STRATEGIES
OF
PROBLEM SOLVING

Third Edition

Maria Nogin

Department of Mathematics

College of Science and Mathematics

California State University, Fresno

2021

Acknowledgements

I am very thankful to my parents for introducing me to the beautiful world of mathematics problem solving. Since my very young age, they constantly asked me intriguing questions answering which required logical thinking, counting, experimenting, and noticing patterns. They bravely gave me very hard problems, but encouraged me when I got stuck. Some problems took me up to two years to solve. But they never just gave answers to me. They believed in my ability to solve those problems on my own.

I would also like to thank my colleagues in the Mathematics Department at CSU Fresno, especially Larry Cusick, Marat Markin, Adnan Sabuwala, and Agnes Tuska, for numerous valuable discussions about interesting problems and solutions as well as teaching and evaluating problem solving. Also thank you to the students in all of my MATH 145 (Problem Solving) classes for their careful reading and catching errors in my materials.

Finally, I am thankful to my family and friends for encouraging me to write this book.

To my daughters Katya and Misha.

I hope you will grow to love mathematics
and enjoy problem solving as much as I do.

Table of Contents

Chapter 0

Preliminaries

0.1 Review of logic

In this section we will recall some basic logic terminology and notations. It will be useful when we discuss types of proofs (see section 0.2).

Definition 0.1. A **proposition** (or statement) is a declarative sentence that is either true or false.

For example, "3 plus 2 is 5" is a true proposition, "3 times 2 is 7" is a false proposition, while "x minus 4 is 8" is not a proposition because the value of x has not been defined, and "is 3 plus 3 equal 6?" is not a proposition because it is an interrogative, not declarative, sentence.

Definition 0.2. Let p and q be propositions.

- The **negation** of p, denoted by $\neg p$ (or $\sim p$), is the proposition "not p." It is true if and only if p is false.

- The **conjunction** of p and q, denoted by $p \wedge q$, is the proposition "p and q." It is true if and only if both p and q are true.

- The **disjunction** of p and q, denoted by $p \vee q$, is the proposition "p or q." It is true if and only if at least one of p and q is true. Note that if both p and q are true, then $p \vee q$ is true, so "or" is not exclusive.

- The **exclusive or** of p and q, denoted by $p \oplus q$, is the proposition "either p or q but not both." It is true if and only if exactly one of p and q is true.

- The **implication** of p and q, denoted by $p \to q$ (or $p \Rightarrow q$), is the proposition "if p then q." It is false when p is true and q is false, and true otherwise.

- The **biconditional** of p and q, denoted by $p \leftrightarrow q$ (or $p \Leftrightarrow q$), is the proposition "p if and only if q." It is true when p and q have the same truth values and is false otherwise.

Below is the so-called truth table that shows the truth values of the compound propositions defined above depending on the truth values of p and q.

p	q	$\neg p$	$p \wedge q$	$p \vee q$	$p \oplus q$	$p \to q$	$p \leftrightarrow q$
T	T	F	T	T	F	T	T
T	F	F	F	T	T	F	F
F	T	T	F	T	T	T	F
F	F	T	F	F	F	T	T

Notice that $p \to q$ is false if and only if p is true and q is false. We will need this observation in section 0.2. Here is some more useful terminology.

- Symbols $\neg$, $\wedge$, $\vee$, $\oplus$, $\to$, and $\leftrightarrow$ are called **logical connectives**.
- A compound proposition that is always true, no matter what the truth values of the propositions that occur in it are, is called a **tautology**.

  For example, $p \vee \neg p$ is a tautology.
- A compound proposition that is always false is called a **contradiction**.

  For example, $p \wedge \neg p$ is a contradiction.
- Propositions p and q are called **logically equivalent** if $p \leftrightarrow q$ is a tautology. The notation $p \equiv q$ (sometimes $p \Leftrightarrow q$) denotes that p and q are logically equivalent. Note that p and q are logically equivalent if and only if they always have the same truth values.

Example 0.3. Show that $\neg(p \vee q)$ and $(\neg p) \wedge (\neg q)$ are logically equivalent.

Solution. *We construct the truth table.*

p	q	$p \vee q$	$\neg(p \vee q)$	$\neg p$	$\neg q$	$(\neg p) \wedge (\neg q)$
T	T	T	F	F	F	F
T	F	T	F	F	T	F
F	T	T	F	T	F	F
F	F	F	T	T	T	T

We see that the truth values of $\neg(p \vee q)$ and $(\neg p) \wedge (\neg q)$ are always the same, therefore these propositions are logically equivalent.

Definition 0.4. A statement $P(x)$ that depends on the value of a variable (x in this case) is called a **propositional function**. Once a value has been assigned to the variable x, the statement $P(x)$ becomes a proposition and has a truth value.

For example, if $P(x)$ is the statement "$x > 3$," then $P(4)$ is true and $P(2)$ is false.

Definition 0.5. Let $P(x)$ be a propositional function. Then

- $\forall x P(x)$ means "for every x, $P(x)$ is true."
- $\exists x P(x)$ means "there exists a value of x for which $P(x)$ is true."
- $\exists! x P(x)$ means "there exists a unique value of x for which $P(x)$ is true."

Symbols $\forall$ and $\exists$ are called **quantifiers**. Namely, $\forall$ is called the **universal quantifier**, and $\exists$ is called the **existential quantifier**.

When interpreting expressions $\forall x P(x)$, $\exists x P(x)$, $\exists! x P(x)$, we need to specify a set S of all possible choices of x. Such a set is called the **domain of discourse**. Unless the domain of discourse has already been specified or is clear from context, we can write $\forall x \in S\ P(x)$, etc. to make it explicit. For example, "the square of every integer x is nonnegative" can be written as $\forall x \in \mathbb{Z}\ x^2 \geq 0$.

Propositional functions can be functions of two or more variables, and then we can use two or more quantifiers with them. It is important to realize that the order of quantifiers makes a difference. For example, in problem 3 below we use the propositional function $F(x, y)$ which means that x and y are friends (the domain of this function can be a set of people). Then e.g. $\forall x \exists y F(x, y)$ means that everybody has at least one friend, while $\exists y \forall x F(x, y)$ means that there is a person who is friends with everybody.

Propositions with negations can always be written so that no negation is outside a quantifier or an expression involving logical connectives, for example:

- $\neg(p \wedge q) \equiv \neg p \vee \neg q$,
- $\neg(p \vee q) \equiv \neg p \wedge \neg q$,
- $\neg \forall x P(x) \equiv \exists x \neg P(x)$,
- $\neg \exists x P(x) \equiv \forall x \neg P(x)$.

Problems

1. Show that the following propositions are logically equivalent.
   (a) $p \rightarrow q$ and $\neg q \rightarrow \neg p$
   (b) $p \rightarrow q$ and $\neg p \vee q$
   (c) $\neg(p \wedge q)$ and $\neg p \vee \neg q$
   (d) $p \vee (q \wedge r)$ and $(p \vee q) \wedge (p \vee r)$
2. Which of the following sentences are statements? For those that are, indicate the truth value.
   (a) Five plus eight is thirteen.
   (b) Five minus eight is three.
   (c) Two times x is 6.
   (d) The number $2n + 6$ is an even integer.
   (e) There are 200 elephants in the San Diego Wild Animal Park.
   (f) I have solved all problems in chapter 1.
   (g) Did you do your homework today?
3. Translate the statement

$$\forall x (C(x) \vee \exists y (C(y) \wedge F(x, y)))$$

   into plain English, where $C(x)$ is "x has a computer," $F(x, y)$ is "x and y are friends," and the domain of discourse is the set of all students at your university.
4. Let $F(x, y)$ be statement "x can fool y." Use quantifiers to express each of the following statements.
   (a) Everybody can fool Amy.
   (b) Mike can fool everybody.

(c) Everybody can fool somebody.

(d) There is no one who can fool everybody.

(e) Everyone can be fooled by somebody.

(f) No one can fool both Kate and Jerry.

(g) Tim can fool exactly two people.

(h) There is exactly one person whom everybody can fool.

(i) No one can fool himself or herself.

5. Let $P(x)$ denote the propositional function "$x = -5$" and let $Q(x)$ denote the propositional function "$x^2 = 25$" and let the domain of discourse be the set of real numbers. Determine the truth values of the following propositions.

(a) $P(4)$

(b) $P(4) \to Q(4)$

(c) $\exists x \neg P(x)$

(d) $\forall x(P(x) \vee Q(x))$

(e) $\exists x(P(x) \wedge Q(x))$

(f) $\forall x(P(x) \to Q(x))$

(g) $\exists x(P(x) \to Q(x))$

(h) $\forall x(P(x) \leftrightarrow Q(x))$

6. Let $P(x)$ denote the propositional function "$(x < 3) \vee (x > 5)$" and let the domain of discourse be the set of real numbers. Determine the truth values of the following propositions.

(a) $P(2)$

(b) $P(4)$

(c) $P(2) \wedge P(4)$

(d) $\forall x P(x)$

(e) $\exists x P(x)$

(f) $\exists! x P(x)$

(g) $\forall x(P(x) \vee P(-x))$

7. Let $Q(x, y)$ denote "$x + y = 0$" and let the domain of discourse be the set of real numbers. What are the truth values of the statements $\forall x \exists y Q(x, y)$ and $\exists y \forall x Q(x, y)$?

8. Rewrite each of the following statements so that negations appear only immediately before propositional functions.

(a) $\neg \forall x \forall y P(x, y)$

(b) $\neg \forall y \exists x P(x, y)$

(c) $\neg \forall y \forall x(P(x, y) \vee Q(x, y))$

(d) $\neg(\exists x \exists y \neg P(x, y) \wedge \forall x \forall y Q(x, y))$

(e) $\neg \forall x(\exists y \forall z P(x, y, z) \wedge \exists z \forall y P(x, y, z))$

(f) $\neg \exists! x P(x)$

9. Let $P(x, y)$ denote the proposition "$x < y$" and let the domain of discourse be the set of real numbers. Determine the truth values of the following propositions.
   (a) $\exists x \exists y P(x, y)$
   (b) $\forall x \forall y P(x, y)$
   (c) $\forall x P(-x, x)$
   (d) $\exists x \forall y P(x, y)$
   (e) $\forall y \exists x P(x, y)$
   (f) $\exists y \forall x P(x, y)$
   (g) $\forall x \exists y P(x, y)$

10. Let $Q(x, y)$ be the statement "$x + y = x - y$" and let the domain of discourse be the set of integers. Determine the truth values of the following propositions.
    (a) $Q(2, 0)$
    (b) $\forall y Q(1, y)$
    (c) $\exists x \forall y Q(x, y)$
    (d) $\forall y \exists x Q(x, y)$
    (e) $\exists y \forall x Q(x, y)$
    (f) $\forall x \exists y Q(x, y)$

11. Let $R(x, y)$ denote the propositional function "$x^2 + y = 5$" where x and y are real numbers. Determine the truth values of the following propositions.
    (a) $\exists x \forall y\ R(x, y)$
    (b) $\forall y \exists x\ R(x, y)$
    (c) $\exists y \forall x\ R(x, y)$
    (d) $\forall x \exists y\ R(x, y)$

12. Determine the truth values of the following statements. Remember to pay attention to the domains.
    (a) $\forall a, b, c \in \mathbb{R}\ (a^2 + b^2 = c^2)$
    (b) $\exists a \in \mathbb{Q}\ \forall b \in \mathbb{Z}\ \exists c \in \mathbb{Q}\ (a^2 + b^2 = c^2)$
    (c) $\exists b \in \mathbb{R}\ \exists c \in \mathbb{R}\ \forall a \in \mathbb{R}\ (a^2 + b^2 = c^2)$
    (d) $\forall c \in \mathbb{Z}\ \exists a \in \mathbb{Z}\ \exists b \in \mathbb{Z}\ (a^2 + b^2 = c^2)$
    (e) $\forall a \in \mathbb{Z}\ \exists b \in \mathbb{R}\ \exists c \in \mathbb{Z}\ (a^2 + b^2 = c^2)$

13. Which of the following compound propositions are logically equivalent, i.e. have the same truth values for any propositional functions $P(x)$ and $Q(x)$? If propositions are logically equivalent, explain why. If not, give an example of propositional functions $P(x)$ and $Q(x)$ (and be sure to specify the domain for each) for which one of the propositions is true and the other one is false.
    (a) $\forall x(\neg P(x))$ and $\neg(\forall x P(x))$
    (b) $\forall x(P(x) \vee Q(x))$ and $(\forall x P(x)) \vee (\forall x Q(x))$
    (c) $\forall x(P(x) \wedge Q(x))$ and $(\forall x P(x)) \wedge (\forall x Q(x))$

(d) $\forall x(P(x) \to Q(x))$ and $(\forall x P(x)) \to (\forall x Q(x))$

(e) $\forall x(P(x) \leftrightarrow Q(x))$ and $(\forall x P(x)) \leftrightarrow (\forall x Q(x))$

(f) $\exists x(\neg P(x))$ and $\neg(\exists x P(x))$

(g) $\exists x(P(x) \vee Q(x))$ and $(\exists x P(x)) \vee (\exists x Q(x))$

(h) $\exists x(P(x) \wedge Q(x))$ and $(\exists x P(x)) \wedge (\exists x Q(x))$

(i) $\exists x(P(x) \to Q(x))$ and $(\exists x P(x)) \to (\exists x Q(x))$

(j) $\exists x(P(x) \leftrightarrow Q(x))$ and $(\exists x P(x)) \leftrightarrow (\exists x Q(x))$

14. Give an example of a propositional function $P(x, y)$ (remember to specify the domain for each variable) such that the statement $\exists! x \exists! y\ P(x, y)$ is true but the statement $\exists! y \exists! x\ P(x, y)$ is false.

15. Give an example of a set S and a propositional function $P(x, y)$ over S such that the statement $\forall x \forall y \forall z\ ((P(x, y) \wedge P(y, z)) \to P(x, z))$ is true. What does this condition remind you of?

16. Express the definition of the limit $\lim_{x \to a} f(x) = L$ using quantifiers.

17. Express the definition of a convergent sequence $a_1, a_2, a_3, \ldots$ using quantifiers.

18. In one country there are two cities, A and B, that are only a few miles apart, and whose residents often visit each other. All residents of city A always say the truth, while all residents of city B always lie. A stranger is passing through one of these cities, but he doesn't know which one. How could he, by asking the first man he sees only one question, determine which city he is passing through?

0.2 Types of proofs

In this chapter we summarize basic types of proofs, and then give a few examples to illustrate them.

Suppose we want to prove a proposition p. Then

- a **direct** proof just shows that p holds;
- a proof **by contradiction** assumes that p is false and derives a contradiction. The contradiction is usually of the form $r \wedge \neg r$ for some proposition r.

If we want to prove an implication "if p, then q", then

- a **direct** proof assumes that p holds and proves q;
- a proof **by contradiction** assumes that $p \to q$ is false, i.e. p is true and q is false, and derives a contradiction;
- a proof **by contrapositive** shows that $\neg q$ implies $\neg p$.

