proposed by Ross Dempsey, student, Princeton University, Princeton, NJ.* For a fixed positive integer k, let $a_0 = a_1 = 1$ and $a_n = a_{n-1} + (k-n)^2 a_{n-2}$ for $n \geq 2$. Show that $a_k = (k-1)!$.

12266. *Proposed by Haoran Chen, Xi'an Jiaotong–Liverpool University, Suzhou, China.* A union of a finite number of squares from a grid is called a *polyomino* if its interior is simply connected. Given a polyomino P and a subpolyomino Q, we write $c(P, Q)$ for the number of components that remain when Q is removed from P. Let $f(k) = \max_P \min_Q c(P, Q)$, where the maximum is taken over all polyominoes and the minimum is taken over all subpolyominoes Q of P of size k. For example, $f(2) \geq 3$, because any domino removed from the pentomino at right breaks the pentomino into 3 pieces. Is f bounded?



12267. *Proposed by Michel Bataille, Rouen, France.* Let x, y, and z be nonnegative real numbers such that $x + y + z = 1$. Prove

$$(1-x)\sqrt{x(1-y)(1-z)} + (1-y)\sqrt{y(1-z)(1-x)} + (1-z)\sqrt{z(1-x)(1-y)} \geq 4\sqrt{xyz}.$$

12268. *Proposed by Samina Boxwala Kale, Nowrosjee Wadia College, Pune, India, Vašek Chvátal, Concordia University, Montreal, Canada, Donald E. Knuth, Stanford University, Stanford, CA, and Douglas B. West, University of Illinois, Urbana, IL.*

(a) Show that there is an easy way to decide whether the edges of a graph can each be colored red or green so that at each vertex the number of incident edges with one color differs from the number having the other color by at most 1.

(b) Show that it is NP-hard to decide whether the vertices of a graph can each be colored red or green so that at each vertex the number of neighboring vertices with one color differs from the number having the other color by at most 1.

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12269. *Proposed by Mehmet Şahin and Ali Can Güllü, Ankara, Turkey.* Let ABC be an acute triangle. Suppose that D, E, and F are points on sides BC, CA, and AB, respectively, such that FD is perpendicular to BC, DE is perpendicular to CA, and EF is perpendicular to AB. Prove

$$\frac{AF}{AB}+\frac{BD}{BC}+\frac{CE}{CA}=1.$$

12270. *Proposed by Moubinool Omarjee, Lycée Henri IV, Paris, France.* Let $a_0=1$, and let $a_{n+1}=a_n+e^{-a_n}$ for $n\ge 0$. Show that the sequence whose nth term is $e^{a_n}-n-(1/2)\ln n$ converges.

12271. *Proposed by Steven Deckelman, University of Wisconsin–Stout, Menomonie, WI.* Let n be a positive integer. Evaluate

$$\int_0^{2\pi}\left|\sin\left((n-1)\theta-\frac{\pi}{2n}\right)\cos(n\theta)\right|d\theta.$$

SOLUTIONS

The Asymptotic Behavior of a Sum

12153 [2020, 85]. *Proposed by Omran Kouba, Higher Institute for Applied Sciences and Technology, Damascus, Syria.* For a real number x whose fractional part is not $1/2$, let $\langle x\rangle$ denote the nearest integer to x. For a positive integer n, let

$$a_n=\left(\sum_{k=1}^{n}\frac{1}{\langle\sqrt{k}\rangle}\right)-2\sqrt{n}.$$

(a) Prove that the sequence $a_1,a_2,\ldots$ is convergent, and find its limit L.
(b) Prove that the set $\{\sqrt{n}(a_n-L)\colon n\ge 1\}$ is a dense subset of $[0,1/4]$.

Composite solution by Jean-Pierre Grivaux, Paris, France, and Eugene A. Herman, Grinnell College, Grinnell, IA.

(a) The limit is -1. Let $I_j=\{n\in\mathbb{N}:\ \langle\sqrt{n}\rangle=j\}$. That is,

$$I_j=\{n:\ j-1/2<\sqrt{n}<j+1/2\}=[(j-1)j+1,\,j(j+1)].$$

Note that $|I_j|=2j$ and that these intervals partition $\mathbb{N}$.