Often, we actually want to prove a statement of the form $\forall x P(x)$ or $\forall x(P(x) \to Q(x))$ rather than just p or $p \to q$. The classification of proofs is similar in these cases. For $\forall x P(x)$,

- a **direct** proof considers an arbitrary value of x in the domain and shows that $P(x)$ holds for it;

- a proof **by contradiction** assumes that $\forall x P(x)$ is false, that is, assumes $\exists x \neg P(x)$, and derives a contradiction.

For $\forall x(P(x) \to Q(x))$,

- a **direct** proof considers an arbitrary value of x for which $P(x)$ holds and proves that $Q(x)$ holds for such a value;
- a proof **by contradiction** assumes that $\forall x(P(x) \to Q(x))$ is false, that is, there exists a value of x for which $P(x)$ is true and $Q(x)$ is false, and derives a contradiction;
- a proof **by contrapositive** shows that for any value of x such that $\neg Q(x)$ holds, $\neg P(x)$ also holds.

Remark. To prove a statement of the form "$\forall x P(x)$" where the domain of discourse is a subset of integer numbers, it is often (but not always!) a good idea to use Mathematical Induction (see chapter 2).

A proof of a statement of the form "$\exists x P(x)$" can be

- **constructive** – when we provide (construct) such a value of x explicitly;
- **existential**, or **nonconstructive** – when we show the existence of such a value of x without actually providing (constructing) it.

To prove a statement of the form "$p \leftrightarrow q$" (resp. "$\forall x(P(x) \leftrightarrow Q(x))$"), we can either

- prove $p \to q$ and $q \to p$ (resp. $\forall x(P(x) \to Q(x))$ and $\forall x(Q(x) \to P(x))$) separately, or
- have each step of our proof of the form "if and only if."

To **disprove** a statement means to show that it is false. To disprove a statement of the form $\forall x P(x)$ it is sufficient to show that there exists at least one **counterexample**, that is, there exists at least one case when the statement does not hold.

Below are some examples of various types of proofs listed above.

Example 0.6. Prove that every odd integer is the difference of two perfect squares.

Direct proof. Every odd integer has the form $2n+1$ for some integer n. Observe that $2n+1 = (n+1)^2 - n^2$.

Example 0.7. Prove that $\sqrt{2}$ is irrational.

Proof by contradiction. Suppose $\sqrt{2}$ is rational. Then there exists an irreducible fraction $\frac{p}{q} = \sqrt{2}$. (Irreducible means that the greatest common divisor of p and q is 1.) Then $\frac{p^2}{q^2} = 2$, thus $p^2 = 2q^2$. If follows that p^2 is even, so p is even (see problem 2 below). Let $p = 2m$, where m is an integer, then $p^2 = 4m^2$. We have $4m^2 = 2q^2$, or $2m^2 = q^2$. Now we see that q^2 is even, therefore q is even. We get a contradiction because we have that on one hand, p and q have the greatest common divisor 1, but on the other hand p and q are both even.

Example 0.8. Prove that if a and b are integers and ab is even, then either a or b is even (or both).

Proof by contrapositive. Suppose that neither a nor b is even, and we will prove that ab is not even. That is, we suppose that both a and b are odd, and we will prove that ab is odd. Any odd numbers a and b can be written in the form $a = 2n + 1$ and $b = 2m + 1$ for some integers n and m. Then we have $ab = (2n + 1)(2m + 1) = 4nm + 2n + 2m + 1 = 2(2nm + n + m) + 1$ is an odd number.

Example 0.9. Prove that for every positive integer n, there exist n consecutive composite numbers.

Constructive proof. We claim that $(n+1)!+2$, $(n+1)!+3$, $\ldots$, $(n+1)!+(n+1)$ are all composite. Indeed, $(n+1)!$ is divisible by 2, by 3, $\ldots$, and by $n+1$. Therefore $(n+1)!+2$ is divisible by 2, $(n + 1)! + 3$ is divisible by 3, $\ldots$, $(n + 1)! + (n + 1)$ is divisible by $n + 1$.

Example 0.10. Prove that $x^3 + x - 1 = 0$ has a real root.

Nonconstructive proof. Let $f(x) = x^3+x-1$. Then $f(-1) = -3 < 0$ and $f(1) = 1 > 0$. Since $f(x)$ is a polynomial, it is continuous. By the Intermediate Value Theorem, there exists c between -1 and 1 such that $f(c) = 0$.

Example 0.11. Prove or disprove that every odd integer is prime.

Counterexample. Observe that 9 is odd but not prime. Thus the statement is false.

Problems

1. Prove that if n is an integer and $3n + 5$ is odd, then n is even. Is your proof direct, by contradiction, or by contrapositive?
2. Prove that an integer a is even if and only if a^2 is even. Did you prove the two implications separately or simultaneously?
3. Let n be an integer. Prove that if $145n - 1$ is odd, then $n + 1000$ is even. Is your proof direct, by contrapositive, by contradiction, or none of these? (If it is none of the listed proof types, then describe/summarize your proof type/strategy.)
4. Prove or disprove that $11n + 5$ is composite for every natural number n.
5. Prove or disprove that $2^n + 1$ is prime for all nonnegative integers n.
6. Prove that for any integer n there is a prime number greater than n. Is your proof constructive?
7. Every odd number is either of the form $4n + 1$ (if it has remainder 1 when divided by 4) or of the form $4n + 3$ (if it has remainder 3) where n is an integer. Prove that if an odd number is a perfect square, then it has the form $4n + 1$. What type of proof did you use? State the converse. Prove or disprove the converse.
8. Prove or disprove each of the following statements. For each part, what type of proof did you use? (Direct, by contradiction, by contrapositive, by counterexample, etc.)
   (a) If a and b are rational numbers, then $a + b$ is also rational.
   (b) If a and b are irrational numbers, then $a + b$ is also irrational.
   (c) If a is rational and b is irrational, then $a + b$ is irrational.

(d) If a is rational and b is irrational, then ab is irrational.

(e) If a and b are rational numbers, then a^b is also rational.

(f) If a and b are irrational numbers, then a^b is also irrational.

9. Prove that the equation $x^{101} + x^{51} + x + 1 = 0$ has exactly one real solution. Split this into two statements:

   (a) the equation has at least one solution. Is your proof constructive or nonconstructive?

   (b) the equation can not have two distinct roots. Is your proof direct, by contradiction, or by contrapositive?

10. Prove that if the sum of two numbers is irrational, then at least one of the numbers is irrational. Is your proof direct, by contradiction, or by contrapositive? State the converse. Prove or disprove the converse.

11. Prove that the equation $4\sin^2 x = 1$ has a real solution. Is your proof constructive?

12. Prove that the equation $x + \sin x = 1$ has a real solution. Is your proof constructive?

13. Prove that the equation $2x^2 + 8x + 7 = 0$ has no rational solutions. Is your proof direct, by contradiction, or by contrapositive?

14. Prove that 0 is a root of the equation $a_n x^n + \cdots + a_1 x + a_0 = 0$ if and only if the free term $a_0 = 0$. Did you prove the two implications separately or simultaneously?

15. Prove that if a positive integer is divisible by 8, then it is the difference of two perfect squares. Is your proof direct, by contradiction, or by contrapositive? Is it constructive or nonconstructive?

16. Prove that for any integers n and m, if $nm + 2n + 2m$ is odd, then both n and m are odd. Is your proof direct, by contradiction, or by contrapositive?

Chapter 1

Introduction

Solving mathematical problems is both a science and an art. It is a science because we need to learn some basic concepts and skills, and use proper terminology when explaining our solution to others. At the same time, it is an art because very often we need to be creative, especially as we strive to find elegant solutions. There are infinitely many types of math problems. While it is important to learn some basic principles of problem solving, it is impossible to learn how to solve every problem in the world, just like it is impossible to learn how to construct every possible model using Lego Blocks. However, after you have constructed a few models following the directions in your Lego set, you can create your own. When you finish school and move on to solving real world problems, you may need some initial training in your field, but no training can ever give you detailed instructions on what to do in every possible situation. You will have to think, make decisions, try things out, and learn from your successes and failures. This is what we are going to do in this book: learn some basics and then explore on our own and learn from our experience.

Below are some problems. Their solutions will illustrate some things mentioned above as well as will introduce a few concepts and principles that will be discussed in later chapters.

1. Eleven children contributed money to buy a present for their classmate. The total amount of money collected was \$30.00. Prove that at least one child gave at least \$2.73.

2. (a) Prove that any two-digit number is divisible by 3 if and only if the sum of its digits is divisible by 3.

   (b) Prove that any natural number is divisible by 3 if and only if the sum of its digits is divisible by 3.

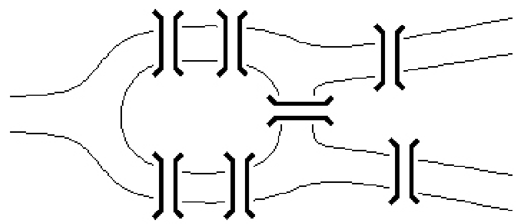
3. Is it true or false that for any natural number n, the number $n^2 + n + 41$ is prime?

4. In a 4×4 table, six cells are marked by a star and all others are blank. Show that it is possible to cross out 2 columns and 2 rows so that the remaining cells are blank.

5. Is it true or false that for any natural number n, the number $n^3 + 2n$ is divisible by 3?

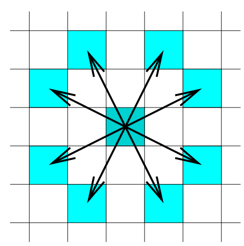
6. Sketch the graph of $f(x) = |x + 2| + |2x - 5|$.

7. Konigsberg is a city which was the capital of East Prussia but now is known as Kaliningrad in Russia. The city is built around the River Pregel where it joins another river. An island named Kniephof is in the middle of where the two rivers join. There are seven bridges that join the different parts of the city on both sides of the rivers and the island.



People tried to find a way to walk all seven bridges without crossing a bridge twice, but no one could find a way to do it. The problem came to the attention of a Swiss mathematician named Leonhard Euler. In 1735, Euler presented the solution to the problem before the Russian Academy. Now, you too try to solve this problem. If such a tour exists, find it. If not, explain why not.

8. (a) Is it possible for a chess knight to start at the upper-left corner and go through every square on an 8×8 chessboard exactly once? (A knight's move is 2 squares up, down, or to the right or left, and 1 square in a perpendicular direction. All allowed moves from a certain square are shown below.)



(b) Is it possible for a knight to start at the upper-left corner, go through every square on an 8×8 chessboard exactly once, and then come back to the starting point in just one additional move?

(c) Is it possible for a knight to start at the upper-left corner and go through every square on an 8×8 chessboard exactly once so that to finish at the lower-right corner?

As mentioned above, learning to solve problems is in part difficult because problems can be very different. However, there are a few basic principles that are good to know. There are a few approaches and methods that can be useful. In this book, we will study some of them. After you study the material of this book you should be able to solve many problems pretty easily. The material and problems in this book are meant to be accessible to motivated high school and college students. They could be used by teachers for a math club or math team training sessions, however, many problems are tricky, and some may present a challenge even to a seasoned mathematician. So don't get discouraged when you struggle!

While using intuition and working out a few examples may help us find an idea, it is also important to write rigorous proofs. Since our intuition is not always correct, we need to justify each step in a solution. We will therefore try to avoid words such as 'obviously.'

In each chapter, we provide basic definitions and facts to get you started. We do not prove most of the facts given in this book, since our main goal is to learn how to solve problems, i.e. use these facts. You will probably prove most, if not all, of the facts given in this book in courses such as Mathematical Analysis, Discrete Mathematics, Abstract Algebra, and Number Theory. Sometimes the idea of a proof of a theorem can be used for solving many problems. In such cases we provide the proof.

Chapter 2

Principle of Mathematical Induction

Theorem 2.1. *(Principle of Mathematical Induction) Let S_n be a statement about a positive integer n. Suppose that*

1. *S_1 is true, and*
2. *if $k \geq 1$ and S_k is true, then S_{k+1} is true.*

Then S_n is true for all positive integers n.

Note. Conditions 1 and 2 in the above theorem are called the basis step and inductive step, respectively.

This principle is easy to understand using the following example: suppose we know how to get to the first floor of a building (e.g. we know where an entrance is), and we also know how to get from any floor to the next one (e.g. we know where an elevator or a staircase is). Then we'll be able to get to any floor in this building. Namely, we'll be able get to the first floor, and then from the first to the second, and then from the second to the third, and so on. The same is true for any statement. If we can check that S_1 is true, then the second condition in theorem 2.1 ensures that S_2 follows from S_1, S_3 follows from S_2, and so on. Thus S_n is true for any natural number n.

Mathematical Induction is used in all areas of mathematics. It can be used to prove summation formulas such as the one in the next example, various number theory, algebraic, and geometric statements.

Example 2.2. Prove that for any natural number n,

$$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}.$$

Proof. We will prove this identity using Mathematical Induction.
Basis step. If $n = 1$, the formula says that $1 = \frac{1 \cdot (1+1)}{2}$ which is true.
Inductive step. Suppose the formula holds for $n = k$, i.e. that

$$1 + 2 + 3 + \cdots + k = \frac{k(k+1)}{2} \tag{2.1}$$

is true. We have to show that the formula holds for $n = k + 1$, i.e. that

$$1 + 2 + 3 + \cdots + (k + 1) = \frac{(k + 1)((k + 1) + 1)}{2}$$

is true. Adding $k + 1$ to both sides of (2.1) gives:

$$\begin{aligned} 1 + 2 + 3 + \cdots + k + (k + 1) &= \frac{k(k + 1)}{2} + (k + 1) \\ &= \frac{k(k + 1) + 2(k + 1)}{2} \\ &= \frac{(k + 2)(k + 1)}{2} \\ &= \frac{((k + 1) + 1)(k + 1)}{2}. \end{aligned}$$

□

Note. For any specific value of n, it is easy to check that the identity holds. For example, for the first four natural numbers we have:

$$1 = \frac{1 \cdot (1 + 1)}{2}, \quad 1 + 2 = \frac{2 \cdot (2 + 1)}{2}, \quad 1 + 2 + 3 = \frac{3 \cdot (3 + 1)}{2}, \quad 1 + 2 + 3 + 4 = \frac{4 \cdot (4 + 1)}{2}.$$

However, remember that it is not sufficient to check *some* values of n. We have to prove the statement for *all* natural numbers n.

Remark. Sometimes we want to prove a statement S_n for all $n \geq 0$, or for all $n \geq 2$, etc., rather than for all $n \geq 1$. In this case, the basis step should check that the statement is valid for the smallest value of n in the desired range, say, $n = 0$, or $n = 2$ in the above cases, and the inequality $k \geq 1$ in the inductive step should be modified accordingly ($k \geq 0$, or $k \geq 2$, etc.).