For $k\in I_j$, the summand in the expression for a_n is $1/j$. Hence I_j contributes 2 to the sum if all of its terms are included. The last interval contributes less when n does not have the form $(j+1)j$. It is the interval I_j such that $\langle\sqrt{n}\rangle=j$. This interval contributes $n-j(j-1)$ terms equal to $1/j$, so it contributes $n/j-j+1$ to the sum. Thus

$$a_n=2(j-1)+\frac{n}{j}-j+1-2\sqrt{n}=-1+j+\frac{n}{j}-2\sqrt{n}.$$

Since $j=\langle\sqrt{n}\rangle$, we have

$$a_n+1=\frac{\left(\sqrt{n}-\langle\sqrt{n}\rangle\right)^2}{\langle\sqrt{n}\rangle}<\frac{(1/2)^2}{\sqrt{n}-1/2}\ \to\ 0$$

as $n\to\infty$. Also $a_n+1\ge 0$, so the sequence converges and $L=-1$.

(b) We prove the density claim in a somewhat stronger form. We prove that in fact the values of $\sqrt{n}(a_n - L)$ for $n \in I_j$ become dense in $[0, 1/4]$ as $j \to \infty$.

Let $b_n = \sqrt{n}(a_n - L) = \sqrt{n}(a_n + 1)$. By the computation of $a_n + 1$ above,

$$b_n = \left(\sqrt{n} - \langle\sqrt{n}\rangle\right)^2 \frac{\sqrt{n}}{\langle\sqrt{n}\rangle}.$$

Thus $b_n \geq 0$, with equality if and only if n is a perfect square (I_j has one perfect square). We first show $b_n < 1/4$.

Consider b_n for $n \in I_j$, where $\langle\sqrt{n}\rangle$ has the constant value j. Let f be the cubic polynomial defined by $f(x) = (x - j)^2 x/j$; note that $f(\sqrt{n}) = b_n$ for $n \in I_j$. We have $f''(x) > 0$ for $x > 2j/3$, which holds when $x = \sqrt{n}$ for $n \in I_j$ since $\sqrt{j(j-1)+1} > j - 1/2$. Hence in order to prove $b_n < 1/4$ on I_j, it suffices to prove this inequality at both endpoints.

We show first that the value is larger at the upper endpoint, that is,

$$\left(j - \sqrt{j^2 - j + 1}\right)^2 \frac{\sqrt{j^2 - j + 1}}{j} \;<\; \left(\sqrt{j^2 + j} - j\right)^2 \frac{\sqrt{j^2 + j}}{j}.$$

Since $j^2 - j + 1 < j^2 + j$, it suffices to prove $j - \sqrt{j^2 - j + 1} < \sqrt{j^2 + j} - j$, which we rewrite as $2j < \sqrt{j(j-1)+1} + \sqrt{j(j+1)}$. Squaring both sides reduces the desired inequality to $4j^2 < 2j^2 + 1 + 2\sqrt{j^4 + j}$, which is true since $j^2 < \sqrt{j^4 + j}$.

To bound the value by $1/4$ at the upper endpoint, where $n = j(j+1)$, the desired inequality $b_n < 1/4$ is

$$\left(j(j+1) - 2j\sqrt{j(j+1)} + j^2\right)\sqrt{j(j+1)}/j \;<\; 1/4,$$

which simplifies to $(2j+1)\sqrt{j(j+1)} \;<\; 2j(j+1) + 1/4$. After dividing by $2j+1$, it suffices to show

$$\sqrt{j(j+1)} \;<\; j + \frac{1}{2} - \frac{1/8}{j + 1/2}.$$

After squaring both sides, this reduces to the true inequality

$$j^2 + j < j^2 + j + \frac{1}{4} - \frac{1}{4} + \frac{1/64}{(j + 1/2)^2}.$$

To establish the density result, consider I_j for large j. An element $n \in I_j$ has the form $j(j-1) + r$ with $1 \leq r \leq 2j$. When $n = j^2$ (that is, $r = j$), we have $b_n = 0$. For $n = j(j+1)$ (that is, $r = 2j$), we expand b_n as a series in $1/j$:

$$\begin{aligned}
b_n &= \left(\sqrt{j(j+1)} - j\right)^2 \sqrt{j(j+1)}/j = j^2(\sqrt{1 + 1/j} - 1)^2\sqrt{1 + 1/j} \\
&= j^2\left(1 + \tfrac{1}{2}j^{-1} - O(j^{-2}) - 1\right)^2\left(1 + \tfrac{1}{2}j^{-1} - O(j^{-2})\right) \\
&= j^2\left(\tfrac{1}{4}j^{-2} + O(j^{-3})\right)\left(1 + O(j^{-1})\right) = \tfrac{1}{4} + O(j^{-1}) \to \tfrac{1}{4}.
\end{aligned}$$

Thus $b_{j(j+1)} \to 1/4$ as $j \to \infty$.

Next consider the difference between b_{n+1} and b_n. We have $b_n = (h(n))^2\sqrt{n}/\langle\sqrt{n}\rangle$, where $h(n) = \sqrt{n} - \langle\sqrt{n}\rangle$. When $\{n, n+1\} \in I_j$ we compute

$$\begin{aligned}
h(n+1) - h(n) &= \sqrt{j(j-1) + r + 1} - j - (\sqrt{j(j-1) + r} - j) \\
&= \sqrt{j(j-1) + r + 1} - \sqrt{j(j-1) + r} \\
&= \frac{1}{\sqrt{j(j-1) + r + 1} + \sqrt{j(j-1) + r}} < \frac{1}{2\sqrt{j(j-1)}},
\end{aligned}$$

which approaches 0 as $j \to \infty$. Hence the gaps between values of $(h(n))^2$ for $n \in I_j$ become arbitrarily small as $j \to \infty$. Also the deviation of $\sqrt{n}/\langle\sqrt{n}\rangle$ from 1 tends to 0. Hence as $j \to \infty$ the differences between successive values of b_n for $n \in [j^2, j^2 + j]$ become arbitrarily small, and so $\{b_n : n \in \mathbb{N}\}$ is dense in $[0, 1/4]$.

Editorial comment. We have seen that the values of $h(n)$ increase through I_j, passing through 0 at $n = j^2$. In fact, the negatives of the values of $h(n)$ for $n < j^2$ interlace with the values for $n > j^2$. That is, $h(j^2 + s) \le -h(j^2 - s) \le h(j^2 + s + 1)$ for $0 \le s \le j - 1$.

Also solved by K. F. Andersen (Canada), A. Berkane (Algeria), R. Chapman (UK), H. Chau, H. Chen, C. Chiser (Romania), A. Deeb & H. Al-Assad (Syria), A. Dixit, G. Fera & G. Tescaro (Italy), O. Geupel (Germany), N. Hodges (UK), W. Janous (Austria), P. Lalonde (Canada), J. H. Lindsey II, O. P. Lossers (Netherlands), M. Omarjee (France), M. A. Prasad (India), C. Schacht, E. Schmeichel, A. Stadler (Switzerland), A. Stenger, R. Stong, R. Tauraso (Italy), T. Wiandt, T. Wilde (UK), Florida State University Problem Solving Group, and the proposer.

Schur's Inequality and Five Triangle Radii

12154 [2020, 85]. *Proposed by Martin Lukarevski, University "Goce Delcev," Stip, North Macedonia.* Let r_a, r_b, and r_c be the exradii of a triangle with circumradius R and inradius r. Prove

$$\frac{r_a}{r_b + r_c} + \frac{r_b}{r_c + r_a} + \frac{r_c}{r_a + r_b} \ge 2 - \frac{r}{R}.$$

Solution by Tamas Wiandt, Rochester Institute of Technology, NY. Let a, b, and c be the lengths of the sides opposite vertices A, B, and C, let s be the semiperimeter, and let t be the area of the triangle. It is well known that $r_a = t/(s - a)$, $r_b = t/(s - b)$, $r_c = t/(s - c)$, $r = t/s$, $R = abc/(4t)$, and $t^2 = s(s - a)(s - b)(s - c)$.