Sometimes to prove S_{k+1}, it is insufficient to assume S_k alone, but S_n for $n \leq k$ is needed. Then we use the so-called Strong Induction formulated below.

Theorem 2.3. *(Strong Mathematical Induction) Let S_n be a statement about a positive integer n. Suppose that*

1. *S_1 is true, and*
2. *if $k \geq 1$ and S_n is true for all $1 \leq n \leq k$, then S_{k+1} is true.*

Then S_n is true for all positive integers n.

Remark. As above, we might want to start with 0 or 2 or something else rather than with 1.

Example 2.4. Prove that any integer $n \geq 2$ can be written in the form $n = 2a + 3b$ for some nonnegative integers a and b (we will say that n is a nonnegative linear combination of 2 and 3).

Proof. Basis step. If $n = 2$, we have $n = 2 \cdot 1 + 3 \cdot 0$.
Inductive step. Suppose that $k \geq 2$ and the statement holds for all $2 \leq n \leq k$. We want to prove it for $n = k + 1$.
Case I: $k = 2$. Then $k + 1 = 3 = 2 \cdot 0 + 3 \cdot 1$.
Case II: $k \geq 3$, then $2 \leq k - 1 \leq k$, thus the statement holds for $n = k - 1$. We have $k - 1 = 2a + 3b$ for some nonnegative integers a and b. Then $k + 1 = k - 1 + 2 = 2a + 3b + 2 = 2(a + 1) + 3b$, so $k + 1$ is a nonnegative linear combination of 2 and 3. □

Remark. Notice that case I above simply checks that the statement holds for $n = 3$. This calculation is often moved to the basis step.

Problems

1. Prove that the following formulas hold for any natural n.

   (a) $1^2 + 2^2 + 3^2 + \cdots + n^2 = \dfrac{n(n+1)(2n+1)}{6}$

   (b) $1^3 + 2^3 + 3^3 + \cdots + n^3 = \left(\dfrac{n(n+1)}{2}\right)^2$

   (c) $1 \cdot 1! + 2 \cdot 2! + \cdots + n \cdot n! = (n+1)! - 1$

   (d) $1 + 3 + 5 + \cdots + (2n-1) = n^2$

   (e) $1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \cdots + n(n+1) = \dfrac{n(n+1)(n+2)}{3}$

   (f) $1 \cdot 3 + 2 \cdot 4 + 3 \cdot 5 + \cdots + n(n+2) = \dfrac{n(n+1)(2n+7)}{6}$

   (g) $2 \cdot 2^1 + 3 \cdot 2^2 + \cdots + (n+1) \cdot 2^n = n2^{n+1}$

2. (a) Prove that for any positive integer n, $n < 2^n$.

   (b) Prove that for any integer $n \geq 4$, $2^n < n!$.

3. Prove that if q is a positive integer, then $3^{2^q} - 1$ is divisible by 2^{q+2}.

4. Prove that for any natural number n, the number

$$\frac{n^7}{7} + \frac{n^5}{5} + \frac{2n^3}{3} - \frac{n}{105}$$

   is an integer.

5. Let $\{F_0,\ F_1,\ F_2,\ \ldots\}$ be the Fibonacci sequence defined by $F_0 = 0$, $F_1 = 1$, and $F_{n+1} = F_n + F_{n-1}$, $n \geq 1$. Prove the following identities and inequalities.

   (a) $F_1F_2 + F_2F_3 + \cdots + F_{2n-1}F_{2n} = F_{2n}^2$

   (b) $F_1^2 + F_2^2 + \cdots + F_n^2 = F_nF_{n+1}$

   (c) $F_{n-1}F_{n+1} = F_n^2 + (-1)^n$

   (d) $\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^n = \begin{bmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{bmatrix}$

   (e) $F_{n-1}^2 + F_n^2 = F_{2n-1}$

   (f) $F_n \geq \left(\dfrac{3}{2}\right)^{n-2}$

6. Prove that any integer $n \geq 4$ can be written as $n = 2a + 5b$ for some non-negative integers a and b.

7. Prove that for every natural number n,

$$1 < \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{3n+1} < 2.$$

8. Let s_n be the number of ways to cover a $2 \times n$ board by n dominoes (2×1 tiles). It is easy to check that $s_1 = 1$, $s_2 = 2$, $s_3 = 3$, $s_4 = 5$, and $s_5 = 8$. Prove that for any natural n, $s_n = F_{n+1}$ (where $\{F_n\}$ is the Fibonacci sequence defined in problem 5).

9. The triangle inequality states that the length of a triangle's side is always smaller than the sum of the lengths of the other two sides of that triangle. Prove that a side length of

   (a) a quadrilateral

   (b) a pentagon

   (c) any polygon

   is less than the sum of all of its other side lengths.

10. Suppose that $n \in \mathbb{N}$ and n lines are given in the plane. They divide the plane into regions. Show that it is possible to color the plane with two colors so that no regions with a common boundary line are colored the same way. (Regions that share only one boundary point may be colored the same way.) Note: such a coloring is called a proper coloring.

11. Suppose that $n \in \mathbb{N}$ and n circles are given in the plane. They divide the plane into regions. Show that it is possible to color the plane with two colors so that no regions with a common boundary line are colored the same way. (Regions that share only finitely many boundary points may be colored the same way.)

12. Suppose that $n \in \mathbb{N}$ and n circles and a chord of each circle are drawn in a plane. They divide the plane into regions. Show that it is possible to color the plane with two colors so that no regions with a common boundary line are colored the same way. (Regions that share only finitely many boundary points may be colored the same way.)

13. Consider a few points in the plane and a few line segments connecting some of them so that (1) no two line segments intersect, and (2) each point is connected with at least two other points (so there are no isolated points and there are no "hanging" line segments). Such line segments divide the plane into several regions. Such a picture is called a map. Prove that a map can be properly colored with two colors if and only if each point is connected with an even number of other points. (See problem 10 for the definition of a proper coloring.)

14. Let α be any real number such that $\alpha + \dfrac{1}{\alpha} \in \mathbb{Z}$. Prove that $\alpha^n + \dfrac{1}{\alpha^n} \in \mathbb{Z}$ for any $n \in \mathbb{N}$.

15. Show that for any $n \geq 3$, there exist distinct positive integers $a_1, a_2, \ldots, a_n$ such that
$$1 = \frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}.$$

16. Prove that for any natural number n, the set $\{1, 2, \ldots, n\}$ has $\dfrac{n(n-1)}{2}$ subsets containing exactly two elements.

17. Prove that the determinant of the $n \times n$ matrix M_n with entries
$$m_{ij} = \begin{cases} 5 \text{ if } i = j \\ 2 \text{ if } |i-j| = 1 \\ 0 \text{ otherwise} \end{cases}$$
is equal to $\dfrac{1}{3}(4^{n+1} - 1)$.

18. Find the determinant of the $n \times n$ matrix A_n with entries

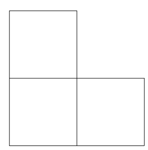
$$a_{ij} = \begin{cases} 2 \text{ if } i = j \\ 1 \text{ if } |i-j| = 1 \\ 0 \text{ otherwise} \end{cases}.$$

Hint: calculate the determinants of A_1, A_2, A_3, and A_4. Notice the pattern. Guess a formula for $\det A_n$, and then prove it by Mathematical Induction.

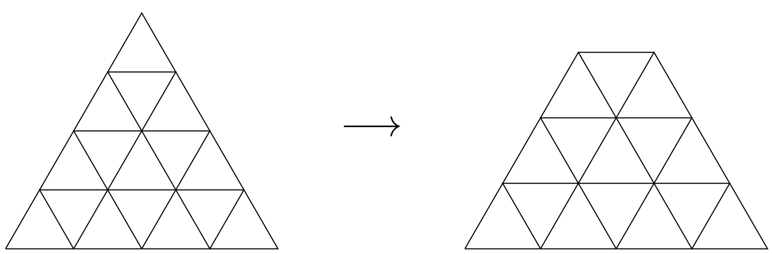
19. Prove that for any integer $n \geq 6$, a square can be subdivided into n (not necessarily congruent) smaller squares.

20. Suppose that $2n$ points are given in space, where $n \geq 2$, none of which are collinear. Altogether $n^2 + 1$ line segments are drawn between these points. Prove that there is at least one triangle (i.e. a set of three points which are joined pairwise by line segments).

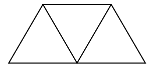
21. Prove that if any one square of a $2^n \times 2^n$ board is removed, then the remaining board can be covered (without overlap) by L-trominoes, i.e. the tiles consisting of 3 squares as shown below.



22. Consider a triangle consisting of 4^n identical smaller triangles. Remove one of the three corner triangles. Here is an example for $n = 2$:



Show that the remaining region can be tiled with pieces shown below.



23. (a) Prove that all numbers in the sequence 1007, 10017, 100117, ... are divisible by 53.
    (b) Prove that all numbers in the sequence 12008, 120308, 1203308, 12033308, ... are divisible by 19.

24. A chess tournament consists of a few players, all playing at least one game with each of the other players. A player is called a champion if they have won at least one game against but lost none to each of the other players (draws are OK). Note that there is at most one champion, however, there may be no champion at the tournament. Your

assignment is to find the champion, if one exists (or determine that there is none), at the tournament, by asking only one type of question — asking a player whether they have won any games against a second player. Everyone must answer your questions truthfully. For example, if Alice and Bob are two players at the tournament, you may ask Alice whether she has won any games against Bob; she must answer truthfully. Prove that if there are n players at the tournament, then you can find the champion, if there is one, with at most $3(n-1)$ questions.

25. Every road in the magical land of Sikinia is one-way. Every pair of cities is connected by exactly one direct road. Show that there exists a city which can be reached from every other city either directly or via at most one other city.

26. Let n be any natural number. Consider all nonempty subsets of the set $\{1, 2, \ldots, n\}$, which do not contain any neighboring elements. Prove that the sum of the squares of the products of all numbers in these subsets is $(n+1)!-1$. (For example, if $n = 3$, then such subsets of $\{1, 2, 3\}$ are $\{1\}$, $\{2\}$, $\{3\}$, and $\{1, 3\}$, and $1^2+2^2+3^2+(1\cdot 3)^2 = 23 = 4!-1$.)

27. Suppose $2n$ dots are placed around the circle, and n of them are colored red while the remaining n are colored blue. Going around the circle clockwise, we keep count of how many red and blue dots we have passed. If at all times the number of red dots we have passed is at least the number of blue dots, we consider it a successful trip around the circle. Prove that no matter how the dots are colored red and blue, it is possible to have a successful trip around the circle if we start at a correct point.

28. There are n identical cars on a circular track. Among all of them, they have just enough gas for one car to complete a lap. Show that there is a car which can complete a lap by collecting gas from other cars on its way around.

Chapter 3

Dirichlet's Box Principle

Theorem 3.1. *(Dirichlet's Box Principle) If $n+1$ or more objects are put into n boxes, then at least one box contains more than one object.*

Proof. Assume to the contrary that each box contains at most one object. Since there are n boxes, altogether there are at most n objects. However, we are given that there are at least $n+1$ objects. We reached a contradiction, therefore, our assumption must be false. $\square$

Dirichlet's Box Principle is often called the **Pigeonhole Principle** and is formulated as follows.
Suppose there are n pigeonholes in the tree, and there are at least $n+1$ pigeons flying into these n holes. Then there is at least one hole containing more than one pigeon.

More formally and more generally, this principle can be formulated in the following way.
If the cardinality of a set S is bigger than the cardinality of a set T, and f is a function from S to T, then f is not one-to-one.

In the above theorem the function

$$S \xrightarrow{f} T$$

is

$$\{n+1 \text{ objects}\} \longrightarrow \{n \text{ boxes}\}$$

or

$$\{n+1 \text{ pigeons}\} \longrightarrow \{n \text{ pigeonholes}\}.$$

Dirichlet's Box Principle is often used to prove statements involving remainders or divisibility. Recall that for any natural number n, there are n possible remainders upon division by n, namely, 0, 1, 2, $\ldots$, $n-2$, and $n-1$. If we are given more than n numbers, then by Dirichlet's Box Principle at least two of them have the same remainder. Note that this implies that the difference of these two numbers is divisible by n. Read more on remainders and divisibility in chapter 4. We should also note that all statements that can be proved using Dirichlet's Box Principle can also be proved by contradiction which mimics the above proof. However, it is considered more elegant to refer to the principle.

Here is another, also very useful, principle.

Theorem 3.2. *If $n-1$ or fewer objects are put into n boxes, then at least one box is empty.*

More formally,
If the cardinality of a set S is smaller than the cardinality of a set T, and f is a function from S to T, then f is not onto.

Below are generalizations of the above principles.

Theorem 3.3. *(Generalized Dirichlet's Box Principle) If $qn+1$ or more objects are put into n boxes, then at least one box contains more than q objects.*

Theorem 3.4. *If $n-k$ or fewer objects are put into n boxes, then at least k boxes are empty.*

Problems

1. Prove that among 13 persons, at least two were born in the same month.

2. Prove that among 50 persons, at least five were born in the same month.

3. A Martian has an infinite number of red, blue, yellow, and black socks in a drawer. He pulls out a few socks without looking, then looks at the socks he got, and decides what to wear. How many socks must the Martian pull out of the drawer to guarantee he has a matching pair?

4. The capital of Sikinia has 300,001 inhabitants, and it is known that none of them has more than 300,000 hairs on his or her head. Can you assert with certainty that there are two persons with the same number of hairs on their heads?

5. Prove that among 120 distinct integers, there are two whose difference ends with 00.

6. Given 50 distinct positive integers less than 99, prove that some two of them add up to 99.

7. Let a_1, a_2, and a_3 be integers. Show that $(a_1 - a_2)(a_1 - a_3)(a_2 - a_3)$ is even.

8. Let a_1, a_2, a_3, and a_4 be integers. Show that $\displaystyle\prod_{1 \le i < j \le 4} (a_i - a_j)$ is divisible by 12.

9. Prove that from any 12 distinct two-digit numbers, we can select two with a two-digit difference of the form aa.

10. A few distinct numbers are randomly selected from the set $\{1, 2, 3, \ldots, 9, 10\}$. What is the smallest number of numbers that must be selected to guarantee that among the selected numbers there are two numbers such that one of them is equal to half of the other?

11. Kevin is paid every other week on Friday. Show that every year, in some month he is paid three times.

12. Five numbers are randomly picked from the following set:

$$\{0, 1, 4, 9, 16, 25, 36, 49, 64, 81, 100\}.$$

Prove that among these numbers there are two whose difference is divisible by 5.

13. Seven points are given inside a regular hexagon with sides of length 1. Prove that there are two among these seven points such that the distance between them is at most 1.

14. Twenty-six points are given inside a 20×15 rectangle. Prove that there are at least two points with distance less than or equal to 5.

15. (a) Seven points are given inside a 9×12 rectangle. Prove that there are two of them such that the distance between them is less than 7.

(b) Is the above assertion also valid for six points?