Now let $x = s - a$, $y = s - b$, and $z = s - c$. The triangle inequality implies $x, y, z > 0$, and we obtain $a = y + z$, $b = x + z$, $c = x + y$, and $s = x + y + z$. With these substitutions, the desired inequality becomes

$$\frac{yz}{x(y+z)} + \frac{zx}{y(z+x)} + \frac{xy}{z(x+y)} \ge 2 - \frac{4s(s-a)(s-b)(s-c)}{abcs} = 2 - \frac{4xyz}{(y+z)(x+z)(x+y)}.$$

After multiplication by the denominators, this turns into

$$x^3y^3 + y^3z^3 + x^3z^3 + 3x^2y^2z^2 \ge x^3y^2z + x^2y^3z + x^3yz^2 + x^2yz^3 + xy^3z^2 + xy^2z^3,$$

which is just Schur's inequality applied to the numbers xy, yz, and xz. Because x, y, and z are positive, the inequality turns into an identity only when $xy = yz = xz$, that is, when $x = y = z$ and $a = b = c$, which means the triangle is equilateral.

Also solved by A. Alt, F. R. Ataev (Uzbekistan), M. Bataille (France), E. Bojaxhiu (Albania) & E. Hysnelaj (Australia), R. Chapman (UK), C. Curtis, P. De (India), M. Dincă (Romania), G. Fera (Italy), S. Gayen (India), O. Geupel (Germany), N. Hodges (UK), E.-Y. Jang (Korea), W. Janous (Austria), M. Kaplan & M. Goldenberg, B. Karaivanov (USA) & T. S. Vassilev (Canada), P. Khalili, K. T. L. Koo (China), O. Kouba (Syria), S. S. Kumar, K.-W. Lau (China), J. H. Lindsey II, O. P. Lossers (Netherlands), D. J. Moore, C. R. Pransesachar (India), V. Schindler (Germany), A. Stadler (Switzerland), R. Stong, R. Tauraso (Italy), M. Tetiva (Romania), D. Văcaru (Romania), M. Vowe (Switzerland), M. R. Yegan (Iran), T. Zvonaru (Romania), Davis Problem Solving Group, and the proposer.

The Area Under a Tractrix

12155 [2020, 85]. *Proposed by Albert Stadler, Herrliberg, Switzerland.* Let $f : [0, \infty) \to [0, 1]$ be the function that satisfies $f(0) = 1$, is differentiable on $(0, \infty)$, and has the following property: If A is a point on the graph of f and B is the x-intercept of the line tangent to the graph of f at A, then $AB = 1$.
(a) Prove $\int_0^\infty f(x)\,dx = \pi/4$.
(b) For $n \in \mathbb{N}$, prove that $\int_0^\infty x^{2n} f(x)\,dx$ is a rational polynomial of π.

Solution to part **(a)** *by Kenneth F. Andersen, Alberta, Canada.* The hypothesis applied when $A = (0, 1)$ indicates that f is continuous from the right at 0. We first show that f is a strictly decreasing map from $[0, \infty)$ onto $(0, 1]$. The hypothesis requires that for each $x > 0$, the tangent line at $(x, f(x))$ has a unique x-intercept. Thus, we must have $f'(x) \neq 0$ for all $x > 0$. The Darboux (intermediate value) property of f' then shows that f' is of fixed sign on $(0, \infty)$. Since $f(0) = 1$ and $f(1) \le 1$, the mean value theorem shows that there is some $x \in (0, 1)$ satisfying

$$0 \ge f(1) - f(0) = f'(x),$$

which shows that the fixed sign of f' must be negative. Hence, the hypothesis $AB = 1$ yields

$$f'(x) = -\frac{f(x)}{\sqrt{1 - f(x)^2}} \tag{1}$$

when $x > 0$. Since f' is negative, f strictly decreases to $m = \lim_{x\to\infty} f(x)$. If we suppose, to draw a contradiction, that $m > 0$, then (1) shows that $f'(x) \le -m/\sqrt{1 - m^2} < 0$ for all $x > 0$. It follows that for $t > 0$,

$$0 \le f(t) = f(0) + \int_0^t f'(x)\,dx \le 1 - \frac{m}{\sqrt{1 - m^2}}t,$$

which is impossible, since the right-hand side tends to $-\infty$ as $t \to \infty$. Thus $m = 0$, so f is a bijection from $[0, \infty)$ to $(0, 1]$.

The inverse function $g : (0, 1] \to [0, \infty)$ is continuous and satisfies $g(1) = 0$, $\lim_{y\to 0^+} g(y) = \infty$, and

$$g'(y) = -\frac{\sqrt{1 - y^2}}{y} \tag{2}$$

when $0 < y < 1$. Using the substitution $x = g(y)$, we obtain

$$\int_0^\infty f(x)\,dx = -\int_0^1 f(g(y))g'(y)\,dy = \int_0^1 \sqrt{1 - y^2}\,dy = \frac{\pi}{4},$$

as claimed.