16. Suppose that fifty-one small insects are placed inside a square of side 1. Prove that at any moment there are at least three insects which can be covered by a single disk of radius 1/7.

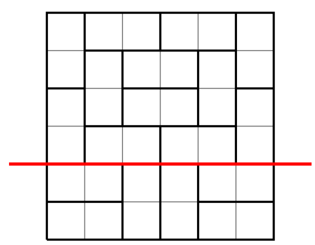
17. Three hundred points are given inside a cube with edge 7. Prove that we can place a small cube with edge 1 inside the big cube such that the interior of the small cube does not contain any of the given points.

18. There are n people in a hiking club. Each person counted with how many other club members they have gone hiking. Prove that there are two people who have gone hiking with the same number of other club members.

19. Inside a circle of radius 4 are chosen 61 points. Show that among them there are two at distance at most $\sqrt{2}$ from each other.

20. Given nine points inside the unit square, prove that some three of them form a triangle whose area does not exceed $\frac{1}{8}$.

21. A 6×6 square is tiled by dominoes (2×1 tiles). Prove that it has at least one fault-line: a straight line cutting the rectangle without cutting any domino. An example of such a line is shown below).



22. (a) Prove that among any 11 distinct natural numbers, there are two numbers $a < b$ such that the difference $b - a$ ends with 0 (i.e. has the units digit 0).

    (b) Is the above statement true for the tens digit? That is, is it true that among any 11 distinct natural numbers, there are two numbers $a < b$ such that the difference $b - a$ has 0 in the tens place?

23. Prove that if $1 = a_1 < a_2 < a_3 < \ldots < a_8 = 100$ and all a_is are integers, then $a_{i+1} - a_i \geq 15$ for some i.

24. (a) Prove that from any 52 positive integers, we can select two such that their sum or difference is divisible by 100.

    (b) Is the above assertion also valid for 51 positive integers?

25. Let n be a positive integer which is not divisible by 2 or 5. Prove that there is a multiple of n consisting entirely of ones.

26. Suppose that five lattice points (i.e. points with integer coordinates) are given in the plane. Prove that we can choose two of these points such that the segment joining these two points passes through another lattice point.

27. Let $\{a_1, a_2, \ldots, a_{n+1}\}$ be numbers from the set $\{1, 2, \ldots, 2n\}$. Prove that at least two of the a_is are relatively prime.

28. Let $\{a_1, a_2, \ldots, a_{n+1}\}$ be numbers from the set $\{1, 2, \ldots, 2n\}$. Prove that one of the a_is is divisible by another.

29. Prove that in any convex $2n$-gon, there is a diagonal not parallel to any side.

30. Prove that any 21-gon has two diagonals (not necessarily having a common point) such that the smaller angle between them is less than 1°.

31. Let f be a one-to-one function from $X = \{1, 2, \ldots, n\}$ onto X. Let $f^k = \underbrace{f \circ f \circ \cdots \circ f}_{k\times}$ denote the k-fold composition of f with itself. Show that for some positive integer m, $f^m(x) = x$ for all $x \in X$.

32. Show that there is a positive term of the Fibonacci sequence that is divisible by 1000.

33. Prove that every set of 10 two-digit numbers has two disjoint nonempty subsets with the same sum of elements.

34. One thousand coins of diameter 1 cm each are placed on a table of size 55 cm by 60 cm. Prove that it is possible to put one more coin of the same size so that it does not touch any of the 1000 original coins. (The coin must lie completely on the surface of the table, i.e. it cannot stick out.)

35. Every block of a 3×7 board is colored either black or white. Prove that no matter how the board is colored, it contains a rectangle consisting of more than one row and more than one column whose four corners have the same color.

36. Every block of a 5×41 board is colored in one of four colors. Prove that, no matter how the board is colored, there exists at least one same-color-corner rectangle (as in problem 35).

37. On each face of a dodecahedron is written a nonnegative integer such that the sum of all 12 integers is 11. Show that there are two faces that share an edge and have the same integer written on them.

38. On each face of an icosahedron is written a nonnegative integer such that the sum of all 20 integers is 39. Show that there are two faces that share a vertex and have the same integer written on them.

39. A chess player trains (indefinitely) by playing at least one game per day, but, to avoid exhaustion, no more than 12 games a week. Prove that there will be a group of consecutive days in which he plays exactly 20 games.

40. Assume in a class of students each of a number of committees contains more than half of all the students.

    (a) Prove that there is a student who is a member in more than half of the committees.

    (b) Prove that if the number of committees is 30, then it is possible to select just 4 students to be committee representatives so as not to leave a committee without a representative. (Committees may be only represented by their own members, but one student may represent any number of committees.)

Chapter 4

Number theory

In this chapter we recall basic properties of integers. (Some of them have been used in previous chapters.)

Definition 4.1. If $a \neq 0$ and $b = aq$, then we say that a **divides** b, or that b is **divisible** by a, and write $a \mid b$.

Theorem 4.2. *(Fundamental properties of divisibility.) Let a, b, and c be integers.*

1. *If $a \mid b$ and $b \mid c$, then $a \mid c$.*
2. *If $a \mid b$ and $a \mid c$, then $a \mid (bx + cy)$ for any integers x and y.*

   Important special cases:

   (a) *If $a \mid b$, then $a \mid bx$ for any integer x.*

   (b) *If $a \mid b$ and $a \mid c$, then $a \mid (b + c)$ and $a \mid (b - c)$.*

Definition 4.3. Let a or b be nonzero integers. The largest number that divides both a and b is called **the greatest common divisor** of a and b, and is denoted by $\gcd(a, b)$ or just (a, b).

Theorem 4.4. *For any nonzero integers a and b, $\gcd(a, a) = a$, $\gcd(a, 1) = 1$, $\gcd(a, 0) = a$, $\gcd(a, b) = \gcd(b, a)$.*

Definition 4.5. An integer greater than 1 is called **prime** if it has exactly two positive divisors, 1 and itself. An integer greater than 1 that is not prime is called **composite**.

Theorem 4.6. *There are infinitely many primes.*

Theorem 4.7. *(Euclid's lemma.) Let a and b be integers. If p is prime and $p \mid ab$, then $p \mid a$ or $p \mid b$.*

Theorem 4.8. *(Fundamental theorem of arithmetic.) Every positive integer larger than 1 has a prime factorization, i.e. can be written as a product of primes, and such a product is unique up to order of the factors.*

Example 4.9. Observe that $12 = 2 \cdot 2 \cdot 3 = 2 \cdot 3 \cdot 2 = 3 \cdot 2 \cdot 2$ are the only prime factorizations of 12. The order of the factors is different, but the set of factors is the same.

Corollary 4.10. *If a is an integer and p and q are distinct primes such as $p \mid a$ and $q \mid a$, then $(pq) \mid a$.*

Definition 4.11. Integers a and b are called **relatively prime**, or coprime, if $\gcd(a, b) = 1$.

Remark. Integers a and b may be relatively prime even if they are both composite. For example, $8 = 2 \cdot 2 \cdot 2$ and $15 = 3 \cdot 5$ are composite but relatively prime since they do not share any factors larger than 1.

Theorem 4.12. *For any pair of nonzero integers a and b, their greatest common divisor $\gcd(a, b)$ is a linear combination of a and b, i.e. there exist integers x and y such that $\gcd(a, b) = ax + by$. Moreover, $\gcd(a, b)$ is the smallest positive integer that can be written in the form $ax + by$ for some integers x and y.*

Corollary 4.13. *Nonzero integers a and b are relatively prime if and only if there exist integers x and y such that $ax + by = 1$.*

Definition 4.14. For any integers a and $b > 0$ there exist unique integers q and r such that

$$a = bq + r, \quad 0 \le r < b.$$

The numbers q and r are called the **quotient** and **remainder** respectively upon division of a by b. The notation $a \mod b$ is commonly used for the remainder.

The following observations are often useful when solving problems involving integers.

- Any integer can be written in the form $10q+r$ for some integers q and r where $0 \le r \le 9$.
- An integer $\underline{a_n a_{n-1} \ldots a_1 a_0}$ (with digits a_n, a_{n-1}, $\ldots$, a_1, a_0) can be written as

$$10^n a_n + 10^{n-1} a_{n-1} + \cdots + 10^2 a_2 + 10 a_1 + a_0.$$

If two integers have the same remainder upon division by a b, then they can be written as $bq_1 + r$ and $bq_2 + r$. Their difference is $b(q_1 - q_2)$, and thus it is divisible by b. As was discussed in chapter 3, since there are b possible remainders upon division by b, given $b + 1$ or more numbers, by Dirichlet's Box Principle at least two of them have the same remainder. Their difference is divisible by b.

Definition 4.15. Let $m > 1$ be an integer. We say that integers a and b are **congruent** modulo m, and we write $a \equiv b \pmod{m}$, if $m \mid (a - b)$.

Example 4.16. We have $22 \equiv 7 \pmod{5}$ because $5 \mid (22 - 7)$. Note that 22 and 7 have the same remainder upon division by 5.

Theorem 4.17. *Let a, b, and $m > 1$ be integers. The following conditions are equivalent:*

- $a \equiv b \pmod{m}$,
- $a - b = mq$ *for some integer* q,
- $a = b + mq$ *for some integer* q,
- $a \mod m = b \mod m$.

Theorem 4.18. *(Properties of congruences.) Let a, b, c, d, and $m > 1$ be integers.*

1. *If $a \equiv b \pmod{m}$ and $c \equiv d \pmod{m}$, then $a \pm c \equiv b \pm d \pmod{m}$ and $ac \equiv bd \pmod{m}$.*
2. *If $a \equiv b \pmod{m}$ then $a^c \equiv b^c \pmod{m}$ for $c \ge 0$.*
3. *If $c \ge 1$ and $ac \equiv bc \pmod{cm}$, then $a \equiv b \pmod{m}$.*

4. If $(c, m) = 1$ and $ca = cb \pmod m$, then $a \equiv b \pmod m$.

Theorem 4.19. *(Fermat's theorem.) If a is an integer and p is prime, then $a^p \equiv a \pmod p$.*

Corollary 4.20. *If a is an integer, p is prime, and p does not divide a, then $a^{p-1} \equiv 1 \pmod p$.*

Example 4.21. Let $p = 5$. Then $1^4 \equiv 1 \pmod 5$, $2^4 \equiv 16 \equiv 1 \pmod 5$, $3^4 \equiv 81 \equiv 1 \pmod 5$, $4^4 \equiv 256 \equiv 1 \pmod 5$, etc.

Finally, we recall two useful formulas.

Theorem 4.22. *Let a and b be integers.*

- *For any natural n,*

$$a^n - b^n = (a-b)(a^{n-1} + a^{n-2}b + \cdots + ab^{n-2} + b^{n-1}).$$

- *If n is odd,*

$$a^n + b^n = (a+b)(a^{n-1} - a^{n-2}b + \cdots + (-1)^{n-2}ab^{n-2} + (-1)^{n-1}b^{n-1}).$$

Problems

1. (a) Prove that a natural number is divisible by 9 if and only if the sum of its digits is divisible by 9.
   (b) Prove that if the sum of the digits of a number is 66, then it is not a perfect square (the square of an integer).
2. (a) The number 8^{2460} is written on a board (it contains over 2,000 digits, so we hope that the reader doesn't mind that we didn't write it out here). The sum of its digits is calculated, then the sum of the digits of the result is calculated, and so on, until we get a single digit. What is this digit?
   (b) Same question as in part (a), for the number 2^{1357}.
3. (a) If an integer number is a perfect square, what are the possible values of its units digit?
   (b) Conclude that a number ending with 3 cannot be a perfect square.
4. (a) If c is a perfect square, what are the possible values of its remainder upon division by 4?
   (b) Conclude from part (a) that a number ending with 66 cannot be a perfect square.
   (c) Prove that the sum of two perfect squares cannot end with 123.
   (d) Prove that the sum of the squares of two odd numbers cannot be a perfect square.
5. Does there exist a perfect square that ends with
   (a) 65?
   (b) 1000?
   (c) 120864?

6. Prove that a natural number that consists of 300 ones, some zeros, and no other digits, cannot be a perfect square.

7. (a) Prove that no perfect square has a remainder of 6 when divided by 7.
   (b) How many perfect cubes have a remainder of 6 when divided by 7?

8. Find the remainder when the sum

$$1! + 2! + 3! + \cdots + 99! + 100!$$

   is divided by 12.

9. Show that $2^{457} + 3^{457}$ is divisible by 5.

10. Show that $A = 3^{105} + 4^{105}$ is divisible by 7. Find $A \bmod 11$ and $A \bmod 13$.

11. Show that if the units digit of a natural number n is 3, then $5|(n^2 + 1)$.

12. Prove that if n is an integer, $n^2 + 5$ is not divisible by 11.

13. Show that for any integer n, $6 \mid (n^3 + 5n)$.

14. Prove that for every integer m, the number $\dfrac{m^3 + 3m^2 + 2m}{6}$ is also an integer.

15. For how many natural numbers n is the number $\dfrac{100}{n+1}$ also natural?

16. Find all integers n for which the number $\dfrac{3n + 24}{n+2}$ is also an integer.

17. Prove that the sum of four consecutive odd integers is divisible by 8.

18. Find all four-digit numbers $\underline{aabb}$ that are perfect squares.

19. Find all quadruples of consecutive integers with the product equal to 3024.

20. Prove that if $p > 3$ is prime, then $p^2 \equiv 1 \pmod{24}$.

21. Find $2^{100} \bmod 5$ (that is, find the remainder of 2^{100} upon division by 5).

22. Find the last two digits of 9^{9^9}.

23. Show that if n is composite, then $2^n - 1$ is composite.

24. Show that for any $n \in \mathbb{N}$, 2^n does not divide $n!$.

25. How many zeros are there at the end of 100!?

26. Prove that if a polynomial $P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ with integer coefficients has odd values at $x = 0$ and $x = 1$, then the equation $P(x) = 0$ has no integer roots.

27. Prove that the following equations have no integral solutions.
   (a) $x^2 + 10y = 5xy + 2$
   (b) $x^2 y = xy^2 + 1$
   (c) $x^2 - 3y^2 = 17$

(d) $x^4 - 5xy - y^2 = 3$

28. Find all primes p and q such that $p^2 - 2q^2 = 1$.

29. Find all integral solutions of the following equations.

    (a) $x + y = xy$
    (b) $xyz + x + y + z = xy + xz + yz$
    (c) $x^2 + y^2 + z^2 + 2 = 2x + 2y + 2z$
    (d) $x + y = x^2 - xy + y^2$

30. Prove that in any Pythagorean triple (a, b, c), that is, integer numbers a, b, and c such that $a^2 + b^2 = c^2$, at least one of the numbers is divisible by 5.

31. How many pairs of positive integers are solutions to the equation

    (a) $2x + 3y = 100$?
    (b) $5x + 7y = 1234$?

32. Do there exist integers m and n such that

    (a) $m^2 + 1234567 = n^2$?
    (b) $m^2 + 12345678 = n^2$?