Composite solution to part **(b)** *by O. P. Lossers, Eindhoven University of Technology, and the Davis Problem Solving Group, Davis, CA.* Integrating (2) yields

$$g(y) = -\sqrt{1 - y^2} + \log\frac{1 + \sqrt{1 - y^2}}{y} = -\sqrt{1 - y^2} + \operatorname{sech}^{-1} y.$$

Thus, again using the substitution $x = g(y)$, we obtain

$$\int_0^\infty x^{2n} f(x)\,dx = \int_0^1 \left(-\sqrt{1 - y^2} + \operatorname{sech}^{-1} y\right)^{2n} \sqrt{1 - y^2}\,dy.$$

Next, we use the substitution $t = \operatorname{sech}^{-1} y$. We have $y = \operatorname{sech} t$, $1 - y^2 = \tanh^2 t$, and $dy = -\operatorname{sech} t \tanh t \, dt$, so

$$\int_0^\infty x^{2n} f(x)\, dx = \int_0^\infty (-\tanh t + t)^{2n} \operatorname{sech} t \tanh^2 t \, dt.$$

Expanding $(-\tanh t + t)^{2n}$ and using the identity $\tanh^2 t = 1 - \operatorname{sech}^2 t$ again, we see that the last integral is a rational linear combination of integrals of the forms

$$I(i, j) = \int_0^\infty t^{2i} \operatorname{sech}^{2j+1} t \, dt \quad \text{and} \quad J(i, j) = \int_0^\infty t^{2i+1} \operatorname{sech}^{2j+1} t \tanh t \, dt$$

for $i, j \geq 0$.

Integration by parts yields

$$\begin{aligned} J(i, j) &= -\int_0^\infty t^{2i+1} \operatorname{sech}^{2j} t \, d(\operatorname{sech} t) \\ &= (2i+1) \int_0^\infty t^{2i} \operatorname{sech}^{2j+1} t \, dt - 2j \int_0^\infty t^{2i+1} \operatorname{sech}^{2j+1} t \tanh t \, dt \\ &= (2i+1) I(i, j) - 2j J(i, j), \end{aligned}$$

so

$$J(i, j) = \frac{2i+1}{2j+1} I(i, j). \tag{3}$$

Similarly, for $i, j \geq 1$,

$$\begin{aligned} I(i, j) &= \int_0^\infty t^{2i} \operatorname{sech}^{2j-1} t \, d(\tanh t) \\ &= -2i \int_0^\infty t^{2i-1} \operatorname{sech}^{2j-1} t \tanh t \, dt + (2j-1) \int_0^\infty t^{2i} \operatorname{sech}^{2j-1} t \tanh^2 t \, dt \\ &= -2i \int_0^\infty t^{2i-1} \operatorname{sech}^{2j-1} t \tanh t \, dt + (2j-1) \int_0^\infty t^{2i} \operatorname{sech}^{2j-1} t (1 - \operatorname{sech}^2 t) \, dt \\ &= -2i J(i-1, j-1) + (2j-1) I(i, j-1) - (2j-1) I(i, j), \end{aligned}$$

so

$$I(i, j) = -\frac{i}{j} J(i-1, j-1) + \frac{2j-1}{2j} I(i, j-1). \tag{4}$$

Combining (3) and (4), we conclude that $\int_0^\infty x^{2n} f(x)\, dx$ is equal to a rational linear combination of integrals of the forms $I(i, 0)$ and $I(0, j)$.

According to equation 3.523.4 in Gradshteyn, I. S., Ryzhik, I. S. (2014), *Table of Integrals, Series, and Products*, 8th ed. Waltham, MA: Academic Press,

$$I(i, 0) = \int_0^\infty t^{2i} \operatorname{sech} t \, dt = \left(\frac{\pi}{2}\right)^{2i+1} |E_{2i}|,$$

where the E_{2i} are the Euler numbers, which are integers. Also, using the substitutions $u = \sinh t$ and $u = \tan\theta$ and then recognizing a well-known Wallis integral, we find that

$$\begin{aligned} I(0, j) &= \int_0^\infty \operatorname{sech}^{2j+1} t \, dt = \int_0^\infty \frac{\cosh t \, dt}{\cosh^{2j+2} t} = \int_0^\infty \frac{\cosh t \, dt}{(1 + \sinh^2 t)^{j+1}} \\ &= \int_0^\infty \frac{du}{(1+u^2)^{j+1}} = \int_0^{\pi/2} \frac{\sec^2\theta \, d\theta}{\sec^{2j+2}\theta} = \int_0^{\pi/2} \cos^{2j}\theta \, d\theta = \binom{2j}{j} \frac{\pi}{2^{2j+1}}. \end{aligned}$$

The desired result follows.

Editorial comment. The graph of f is known as a *tractrix*. It is the path of an object that begins at the point (0, 1) and is dragged by a rope of length 1 attached to a point moving along the positive x-axis.

The solution to part **(b)** gives an alternative solution to part **(a)**:

$$\int_0^\infty f(x)\,dx = \int_0^\infty \operatorname{sech} t \tanh^2 t\,dt = \int_0^\infty \operatorname{sech} t(1-\operatorname{sech}^2 t)\,dt$$
$$= I(0,0) - I(0,1) = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}.$$

Yet another approach to part **(a)** was suggested by the Davis Problem Solving Group: The value $\pi/4$ for the area can be seen directly geometrically with the differential geometric observation that as A moves from (0, 1) to ∞, the tangent segments AB, when translated so that A is at the origin, rotate 90°, sweeping out a quarter of a unit circle. This is an application of Mamikon's sweeping tangent theorem; see Apostol, T. M., Mnatsakanian, M. A. (2012), *New Horizons in Geometry*, Dolciani Mathematical Expositions No. 47, Washington, DC: Mathematical Association of America. For a dynamic illustration of this proof, see demonstrations.wolfram.com/AreaUnderTheTractrix.

Also solved by R. Chapman (UK), G. Fera & G. Tescaro (Italy), N. Hodges (UK), O. Kouba (Syria), B. Lai & R. Wang (China), R. Stong, T. Wilde (UK), and the proposer. Part **(a)** also solved by A. Dixit (Canada) & S. Pathak (USA), E. A. Herman, J. H. Lindsey II, E. I. Verriest, M. Vowe (Switzerland), Missouri State University Problem Solving Group.

A Symmetric Identity

12156 [2020, 85]. *Proposed by Hideyuki Ohtsuka, Saitama, Japan.* For positive integers m and n and nonnegative integers r and s, prove

$$\sum_{0\le j_1\le\cdots\le j_m\le r} \frac{\binom{n+s}{n}\binom{n+j_1}{n}\binom{s+j_1}{s}}{\prod_{i=1}^m (n+j_i)} = \sum_{0\le j_1\le\cdots\le j_m\le s} \frac{\binom{n+r}{n}\binom{n+j_1}{n}\binom{r+j_1}{r}}{\prod_{i=1}^m (n+j_i)}.$$

Solution by Roberto Tauraso, Università di Roma "Tor Vergata," Rome, Italy. We first prove by induction on m that, for every nonnegative integer k,

$$\sum_{0\le j_1\le\cdots\le j_m\le s} \frac{\binom{n+j_1}{n}\binom{j_1}{k}}{\prod_{i=1}^m (n+j_i)} = \frac{\binom{n+s}{n}\binom{s}{k}}{(n+k)^m}.$$

Let $F_s(m,n,k)$ be the left side of this identity.

When $j_1 > k$,

$$\binom{n+j_1-1}{n}\binom{j_1-1}{k} = \frac{j_1}{n+j_1}\binom{n+j_1}{n}\frac{j_1-k}{j_1}\binom{j_1}{k} = \frac{j_1-k}{n+j_1}\binom{n+j_1}{n}\binom{j_1}{k}.$$

The left and right ends of this display are also equal when $j_1 \le k$, since both are 0. Thus for the base case $m=1$, we obtain the telescoping sum

$$F_s(1,n,k) = \sum_{0\le j_1\le s} \frac{\binom{n+j_1}{n}\binom{j_1}{k}}{n+j_1} = \sum_{0\le j_1\le s} \frac{\binom{n+j_1}{n}\binom{j_1}{k}}{n+k}\cdot\frac{(n+j_1)-(j_1-k)}{n+j_1}$$
$$= \sum_{0\le j_1\le s} \frac{\binom{n+j_1}{n}\binom{j_1}{k} - \binom{n+j_1-1}{n}\binom{j_1-1}{k}}{n+k} = \frac{\binom{n+s}{n}\binom{s}{k}}{n+k}.$$