33. Does there exist a positive integer that

    (a) starts with 123
    (b) ends with 123

    and is divisible by 4567? If so, find it.

34. Suppose we write a natural number at each vertex of a cube. Then at the midpoint of each edge we write the sum of the two numbers that are at the ends of this edge. Finally, in the middle of each face we write the sum of the four numbers that are at the vertices of this face. Could the sum of all 26 numbers be equal to 1234?

35. When 3 was appended to a three-digit number on the left (i.e. written before the number), the number increased by 9 times. What was the number?

36. When 6 was appended to a number on the right (i.e. as the rightmost digit), it increased by 13 times. What was the number?

37. When 36 was appended to a number on the right, it increased by 103 times. What was the number?

38. The first digit of a six-number digit number is 1. When this digit 1 is moved from the first digit position to the end so that it becomes the last digit, the new 6 digit number is 3 times larger than the original number. What is the original number?

39. Crookshanks multiplied two 3-digit numbers. Then he covered three of the digits, so the equation looked like

    $$\square 12 \times 30\square = 33,8\square 4.$$

    What is the product of the covered digits?

40. Jerry came up with an encrypted problem:

$$AB - BA = 7.$$

(As always in such puzzles, he wants the same letter to always stand for the same digit, and different letters to stand for different digits.) However, no matter how hard he tried, he couldn't solve it. Explain why.

41. If 792 divides the integer $13xy45z$, find the digits x, y, and z.

42. How many positive three-digit integers have a remainder of 2 when divided by 6, a remainder of 5 when divided by 9, and a remainder of 7 when divided by 11?

43. Prove that the following numbers are irrational.

    (a) $\sqrt[3]{25}$

    (b) $\log_2 5$

44. There is a row of one hundred light bulbs, numbered from 1 to 100. Each one is controlled by an ordinary switch that has two positions: ON and OFF. Initially all the switches are in the OFF position. Cameron turns every switch on. Then he starts from number 1, and turns all the even-numbered switches off. Starting from number 1 again, he changes the position of every switch whose number is divisible by 3 (turns it off if it's on, and turns it on if it's off). Starting from number 1 again, he changes the position of every switch whose number is divisible by 4 (turns it off if it's on, and turns it on if it's off). He repeats this procedure 96 more times, every time changing the switches whose numbers are divisible by 5, 6, and so on, all the way to 100. (So, his last step is to change just the switch with number 100). After he is done, how many lights are on, and what are they?

Chapter 5

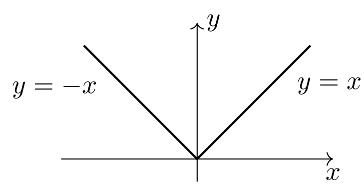
Case analysis

We have seen in previous chapters that some problems can be solved by considering a few different cases. For example, in number theory it was sometimes helpful to consider separately the cases of odd and even integers, or, more generally, consider all possible remainders upon division by a certain number. Below are three more examples of situations where it is helpful to consider two or more cases. As we will see in the future chapters, this technique can be used in many other types problems.

Recall that the absolute value of a real number x is denoted $|x|$ and is defined by

$$|x| = \begin{cases} x & \text{if} \quad x \geq 0, \\ -x & \text{if} \quad x < 0. \end{cases}$$

Here is the graph of $y = |x|$.



To solve problems involving an absolute value, we consider the following two cases: when the expression inside the absolute value is positive or 0, and when it is negative.

Example 5.1. Solve $3|x^2 - 9| - 11x + 7 = 0$.

Solution. *Case I:* $x^2 - 9 \geq 0$*, then* $|x^2 - 9| = x^2 - 9$*, and the equation becomes* $3(x^2 - 9) - 11x + 7 = 0$*, or* $3x^2 - 11x - 20 = 0$*. Using the quadratic formula, we find* $x = 5$ *or* $x = -4/3$*. The root* $x = 5$ *satisfies the condition* $x^2 - 9 \geq 0$*. However,* $x = -4/3$ *does not, so we throw it away.*

Case II: $x^2 - 9 < 0$*, then* $|x^2 - 9| = -(x^2 - 9)$*, and the equation becomes* $-3(x^2 - 9) - 11x + 7 = 0$*, or* $-3x^2 - 11x + 34 = 0$*. The roots are* 2 *and* $-17/3$*, however, the second root does not satisfy the condition* $x^2 - 9 < 0$*, so we throw it away.*

Thus the only roots are 5 *and* 2*.*

If there are several absolute values, we can consider two cases for each absolute value. Note that if there are n absolute values, then we have 2^n cases total. An alternative, often shorter, approach is to find all the points at which the expressions inside absolute values change sign. These points divide the real number line into several intervals (two of which are infinite). Then we can consider each interval separately.

Theorem 5.2. *If a and b are integers, then $a^b = 1$ if and only if at least one of the following holds:*

- $a = 1$,
- $a \neq 0$, $b = 0$,
- $a = -1$ *and b is even.*

Example 5.3. Find all integer values of x for which $(x^2 - 5x + 5)^{x^2-9x+20} = 1$.

Solution. *Let $a = x^2 - 5x + 5$ and $b = x^2 - 9x + 20$. Then we have $a^b = 1$, where $a, b \in \mathbb{Z}$.*

Case I: $a = 1$, then $x^2 - 5x + 5 = 1$. Therefore $x = 1$ or $x = 4$.

Case II: $a \neq 0$, $b = 0$. First solve $b = 0$, i.e. $x^2 - 9x + 20 = 0$. We have $x = 4$ or $x = 5$. Note that for both roots $a \neq 0$.

Case III: $a = -1$ and b is an even integer. First solve $a = -1$, i.e. $x^2 - 5x + 5 = -1$. We have $x = 2$ or $x = 3$. Note that for both roots b is even.

Thus solutions are 1, 2, 3, 4, and 5.

Many logic puzzles can be solved by considering all possible scenarios.

Example 5.4. There are three hats, each with an accompanying statement.

Hat One: "The cat is in this hat."
Hat Two: "The cat is not in this hat."
Hat Three: "The cat is not in Hat One."

Exactly one of the statements is true. Exactly one hat contains a cat. Do we have enough information to deduce which hat contains the cat?

Solution. *Let's consider all possible cases for the cat.*
Case I: the cat is in Hat One. Then the statements of Hat One and Hat Two are both true, which contradicts the given information, so this case is not possible.
Case II: the cat is in Hat Two. Then the statement of Hat Three is the only correct one, thus this case is possible.
Case III: the cat is in Hat Three. Then the statements of Hat Two and Hat Three are both true, so this case is not possible.
Thus the given information is sufficient to state with certainty that the cat is in Hat Two.

Problems

1. Solve the following equations over $\mathbb{R}$.
   (a) $x^2 + |2x - 2| = 1$
   (b) $2x^2 + |4x + 3| = 3$
2. Solve the following equations over $\mathbb{R}$.
   (a) $|2x + 3| - |x| = 3$
   (b) $|2x - 1| - |x + 5| = 3$
   (c) $|3x + 6| + |x - 1| = 2$
   (d) $|x + 1| + |3x + 1| = |2x + 1| + |4x + 1|$

3. Solve the following inequalities over $\mathbb{R}$.
   (a) $x^2 - |5x-6| \le 0$
   (b) $|x+1| + 5 - x^2 > 0$
   (c) $x^2 - |7x+15| \ge 3$
   (d) $x^2 - 2x + |x-1| < 5$
   (e) $x^2 + |x-3| \le 2$
4. Solve the following inequalities over $\mathbb{R}$.
   (a) $|x-5| + |2x-4| \le 6$
   (b) $|x-5| - |x-2| \ge 1$
   (c) $|x-1| - |x-3| > 5$
5. Sketch the graph of the following equations.
   (a) $y = |x^2-4| + 2$
   (b) $y = |x^2-1| - |x^2-4|$
   (c) $y = |x + |x+2||$
   (d) $y = |x^2 - 4|x| + 3|$
6. Sketch the graph of $|x| + |y| = 1 + |xy|$.
7. Sketch the following regions.
   (a) $\{(x,y) \mid |x| + |y^3| < 8\}$
   (b) $\{(x,y) \mid ||x| + |y| - 3| > 1\}$
   (c) $\{(x,y) \mid 2|y-x| + |y+x| \le 1\}$
   (d) $\{(x,y) \mid |x-y| + |x| - |y| \le 2\}$
8. How many ordered pairs of real numbers (x,y) satisfy the following system of equations?

$$\begin{cases} x+3y = 3 \\ \big||x|-|y|\big| = 1 \end{cases}$$

9. Solve the following equations over $\mathbb{Z}$.
   (a) $x^{x^2-7x+12} = 1$
   (b) $(x-3)^{x^2-8x+15} = 1$
   (c) $(x^{x+1})^{x^2} = 1$
   (d) $(5x+2)^{x+3} = 1$
10. Solve the following equations over $\mathbb{Z}$.
    (a) $x^{(x^2)} = x^2$
    (b) $x^{((x+1)^2)} = x^{16}$
    (c) $x^{(x^x)} = (x^x)^x$
    (d) $\sqrt{x^{x+1}} = x^{\sqrt{x+1}}$

11. Solve the following systems over $\mathbb{Z}$.

    (a) $\begin{cases} x^{2x} &= y+1 \\ x^y &= 1 \end{cases}$

    (b) $\begin{cases} x^{x+y} &= y^4 \\ y^{x+y} &= x \end{cases}$

12. Find all integral solutions of the following equations.

    (a) $a^b = 625$

    (b) $(a^b)^c = 64$

13. If exactly one of the following statements is false, which statement is false?

    A. "Statement D is true."
    B. "Statement A is false."
    C. "Statement B is false."
    D. "Statement C is true."

14. Four men are taken to police headquarters following a bank robbery. The police are certain that one of the men is guilty, but they don't know for sure which one. Here's what the four men had to say for themselves:

    Alan: "Bill did it."
    Bill: "Don did it."
    Charlie: "I didn't do it."
    Don: "Bill lied when he said I did it."

    If only one of the four statements is true, who is the guilty man?

15. If an octopus has an even number of legs, it always tells the truth, but if it has an odd number of legs, it always lies.

    One day a green octopus said to a blue one, "I have eight legs. You only have six."
    "I have eight legs. You have just seven," indignantly replied the blue octopus.
    "The blue octopus really does have eight legs," agreed the purple octopus. "I have nine," added he.
    A striped octopus joined in, saying, "None of you have eight legs. I'm the only one who does!"

    Which of these octopuses actually has eight legs?

16. On an Island of Knights and Knaves, there are two kinds of people: knights, who always tell the truth, and knaves, who always lie. One day you meet an islander and they tell you: "I met two fellow islanders yesterday. One of them said that they were both knights, and the other said that they were both knaves." Is this islander a knight or a knave?

17. I told my friend a riddle. First I told her that the product of my three siblings' ages was 36, and the sum of their ages was the day of the month on which her birthday fell. After a bit of thinking, she told me that I haven't given her enough information. So I told her that I had to pick up my oldest sibling from soccer practice. Then she was able to determine their ages. How old are my three siblings?

18. How many ways are there to "paint" all natural numbers, either red or green each, so that the sum of any two distinct red numbers is red and the sum of any two distinct green numbers is green?

Chapter 6

Finding a pattern

Sometimes (e.g. in problems such as number 18 in chapter 2), a formula can be guessed after computing a few values. Although a guess is not sufficient and a rigorous proof is needed to ensure correctness of a formula, guessing often is a powerful technique. In this chapter we will consider problems in which we can find the first few values of a certain sequence, guess a formula for a general case, and then prove it (e.g. using Mathematical Induction or some other proof technique).

Example 6.1. Guess a closed formula for the n-th term of the sequence

$$1,\ 3,\ 6,\ 10,\ 15,\ 21,\ \ldots.$$

(A closed formula means that the n-th term of the sequence is expressed in terms of n and not in terms of the previous terms of the sequence.)

Solution. *Notice that the difference between the first and the second terms is 2, the difference between the second and the third terms is 3, and then the differences are 4, 5, 6, and so on. Thus*

$$\begin{aligned} a_1 &= 1, \\ a_2 &= 1+2, \\ a_3 &= 1+2+3, \\ a_4 &= 1+2+3+4, \\ a_5 &= 1+2+3+4+5, \\ a_6 &= 1+2+3+4+5+6. \end{aligned}$$

It appears that $a_n = 1+2+\cdots+n = \dfrac{n(n+1)}{2}$.

Note. For problems like this one, since only a few terms of a sequence are given, there are many different ways to continue the sequence, thus there are many different formulas valid for these few terms. For example, $a_n = 2^n - \frac{1}{60}n^5 + \frac{1}{6}n^4 - \frac{11}{12}n^3 + \frac{7}{3}n^2 - \frac{77}{30}n$ is another valid formula for this sequence. Note that it gives $a_7 = 30$ while the formula we have in the above solution gives $a_7 = 28$, so these two formulas really define different sequences. Above, we tried to find a simple one. However, in problems such as in the next example and all problems except the first one in this chapter, it is important to prove the formula that we guess based on our observation.

Example 6.2. Find a closed formula for the n-th derivative of $f(x) = 5^x$.

Solution. *Find the first few derivatives (until you can see a pattern):*

$$\begin{aligned} f'(x) &= \ln 5 \cdot 5^x, \\ f''(x) &= \ln 5 \cdot \ln 5 \cdot 5^x = (\ln 5)^2 \cdot 5^x, \\ f'''(x) &= (\ln 5)^2 \cdot \ln 5 \cdot 5^x = (\ln 5)^3 \cdot 5^x. \end{aligned}$$

We notice that $f^{(n)}(x) = (\ln 5)^n \cdot 5^x$. *It is easy to prove this formula using Mathematical Induction. The basis step is* $f'(x) = \ln 5 \cdot 5^x$. *For the inductive step, suppose* $f^{(k)}(x) = (\ln 5)^k \cdot 5^x$ *is true. Then*

$$f^{(k+1)}(x) = (f^{(k)}(x))' = ((\ln 5)^k \cdot 5^x)' = (\ln 5)^k \cdot \ln 5 \cdot 5^x = (\ln 5)^{k+1} \cdot 5^x.$$

Problems

1. Guess a closed formula for the n-th term of the sequence $a_1, a_2, a_3, \dots$ whose first few terms are given. Try to find the simplest possible formula, but any correct formula (that is, any formula that works for the given terms) is acceptable.
   (a) 1, 4, 9, 16, 25, 36, 49, ...
   (b) -1, 0, 1, 2, 3, 4, 5, ...
   (c) 5, 7, 9, 11, 13, 15, ...
   (d) 8, 10, 12, 14, 16, 18, ...
   (e) $3, 1, -1, -3, -5, -7, \dots$
   (f) 1, 2, 1, 4, 1, 6, 1, 8, ...
   (g) 1, 3, 4, 6, 7, 9, 10, 12, ...
   (h) 0, 1, 3, 7, 15, 31, ...
   (i) $\frac{1}{2}, \frac{1}{2}, \frac{3}{8}, \frac{1}{4}, \frac{5}{32}, \frac{3}{32}, \frac{7}{128}, \dots$

2. Compute $S_n = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \cdots + \frac{1}{(n-1)n}$ for some small values of n. Notice the pattern. Write a closed formula for S_n and prove it using Mathematical Induction.