Now consider $m \geq 2$. For $0 \leq t \leq s$, all terms in the difference

$$F_t(m, n, k) - F_{t-1}(m, n, k)$$

cancel except those where $j_m = t$. Note that $F_{-1}(m, n, k)$ here is an empty sum and is therefore 0. Thus in the computation below, we begin by grouping the terms according to the value of j_m, taken as t. With t then fixed within each term, we can factor out $n + t$ from the denominator of the summand, allowing us to reduce m and apply the induction hypothesis. The computation then completes the proof of the claim:

$$\begin{aligned} F_s(m, n, k) &= \sum_{t=0}^{s} \big(F_t(m, n, k) - F_{t-1}(m, n, k)\big) = \sum_{t=0}^{s} \frac{F_t(m-1, n, k)}{n+t} \\ &= \sum_{t=0}^{s} \frac{1}{n+t} \cdot \frac{\binom{n+t}{n}\binom{t}{k}}{(n+k)^{m-1}} = \frac{F_s(1, n, k)}{(n+k)^{m-1}} = \frac{\binom{n+s}{n}\binom{s}{k}}{(n+k)^m}. \end{aligned}$$

Letting $R(m, n, r, s)$ be the right side of the desired identity, it remains to show $R(m, n, s, r) = R(m, n, r, s)$. By Vandermonde's identity,

$$\binom{r + j_1}{r} = \sum_{k=0}^{r} \binom{r}{r-k}\binom{j_1}{k} = \sum_{k=0}^{r} \binom{r}{k}\binom{j_1}{k}.$$

Using the identity for $F_s(m, n, k)$, we obtain

$$\begin{aligned} R(m, n, r, s) &= \sum_{0 \leq j_1 \leq \cdots \leq j_m \leq s} \frac{\binom{n+r}{n}\binom{n+j_1}{n} \sum_{k=0}^{r} \binom{r}{k}\binom{j_1}{k}}{\prod_{i=1}^{m}(n + j_i)} \\ &= \binom{n+r}{n} \sum_{k=0}^{r} \binom{r}{k} \sum_{0 \leq j_1 \leq \cdots \leq j_m \leq s} \frac{\binom{n+j_1}{n}\binom{j_1}{k}}{\prod_{i=1}^{m}(n + j_i)} \\ &= \binom{n+r}{n} \sum_{k=0}^{r} \binom{r}{k} \frac{\binom{n+s}{n}\binom{s}{k}}{(n+k)^m} = \binom{n+r}{n}\binom{n+s}{n} \sum_{k=0}^{\min(r,s)} \frac{\binom{r}{k}\binom{s}{k}}{(n+k)^m}. \end{aligned}$$

The resulting formula is symmetric in r and s, which completes the proof.

Also solved by R. Stong and the proposer.

Sums and Differences of a Cube and a Prime

12157 [2020, 85]. *Proposed by Nick MacKinnon, Winchester College, Winchester, UK.* Show that there are infinitely many positive integers that are neither the sum of a cube and a prime nor the difference of a cube and a prime (in either order).

Solution I by Joel Schlosberg, Bayside, NY. Such an integer is given by m^3 for any positive integer m congruent to 8 modulo 91. For such m,

$$m^3 - (m-1)^3 = 3m^2 - 3m + 1 \equiv 3 \cdot 8^2 - 3 \cdot 8 + 1 \equiv 0 \pmod{13}$$

and

$$|m^3 - (m+1)^3| = 3m^2 + 3m + 1 \equiv 3 \cdot 8^2 + 3 \cdot 8 + 1 \equiv 0 \pmod{7},$$

which show that $|m^3 - n^3|$ is not prime when $n = m \pm 1$. Also,

$$m^3 \pm n^3 = (m \pm n)(m^2 \mp mn + n^2)$$

for any positive integer n, which shows that $|m^3 - n^3|$ is not prime when $|m - n| > 1$. Therefore, $|m^3 \pm n^3|$ is never prime, and m^3 has the desired property.

Solution II by Li Zhou, Polk State College, Winter Haven, FL. We prove a broad generalization: Let P be the set of primes, and for any $k \geq 3$ let $S_k = \{m^i : m \geq 0 \text{ and } 3 \leq i \leq k\}$. We show that $(P - S_k) \cup (P + S_k) \cup (S_k - P)$ avoids infinitely many positive integers.

Let $(p_1, p_2, p_3, \ldots)$ be the sequence of odd primes. Every element of S_k can be expressed as m^i where $i \in \{4, p_1, p_2, \ldots, p_s\}$, where p_s is the largest prime less than or equal to k. By the Chinese remainder theorem, there exists u such that $u \equiv 2 \pmod 8$ and $u \equiv 0 \pmod{p_1 \cdots p_s}$. For $1 \leq i \leq s$, let

$$f_i(x) = \sum_{j=1}^{p_i} x^{p_i - j}(x-1)^{j-1} \qquad \text{and} \qquad g_i(x) = \sum_{j=1}^{p_i} x^{p_i - j}(x+1)^{j-1}.$$

Let q_i be a prime factor of $f_i(2^{u/p_i})$ and r_i be a prime factor of $g_i(2^{u/p_i})$. Let $v = 8p_1 \cdots p_s$ and $a = 2^u n^v$ with $n \geq 2$ and $n \equiv 1 \pmod{q_1 \cdots q_s r_1 \cdots r_s}$. We show that a is not an element of $(P - S_k) \cup (P + S_k) \cup (S_k - P)$.

First, with $b = (a/4)^{1/4}$, we have

$$a + m^4 = 4b^4 + m^4 = (2b^2 + 2bm + m^2)(b^2 + (b - m)^2);$$

thus $a + m^4$ is not prime.

Next, let $c = a^{1/2}$, so $a - m^4 = (c - m^2)(c + m^2)$. If $a - m^4$ is prime, then $c - m^2 = 1$. Letting $d = (2c)^{1/2}$, we have $c + m^2 = 2c - 1 = (d-1)(d+1)$. Hence $a - m^4$ is not prime.

Next, consider $m^4 - a = (m^2 - c)(m^2 + c)$. If $m^4 - a$ is prime, then $m^2 - c = 1$. With $e = (c/2)^{1/4}$, we have

$$m^2 + c = 2c + 1 = (2e^2 + 2e + 1)(2e^2 - 2e + 1);$$

hence $m^4 - a$ is not prime.

Now take any p_i from $\{p_1, \ldots, p_s\}$. Since $a + m^{p_i}$ is greater than its factor $a^{1/p_i} + m$, it cannot be prime. Next, if $a - m^{p_i}$ is prime, then its factor $a^{1/p_i} - m$ is 1, and its other factor becomes

$$f_i(2^{u/p_i} n^{v/p_i}) \equiv f_i(2^{u/p_i}) \equiv 0 \pmod{q_i},$$

which cannot be prime. Similarly, if $m^{p_i} - a$ is prime, then its factor $m - a^{1/p_i}$ is 1 and its other factor becomes

$$g_i(2^{u/p_i} n^{v/p_i}) \equiv g_i(2^{u/p_i}) \equiv 0 \pmod{r_i},$$

which cannot be prime. Therefore all such $2^u n^v$ form an infinite set with the desired property.

Editorial comment. Carl Pomerance and David Stone remarked that if Bunyakovsky's conjecture holds, then for any noncube N, the polynomial $x^3 - N$ assumes infinitely many prime values, and therefore a noncube N can always be written as a cube minus a prime. Conjecturally, then, the only solutions to the problem are values of m^3 for which both $3m^2 - 3m + 1$ and $3m^2 + 3m + 1$ are composite.

Also solved by H. Al-Assad (Syria), A. Avagyan (Armenia), R. Chapman (UK), J. Christopher, C. Curtis, D. Fleischman, O. Geupel (Germany), N. Hodges (UK), O. Kouba (Kyria), S. S. Kumar O. P. Lossers (Netherlands), C. Pomerance & D. Stone , M. A. Prasad (India) C. Schacht, A. Stadler (Switzerland), A. Stenger, R. Stong, T. Wilde (UK), The Missouri State University Problem Solving Group, and the proposer.