3. Find a closed formula for $\frac{1}{1\cdot 3} + \frac{1}{3\cdot 5} + \cdots + \frac{1}{(2n-1)(2n+1)}$.

4. Find a closed formula for $\frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \cdots + \frac{n-1}{n!}$.

5. Find a closed formula for
$$\prod_{i=1}^{2n-1}\left(1 - \frac{(-1)^i}{i}\right) = \left(1 - \frac{-1}{1}\right)\left(1 - \frac{1}{2}\right)\left(1 - \frac{-1}{3}\right)\cdots\left(1 - \frac{-1}{2n-1}\right).$$

6. How many binary sequences (i.e. consisting of 0s and 1s) of length n have no consecutive 0s?

7. Let the sequence $\{a_n\}$ be defined by $a_1 = 1$, $a_2 = 2$, and $a_n = a_{n-1} + 2a_{n-2}$ for $n \geq 3$. Find a closed formula for a_n.

8. Let the sequence $\{a_n\}$ be defined by $a_1 = 1$, $a_2 = 3$, and $a_n = 4a_{n-1} - 3a_{n-2}$ for all $n \geq 3$. Find a closed formula for a_n.

9. Let $f_1(x) = 2x + 1$ and $f_n = f_1 \circ f_{n-1}$ for $n \geq 2$. Find a closed formula for $f_n(x)$.

10. Let $f_1(x) = \dfrac{1}{2-x}$ and $f_n = f_1 \circ f_{n-1}$ for $n \geq 2$. Find a closed formula for $f_n(x)$.

11. The units digit of a number a^b can be found by computing the units digits of the first few powers of a, i.e. a^1, a^2, a^3, etc. and noticing a pattern. Find the units digit of the following numbers.

    (a) 107^{107}

    (b) 1234^{5678}

12. Find the last two digits of 7^{50}.

13. Find the remainder of 2^{100} upon division by 12.

14. Find the remainder of 5^{4321} upon division by 11.

15. Find the n-th derivative of

    (a) $f(x) = \sin(x)$,

    (b) $g(x) = \ln(x)$,

    (c) $h(x) = 2e^{5x}$.

16. Suppose that n lines in general position are given in a plane. (General position means that no two lines are parallel, and no three lines have a common point.) Into how many regions do they divide the plane?

17. Suppose that n circles are given in a plane, such that every pair of circles has 2 intersection points, but no 3 circles have a common point. Into how many regions do they divide the plane?

18. Suppose we are given ten planes in a general position (i.e. no two are parallel, no three are parallel to the same line, no four have a common point). Into how many (3-dimensional) regions do they divide $\mathbb{R}^3$?

19. Let $F_0 = 0$, $F_1 = 1$, $F_2 = 1$, $\ldots$, F_{99} be the first 100 Fibonacci numbers (recall that $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$).

    (a) How many of them are even?

    (b) How many of them are divisible by 3?

    (c) How many of them are divisible by 5?

20. Amanda is training her rabbit to climb a flight of 10 steps. The rabbit can hop up 1 or 2 steps each time he hops. He never hops down, only up. In how many different ways can he hop up the flight of 10 steps?

Chapter 7

Working backwards

"Working backwards" is a very powerful tool that can be used to solve many different problems.

Theorem 7.1. *Let a and b be nonzero integers, and let $d = \gcd(a, b)$. Then there exist integer numbers x and y such that $x \cdot a + y \cdot b = d$. When we find such integers x and y, we say that we wrote d as a linear combination of a and b.*

Euclid's algorithm. Given integers a and b, where $b \neq 0$, we can divide a by b and obtain quotient q and remainder r. Notice that since $a = qb + r$, the greatest common divisor of a and b is equal to the greatest common divisor of b and r. Euclid's algorithm of finding $\gcd(a, b)$ and expressing it as a linear combination of a and b is based on this fact:

$$
\begin{array}{lll|l}
a = q_1 \cdot b + r_1, & r_1 < b, & (a, b) = (b, r_1), & r_1 = a - q_1 \cdot b, \\
b = q_2 \cdot r_1 + r_2, & r_2 < r_1, & (b, r_1) = (r_1, r_2), & r_2 = b - q_2 \cdot r_1, \\
r_1 = q_3 \cdot r_2 + r_3, & r_3 < r_2, & (r_1, r_2) = (r_2, r_3), & r_3 = r_1 - q_3 \cdot r_2, \\
\ldots \downarrow & \ldots & \ldots & \ldots \uparrow \\
r_{n-2} = q_n \cdot r_{n-1} + r_n, & r_n < r_{n-1}, & (r_{n-2}, r_{n-1}) = (r_{n-1}, r_n), & r_n = r_{n-2} - q_n \cdot r_{n-1}, \\
r_{n-1} = q_{n+1} \cdot r_n, & \text{rem.} = 0, & (r_{n-1}, r_n) = r_n. &
\end{array}
$$

Thus $\gcd(a, b) = r_n$. Now we can work backwards using the equations on the right to express r_n as a linear combination of a and b.

Example 7.2. Find the greatest common divisor d of $a = 115$ and $b = 80$, and find x and y such that $x \cdot a + y \cdot b = d$.

Solution. *First perform divisions to find the greatest common divisor:*

$$
\begin{aligned}
115 &= 1 \cdot 80 + 35, \\
80 &= 2 \cdot 35 + 10, \\
35 &= 3 \cdot 10 + 5, \\
10 &= 2 \cdot 5.
\end{aligned}
$$

Therefore $\gcd(a, b) = 5$.

Note. To find $\gcd(a, b)$*, we could simply factor* $115 = 5 \cdot 23$, $80 = 2^4 \cdot 5$*, so* $\gcd(115, 80) = 5$*. However, we need the above divisions to find the desired linear combination:*

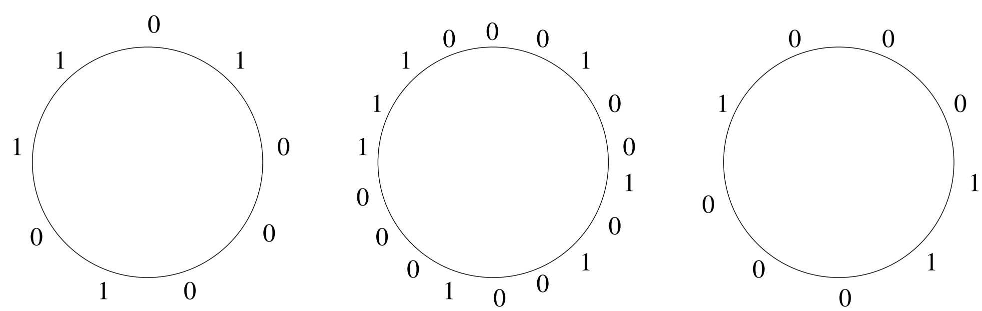
$$\begin{aligned} 5 &= 35 - 3 \cdot 10, \\ 10 &= 80 - 2 \cdot 35, \\ 35 &= 115 - 1 \cdot 80, \end{aligned}$$

so

$$\begin{aligned} 5 &= 35 - 3 \cdot 10 \\ &= 35 - 3(80 - 2 \cdot 35) = 35 - 3 \cdot 80 + 6 \cdot 35 = 7 \cdot 35 - 3 \cdot 80 \\ &= 7(115 - 1 \cdot 80) - 3 \cdot 80 = 7 \cdot 115 - 7 \cdot 80 - 3 \cdot 80 = 7 \cdot 115 - 10 \cdot 80. \end{aligned}$$

Thus $x = 7$ and $y = -10$.

Example 7.3. Suppose four 1s and five 0s are written along a circle. Between every two equal numbers we write 1 and between two distinct numbers we write 0. Then the original numbers are wiped out. For example, a possible initial distribution of ones and zeros and the first step are shown below:



This step is repeated indefinitely. Show that we can never reach nine 1s.

Solution. *Suppose that nine 1s are attainable. Look at the first time we have nine 1s. One step before that we must have nine equal numbers. Since it was the first time we got nine 1s, one step before we must have nine 0s. Still one step before we have nine changes from 0 to 1 or from 1 to 0. In other words, the 0s and 1s are alternating. However, with an odd number of integers, this is not possible. We have a contradiction.*

Example 7.4. Two players play the following game. There are initially 10 counters in a pile. The players take turns removing 1 or 2 counters. The game ends when all counters have been removed. The player who takes the last counter loses. Find a winning strategy for one of the players.

Solution. *We do not want to take the last counter as then we would lose. On our last turn we want to leave one counter so that to force our opponent to take it. We will call leaving one counter a good position. We would not want to leave two counters after our move as then our opponent can take one and get into a winning positions. We will call leaving two counters a bad position. Similarly, we would not want to leave three counters after our move as our opponent can take two and leave just one. So leaving three counters is also a bad position. However, leaving four counters is a good position: our opponent can take one or two, leaving three or two, respectively, and we already know that these are bad positions (and we want to force our opponent to go to such). So leaving four counters is a good position. Continuing in this manner, we will find that 5 and 6 are bad positions, 7 is good, 8 and 9 are bad, and 10* *is good. This means that we want to go second. Our opponent will make the first move and be forced to go to a bad position. Then, we make sure that we go to a good position on every move.*

Note. In the above game, the initial number of counters was rather low and it was quick to classify all possible positions. If the initial number was higher, we would want to find a more general approach. Often, we can work out a few positions and notice a pattern. In the above case, notice that no matter how our opponent plays, we can always play in such a way that the number of counters our opponent takes plus the number of counters we take is equal to 3 (namely, if they take 1, we can take 2; if they take 2, we can take 1). Since we want to leave just one counter in the end, we should always leave a number that is congruent to 1 modulo 3. Since the initial amount, 10, is congruent to 1 modulo 3, we want to go second.

Problems

1. Use Euclid's algorithm to find the greatest common divisor d of the given numbers a and b, and find numbers x and y such that $xa + yb = d$.

   (a) $a = 46$, $b = 32$

   (b) $a = 24$, $b = 10$

   (c) $a = 96$, $b = 54$

   (d) $a = 219$, $b = 51$

2. Find integer numbers a and b such that $6 = 67a + 25b$.

3. Find a and b such that in Euclid's algorithm $r_7 = (a, b)$. Write out all the divisions.

4. Find $a > 1000$ and $b > 1000$ such that it will take exactly 5 divisions to reach the greatest common divisor of a and b.

5. Find positive a and b such that it will take at least 5 divisions to reach the greatest common divisor of a and b with the smallest possible number a and $b < a$.

6. In the sequence a_n, each term starting with the third is the sum of the two previous terms. Find a_1 and a_2 such that $a_5 = 17$ and $a_7 = 76$.

7. Consider the function $f : \mathbb{Z} \to \mathbb{Z}$ defined by

$$f(x) = \begin{cases} \dfrac{x}{2} & \text{if } x \text{ is even,} \\ 3x + 1 & \text{if } x \text{ is odd.} \end{cases}$$

   Find all values of x for which $f^5(x) = 1$.

8. Consider the function $f : \mathbb{Z} \to \mathbb{Z}$ defined by

$$f(x) = \begin{cases} 2x - 11 & \text{if } x \text{ is even,} \\ 3x - 11 & \text{if } x \text{ is odd.} \end{cases}$$

   Prove that there is no value of x for which $f^5(x) = 141$.

9. The integers 1, 2, $\ldots$, n are placed in order, so that each value is either bigger than all preceding values or is smaller than all preceding values. In how many ways can this be done?

10. Consider all sequences $\{a_1, a_2, a_3, \ldots, a_7\}$ that have the following property: the difference between any two consecutive terms is 1, that is, $a_{k+1} = a_k \pm 1$. How many sequences with this property have $a_7 = 10$?

11. Starting with 1, 2, 3, 4, we construct the sequence 1, 2, 3, 4, 0, 9, 6, $\ldots$, where each new digit is the mod 10 sum of the preceding four terms. Prove that the 4-tuple 0, 5, 0, 5 will never occur.

12. Starting with 1, 2, 3, 4, we construct the sequence 1, 2, 3, 4, 0, 9, 6, $\ldots$, where each new digit is the mod 10 sum of the preceding four terms. Will the 4-tuple 0, 6, 5, 0 ever occur?

13. Two players play the following game. There are initially 27 counters in a pile. The players take turns removing 1, 2, 3, or 4 counters. The game ends when all counters have been removed. The player who takes the last counter loses. Find a winning strategy for one of the players.

14. Two players play the following game. There are initially 27 counters in a pile. The players take turns removing 1, 2, 3, or 4 counters. The game ends when all counters have been removed. The player who takes the last counter wins. Find a winning strategy for one of the players.

15. Two players play the following game. There are initially 10 counters in a pile. The players take turns removing 1, 2, or 4 counters. The game ends when all counters have been removed. The player who takes the last counter wins. Find a winning strategy for one of the players.

16. Two players play the following game. There are initially 15 counters in a pile. The players take turns removing 2, 3, or 4 counters. The game ends when a player cannot make a move. The player who made the last move wins. Find a winning strategy for one of the players.

17. Two players play the following game. There are initially 50 counters in a pile. The players take turns removing 1, 2, 4, 8, 16, or 32 counters. The game ends when all counters have been removed. The player who takes the last counter wins. Find a winning strategy for one of the players.

18. Two players play the following game. Initially $X = 0$. The players take turns adding any number between 1 and 10 (inclusive) to X. The game ends when X reaches 100. The player who reaches 100 wins. Find a winning strategy for one of the players.

19. Two players play the following game. A box initially contains 300 matches. The players take turns removing some (more than zero) but no more than half of the matches in the box. The player who cannot take any matches loses. Find a winning strategy for one of the players.

20. Two players play the following game. There are two piles of candy. Initially one pile contains 20 pieces, and the other 21. The players take turns eating all of the candy in one pile and separating the remaining candy into two (not necessarily equal) piles. (A pile may have 0 candies in it.) The player who cannot eat a candy on his/her turn loses. Find a winning strategy for one of the players.

21. Two players play the following game. The number 55 is written on the board. The players take turns subtracting from the number on the board any of its positive divisors, and replacing the original number with the result of this subtraction. The player who writes the number 0 loses. Find a winning strategy for one of the players.

22. The game begins with the number 1000. Two players take turns subtracting from the current number any natural number that is a power of 2 and less than the current number. (Note that 1 is a power of 2, since $1 = 2^0$.) The player who reaches the number 1 wins. Find a winning strategy for one of the players.

23. The game begins with the number 2. Two players take turns adding to the current number any natural number smaller than it. The player who reaches the number 1000 wins. Find a winning strategy for one of the players.

24. Initially $X = 1$. Two players take turns multiplying X by any whole number from 2 to 9 (inclusive). The player who first names a number greater than 1000 wins. Find a winning strategy for one of the players.

25. A counter is placed in the upper right corner of a 7×7 board. Two players take turns moving the counter. It can be moved only one square to the left, one square down, or diagonally one to the left and one down. The winner is the player who puts the counter in the bottom left corner. Determine all winning positions for this game (that is, the "good" positions: the positions to which you would want to move the counter on your move). Find a winning strategy for one of the players.

26. A counter is placed in the upper right corner of a 7×8 board. Two players take turns moving the counter. It can be moved any number of spaces to the left, down, or diagonally to the left and down. The winner is the player who puts the counter in the bottom left corner. Determine all winning positions for this game (that is, the positions to which you would want to move the counter on your move). Find a winning strategy for one of the players.

27. A counter is placed in one of the squares of an 8×8 board. Two players take turns moving the counter. It can be moved one space to the left, or one space down, or any number of spaces diagonally to the left and down. The winner is the player who puts the counter in the bottom left corner. Determine all winning positions for this game (that is, the positions to which you would want to move the counter on your move). Which player (first or second) has a winning strategy, depending on where the counter is placed initially?

28. A counter is placed in one of the squares of an 8×8 board. Two players take turns moving the counter. It can be moved any number of spaces to the left, any number of spaces down, or just one space diagonally to the left and down. The winner is the player who puts the counter in the bottom-left corner. Determine all winning positions for this game (that is, the positions to which you would want to move the counter on your move). Which player (first or second) has a winning strategy, depending on where the counter is placed initially?

29. There are two piles of counters, both containing 6 counters. Two players take turns taking either one counter from one of the piles, or one counter from each pile. The player who cannot make a move loses. Determine all winning positions for this game (i.e. how many counters you would want to leave in each pile after your move) and which player has a winning strategy.

30. There are two piles of counters. One contains 6 counters and the other one contains 7 counters. Two players take turns taking either any number of counters from one of the piles, or an equal number from each pile. The player who cannot make a move loses. Determine all winning positions for this game (i.e. how many counters you would want to leave in each pile after your move) and which player has a winning strategy.

31. There are two piles of 7 counters each, pile A and pile B. Two players take turns taking either a single counter from one of the piles, or one counter from each pile, or taking one counter from pile A and placing it in pile B. The player who cannot move loses. Determine all winning positions for this game (i.e. how many counters you would want to leave in each pile after your move) and which player has a winning strategy.

32. A regular 6-sided die, in which the sum of the numbers on any two opposite faces is 7, is rolled. The game begins with the number on the top face. Two players take turns turning the die to the side and adding the number on the new top face to the previous sum. Note: the die must be turned at each move, but only by 90 degrees (it cannot be turned over). The player who first obtains a sum larger than 31 loses. Determine all winning positions for this game and which player has a winning strategy.

33. Suppose you are writing a set of calculus exercises. You want to find a few cubic polynomials $f(x) = ax^3 + bx^2 + cx + d$ (preferably with integer coefficients) whose critical numbers are integers. (Recall that a critical number is a value of x at which the derivative is equal to 0.) How would you find such polynomials? Use your strategy to find a couple of polynomials.

34. Suppose you want to give your high school students a system of two linear equations with two variables. You would like the system to have one solution which is a pair of integers. You could, of course, try random coefficients, say

$$\begin{cases} 2x + 3y = 4 \\ 5x - 6y = 7 \end{cases},$$

solve your systems, and hope that sooner or later you will find a system with one integer pair solution, but is there a better strategy?

35. Suppose you are teaching linear algebra, and you need to find matrices with integer entries whose reduced echelon forms also have integer entries. How would you find such matrices?

Chapter 8

Invariants

Definition 8.1. An **invariant** is a quantity or a property that does not change.

Example 8.2. The numbers $1, 2, \ldots, 10$ are written on a board. We pick any two numbers, let us call them a and b. We erase these numbers, and write $a + 1$ and $b - 1$ instead. Is it possible to get ten 5s by a sequence of such operations?

Solution. *Notice that when we increase* a *by* 1 *and decrease* b *by* 1*, their sum does not change. Therefore, the sum of all ten numbers does not change (so in this example the sum of the ten numbers is an invariant). Initially the sum is* $1 + 2 + \cdots + 10 = 55$*, however,* $10 \cdot 5 = 50$*, therefore it is not possible to get ten 5s.*

Here are a few quantities that are very often invariants in problems involving sets of numbers and allowed operations, so you may want to try to look at them. Sometimes, of course, you have to be very creative!

- The sum or the product of all given numbers.
- The number of positive or negative numbers.
- The number of even or odd numbers, or, more generally, the number of numbers congruent to a modulo b for some integers a and b.
- One of the above modulo a positive number (e.g. the sum modulo 2, i.e. the parity of the sum; the product modulo 3; the number of positive numbers modulo 4; etc.).

Sometimes it is useful to introduce a change even if none was given in the problem itself.

Example 8.3. Each of the numbers a_1, a_2, $\ldots$, a_n is either 1 or -1, and

$$a_1a_2a_3a_4 + a_2a_3a_4a_5 + \cdots + a_{n-3}a_{n-2}a_{n-1}a_n + a_{n-2}a_{n-1}a_na_1 + a_{n-1}a_na_1a_2 + a_na_1a_2a_3 = 0.$$

Prove that $4 \mid n$.

Solution. *Let*

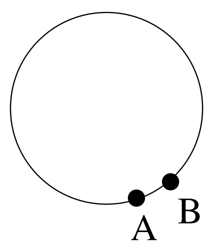
$$S = a_1a_2a_3a_4 + a_2a_3a_4a_5 + \cdots + a_{n-3}a_{n-2}a_{n-1}a_n + a_{n-2}a_{n-1}a_na_1 + a_{n-1}a_na_1a_2 + a_na_1a_2a_3.$$

Observe that if we replace a_i *by* $-a_i$*, then* S *does not change modulo* 4 *since four terms (the ones containing* a_i*) change their sign. Indeed, if all four terms have the same sign, then their sum changes either from* -4 *to* 4 *or from* 4 *to* -4*, thus* S *changes by* ± 8*. If one or three* *of these four terms are positive, then the sum of the four terms changes either from* -2 *to* 2 *or from* 2 *to* -2*, thus* S *changes by* ± 4*. Finally, if two of these four terms are positive and two are negative, then the sum does not change. Initially, we have* $S = 0$ *which implies* $S \equiv 0 \pmod 4$*. Now, step-by-step, we can change each* -1 *into a* 1*. At the end, we have* $S = n$*, and we must still have* $S \equiv 0 \pmod 4$*, so* $4 \mid n$*.*

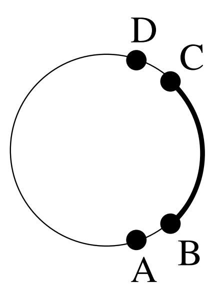
Sometimes a quantity that may change is useful as well. Especially if it can change only in a certain way, say, it can only increase or decrease. A quantity that only changes monotonically is called a monoinvariant. For instance, in the example below we find a positive decreasing function rather than a constant function. The idea is that the value of that function must be nonnegative. We apply a series of steps each of which decreases the value of the function. Since the value cannot become negative, sooner or later it will reach 0.

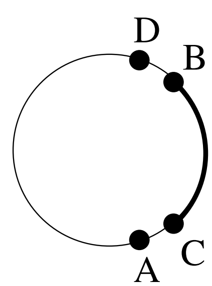
Example 8.4. Suppose that $2n$ ambassadors are invited to a banquet. Every ambassador has at most $n-1$ enemies. Prove that the ambassadors can be seated around a round table so that nobody sits next to their enemy.

Solution. *First, we seat the ambassadors randomly. Let* H *be the number of neighboring hostile couples. We want to find an algorithm which reduces this number whenever* $H > 0$*. Let* $\{A, B\}$ *be a hostile couple with* B *sitting to the right of* A*:*



We want to separate them so as to not gain any new neighboring hostile couples. This can be achieved by reversing some arc BC *as shown below. The number* H *will be reduced if* $\{A, C\}$ *and* $\{B, D\}$ *are friendly couples.*





It remains to be shown that such a couple $\{C, D\}$ *always exists. We start at* A *and go around the table counterclockwise. We will encounter at least n friends of* A*. To their right, there are at least n seats. They cannot all be occupied by enemies of* B *since* B *has at most* $n-1$ *enemies. Thus there is a friend* C *of* A *with the right neighbor* D *being a friend of* B*.*

Problems

Some problems below deal with a list (e.g. of numbers). By a list we mean an unordered collection, with repetition allowed.

1. We start with the list $\{-3,-2,-1,1,2,3\}$. In each step we may choose any two of these numbers and change their signs. We may repeat this step as many times as we want. Show that it is not possible to reach the list $\{3,2,1,1,2,3\}$.

2. We start with the list $\{-3,-2,-1,1,2,3\}$. In each step we may multiply or divide any of these numbers by any positive number. We may repeat this step as many times as we want. Show that it is not possible to reach the list $\{-2,-1,1,2,3,4\}$.

3. We start with the list $\{1,1,1,1\}$. In each step we may either multiply one of the numbers by 3, or subtract 2 from it. We may repeat this step as many times as we want. Is it possible to reach the list $\{1,2,3,4\}$?

4. We start with the list $\{1,4,32,128,256\}$. In each step we may divide one number by 2 and multiply another number by 2. We may repeat this step as many times as we want. Is it possible to reach the list $\{512,32,16,16,2\}$?

5. Initially 1 is written in every cell of a 5×5 table. In each step we may change the signs of the numbers in any two adjacent cells. We may repeat this step as many times as we want. Is it possible to make all of the numbers -1?

6. We start with the list $\{1,2,3,4,5,6\}$. In each step we may either add 2 to any 5 numbers or subtract 1 from any 5 numbers. We may repeat this step as many times as we want. Can we reach the list $\{1,2,4,8,16,32\}$?

7. We start with the list $\{1,2,3,4\}$. In each step we may either add 2 times one of the numbers to another number or subtract 2 times one number from another number. For example, we may replace 1 by $1+2\cdot 2$, or by $1-2\cdot 2$, or by $1+2\cdot 3$, etc. We may repeat this step as many times as we want. Can we reach the list $\{10,20,30,40\}$?

8. We start with the list $\{1,2,3,5,8,13\}$. At each step, we may choose any two or three numbers from this list and replace each of them by their average. For example, we could choose 2, 3, and 13, and replace each of them by $\dfrac{2+3+13}{3}=6$ to obtain $\{1,5,6,6,6,8\}$. We may repeat this step as many times as we want. Is it possible to obtain the list $\{5,5,5,5,5,5\}$?

9. Six stacks of coins are placed on the table. The first has 1 coin, the second 2, and so on, until the last, which has 6 coins. We are allowed to select any two stacks and add one coin to each. We may repeat this operation as many times as we want. Can we make all the stacks equal?

10. We start with the table shown below. In one step we may either add 1 to all the numbers in any row or column, or subtract 1 from all the numbers in any row or column. We may repeat this step as many times as we want.

0	0	0
0	1	0
0	0	0

   (a) Prove that it is not possible to reach nine 0s.
   (b) Prove that it is not possible to reach nine odd numbers.

11. There are several $+$ and $-$ signs written on a board. We may erase any two signs and write, instead, $+$ if they are equal and $-$ if they are unequal. We do this until only one sign remains. Prove that the last sign on the board does not depend on the order of erasure.

12. Assume we have an 8×8 chessboard with the usual coloring. We may switch the colors of all the squares in any row or column. We may repeat this step as many times as we want. The goal is to attain just one black square. Can we reach the goal? What if we are allowed to switch the colors of all the squares in any 2×2 square?

13. Each of the numbers from 1 to 1,000,000 is repeatedly replaced by its digital sum until we reach 1,000,000 single-digit numbers. For example, 987654 is replaced by 39 (since $9+8+7+6+5+4=39$), then 39 is replaced by 12 (since $3+9=12$), and finally, 12 is replaced by 3 (since $1+2=3$). Among these 1,000,000 single-digit numbers, will we have more 1s or 2s?

14. We may write all the digits from 1 to 9 in a row in any order we like, and then we write plus signs between some digits (as many plus signs as we like). For example, we could write $7+35+19+4+286$. Finally, we evaluate the obtained expression. Prove that there is no way to get the value of 100. Or 101. Or 102. Or 103... What is the smallest three-digit number that can be obtained in this process?

15. We compute and write out the number 7^{1234}. Then we strike its first digit and add it to the remaining number. Suppose that this step is repeated until a number with ten digits remains. Prove that this number has at least two equal digits.

16. Alice and Bob have a large chocolate bar in the shape of a rectangular grid. They play a game where Alice goes first, then Bob, and then their turns alternate. At each turn a player may either eat an entire bar of chocolate or break any chocolate bar into two smaller rectangular chocolate bars along a grid line. The player who moves last loses. Who wins this game?

17. Let n be any positive integer. We start with the integers $1, 2, \ldots, 4n-1$. In each step we may replace any two integers by their difference. We do this until only one number remains. Prove that an even integer will be left after $4n-2$ steps.

18. Let n be an odd positive integer. We start with the numbers $1, 2, 3, \ldots, 2n$. In each step we may replace any two integers by their difference. We do this until only one number remains. Prove that an odd number will remain at the end.

19. The numbers from 0 to 9 are written along a circle in random order. Between every two neighboring numbers a and b (in the clockwise order) we write $2b-a$. Then we erase the original ten numbers. (For example, the numbers could be written in the following order: 1, 5, 3, 9, 0, 2, 4, 6, 8, 7. Then the new numbers would be 9, 1, 15, -9, 4, 6, 8, 10, 6, -5.) This step can be repeated as many times as we want. Show that it is not possible to reach ten 5s.

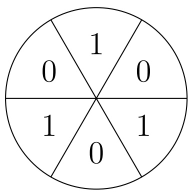
20. We start with the list $\{1, 3, 6\}$. In each step we may choose any two of these numbers, let's call them a and b, and replace them by $0.6a - 0.8b$ and $0.8a + 0.6b$. We may repeat this step as many times as we want. Can we reach the list $\{2, 4, 5\}$?

21. We arrange the integers $1, 2, 3, 4, 5, 6$ in any order on six places numbered 1 through 6. Next we add each number to its place. Prove that at least two of the sums have the same remainder upon division by 6.

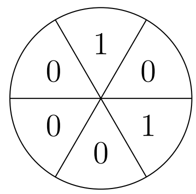
22. There are seven 1s and eight -1s on a board. In each step we may erase any two numbers, say, a and b, and write $-ab$ instead. We do this until only one number remains. Show that no matter in what order we erase the numbers, 1 will remain in the end.

23. A circle is divided into six sectors. Some 0s and 1s are written into the sectors as shown below. We may increase any two neighboring numbers by 1. We may repeat this step as many times as we want. Is it possible to equalize all the numbers?

(a)



(b)



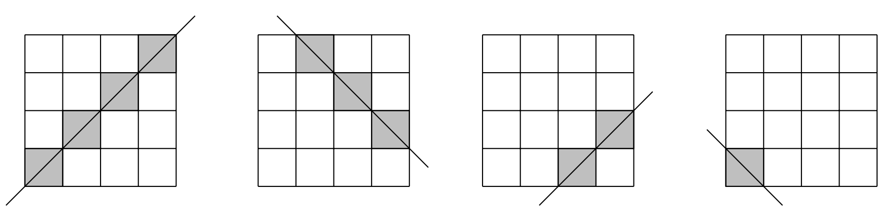
24. In each table below, we may switch the signs of some numbers as indicated. We may repeat this step as many times as we want. Prove that at least one -1 will always remain in the table.

   (a) We may switch the signs of all the numbers in any row, any column, or any diagonal.

-1	1	-1	1
1	1	1	1
-1	1	-1	1
1	1	1	1

   (b) We may switch the signs of all the numbers in any row, any column, or on any line parallel to a diagonal (see pictures below; note that in particular, we may switch the sign of each corner square).

-1	1	-1	1
1	1	1	1
-1	1	-1	1
1	1	-1	1



Examples of lines parallel to a diagonal.

25. There are a white, b black, and c red chips on a table. In one step we may choose two chips of different colors and replace them by one chip of the third color. We do this until all remaining chips are of the same color. If only one chip remains, prove that its color does not depend on the evolution of the game, but it only depends on the numbers a, b, and c.

26. (a) On the island of Camelot live 12 gray, 17 green, and 19 crimson chameleons. If two chameleons of different colors meet, they both simultaneously change color to the third color (e.g., if a gray chameleon and a green chameleon meet each other, they both change to crimson). Is it possible that at some point they all become gray? How about all green? All crimson? Is it possible that there will eventually be the same numbers of gray, green, and crimson chameleons?

    (b) Same questions as above for 13 gray, 17 green, and 19 crimson chameleons.

27. There is a mob of 3000 people, one of whom is a zombie while the other 2999 are not. Every minute, all 3000 people form 1000 groups of three. If there are any zombies in a group of three, all three end up zombies. What is the probability that after five minutes there will be exactly 100 zombies in the mob?

28. There are several letters a, b, and c on a board. We may replace: two a's by one b, two b's by one a, two c's by one c, an a and a b by one c, an a and a c by one a, or a b and a c by one b. We do such replacements until only one letter remains. Prove that this last letter does not depend on the order of erasure.

29. A dragon has 100 heads. A knight can cut off 5, 15, 17, or 20 heads, respectively, with one blow of his sword. In each of these cases 17, 24, 2, or 14, new heads grow on its shoulders, respectively. If all heads are blown off, the dragon dies. Can the dragon ever die?

30. There is an integer in each square of an 8×8 board. In one move, we may choose any 4×4 or 3×3 square and add 1 to each integer in the chosen square. Can we always get a board with each entry being even?

31. (a) In a regular pentagon all diagonals are drawn. Initially each vertex and each point of intersection of the diagonals is labeled by the number 1. In one step it is permitted to change the signs of all numbers on a side or a diagonal. Is it possible to change all labels to -1 by a sequence of such steps?

    (b) Same problem for a regular hexagon.

32. We start with the list of numbers $\{1, 5, 25, 125, 625\}$. In each step, we may choose any two of these numbers, let's call them a and b, and replace them either by $a+1$ and $b-1$, or by $a+3$ and $b-2$. We may repeat this step as many times as we want. Is it possible to obtain the list $\{145, 150, 155, 160, 165\}$?

33. Nine 1×1 cells of a 10×10 square are infected. Two cells are called neighbors if they have a common side. In one time unit, the cells with at least two infected neighbors become infected. Can the infection spread to the whole square (in any amount of time)?

34. Twelve 1×1 cells of a 10×10 square are infected. Two cells are called neighbors if they share at least one vertex (thus an inner cell has 8 neighbors). In one time unit, the cells with at least four infected neighbors become infected. Can the infection spread to the whole square (in any amount of time)?

35. In the Parliament of Sikinia each member has at most three enemies. Prove that the parliament can be separated into two houses so that each member has at most one enemy in their own house.

Chapter 9

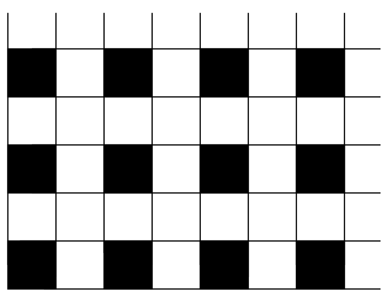
Coloring

Example 9.1. In 1961, the British theoretical physicist M. E. Fisher solved a famous and very tough problem. He showed that an 8×8 chessboard can be covered by 2×1 dominoes (with no dominoes overlapping or overhanging) in $2^4 \cdot 901^2 = 12{,}988{,}816$ ways. Now let us cut out two diagonally opposite corners of the board. In how many ways can we cover the 62 squares of the mutilated chessboard with 31 dominoes?

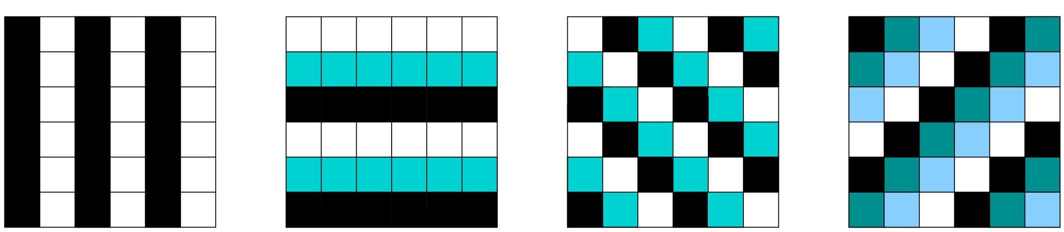
Solution. *Zero. There is no way to cover the mutilated chessboard. Each domino covers one black and one white square. If a covering of the board existed, the 31 dominoes would cover* 31 *black and* 31 *white squares. However, the mutilated chessboard has* 30 *squares of one color and* 32 *squares of the other color.*

Example 9.2. A rectangular floor is covered by 2×2 and 4×1 tiles. One tile got smashed. There is a tile of the other kind available. Show that the floor cannot be covered by rearranging the tiles.

Solution. *Let us color the floor as shown in the picture below. A* 4×1 *tile always covers either* 0 *or* 2 *black squares. A* 2×2 *tile always covers one black square. Therefore it is impossible to exchange one tile for a tile of the other kind.*



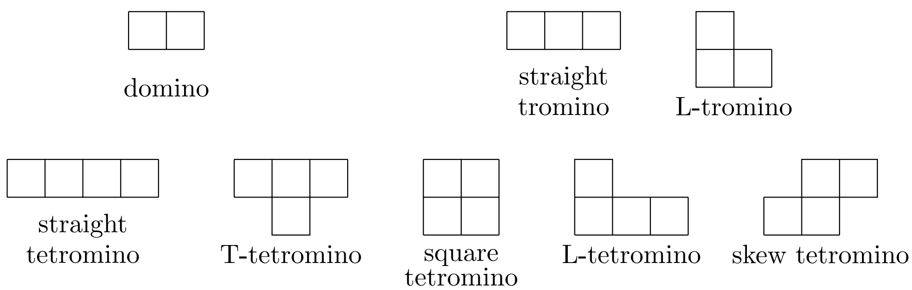
Besides the colorings used in the above examples, the “stripe colorings” and the ”diagonal colorings” shown below are often helpful.



Also, you can use the stripe or diagonal pattern with more colors; or the stripe pattern with one stripe of one color followed by several stripes of another color; and so on. Of course, sometimes you have to be creative and find your own coloring that would work for a particular problem!

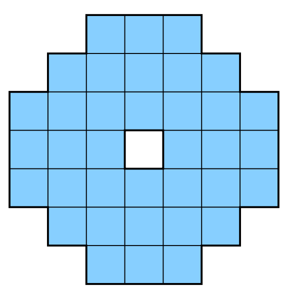
Problems

Some problems below deal with tiles. A tile consisting of 2 squares is called a domino, a 3-square tile is a tromino, a 4-square tile is a tetromino, and so on. Starting with trominoes, there are different shapes possible, and we will use the following names for trominoes and tetrominoes.



By covering a board with tiles we mean completely covering with no tiles overlapping or overhanging.

1. Prove that the figure shown below (with the center square removed) cannot be covered by dominoes.

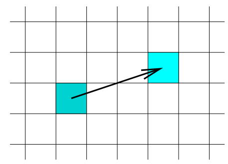


2. Each square of a 9×9 board has a bug sitting on it. On a signal, each bug crawls onto one of the squares which shares a side with the one the bug was on. Is it possible for all the bugs to be on different squares now?

3. King Dodon announced a challenging chess problem with a prize of 1000 shmollars: "A chess knight is at the bottom left corner of an 8×8 chessboard. Invent a sequence of eleven chess-knight moves that brings the knight to the same spot where it started." This prize has not yet been claimed. Explain why.

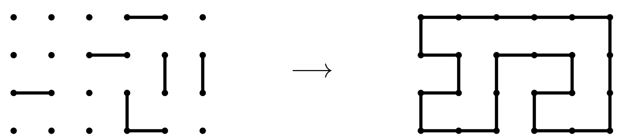
4. A counter is placed in the upper-left square of an 8×8 chessboard. In one move, it can be moved 1 square up, down, to the right, or to the left. Is it possible to move it so that the counter visits every square of the board exactly once and ends in the lower-right square?

5. A new chess piece called camel can move similarly to a knight, but in a $(3, 1)$ pattern instead of $(2, 1)$. One allowed move is shown in the picture below. Can the camel get from the top-left to the top-right corner of an 8×8 chessboard in a few moves?

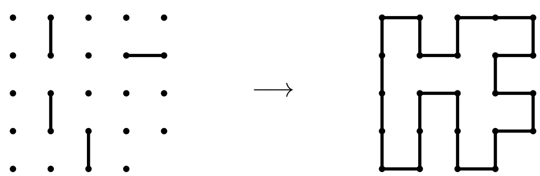


6. In a game called Monorail, a rectangular grid (sometimes with one corner removed) of points and some pieces of a "monorail" are given. The goal is to construct a monorail (that is, a path) that goes through every point once, uses the given pieces, and makes a closed loop. Below are two examples of given starting positions and solutions.

   Example 1.



   Example 2.



   Why is a corner sometimes removed from the grid?

7. Prove that a 10×10 board cannot be covered by 25 T-tetrominoes.

8. Prove that a 10×10 board cannot be covered by 20 square tetrominoes and 5 T-tetrominoes.

9. Prove that a 10×10 board cannot be covered by 15 T-tetrominoes and 10 L-tetrominoes.

10. Prove that a 10×10 board cannot be covered by 25 straight tetrominoes.

11. Prove that a 10×10 board cannot be covered by 25 L-tetrominoes.

12. Prove that an 8×8 board cannot be covered by 11 straight tetrominoes and 5 L-tetrominoes.

13. Prove that an 8×8 board with one corner square removed (so, 63 squares remain) cannot be covered by 21 straight trominoes.

14. Prove that a 15×8 board cannot be covered by 2 L-tetrominoes and 28 skew tetrominoes.

15. Prove that a 23×23 board cannot be covered by 2×2 and 3×3 tiles.

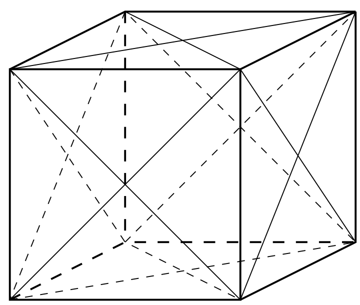
16. A 7×7 board is covered by sixteen 3×1 and one 1×1 tiles. What are the possible positions of the 1×1 tile?

17. Is it always possible to cover a chessboard with two squares removed, one black and one white, by 31 dominoes?

18. A rectangular board is covered by tetrominoes. Prove that the number of T-tetrominoes is even.

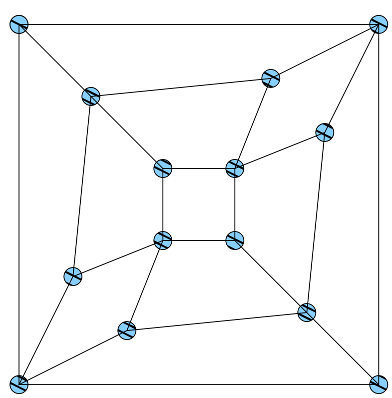
19. The vertices and midpoints of the faces are marked on a cube, and all face diagonals are drawn.

    (a) Prove that there is no path along the face diagonals that visits each marked point exactly once.



    (b) Show that if one walk along an edge is allowed, then there is a path visiting all the marked points. (Find such a path.)

20. The figure below shows a road map connecting 14 cities. Is there a path passing through each city exactly once?



21. Prove that an $a \times b$ rectangle can be covered by $1 \times n$ rectangles if and only if $n|a$ or $n|b$.

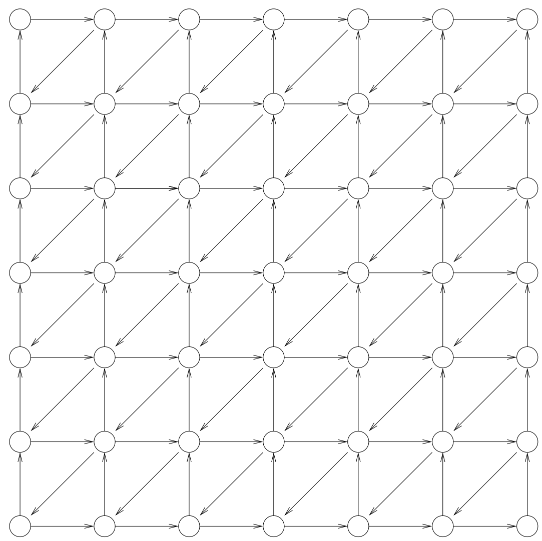
22. The map below shows the cities and one-way roads in Sikinia.

    (a) Prove that there is no closed path (a path is closed if it starts and quits in the same city) that visits every city exactly once.

    (b) Is there a closed path that visits every city exactly twice?

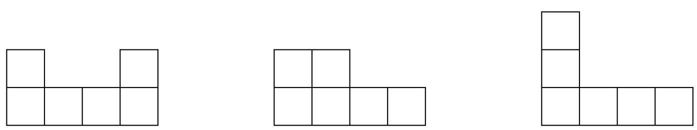
(c) Is there a path, not necessary closed, that starts in the upper left corner and visits every city exactly once?

(d) Is there a path, not necessarily closed, that starts in the upper left corner and visits every city exactly twice?



23. Is it possible for a chess knight to pass through all the squares of a 123×4 board having visited each square exactly once, and return to the initial square?

24. Show that it is not possible to cover any rectangle by one tile of type 1 shown below, one tile of type 2, and any number of tiles of type 3.

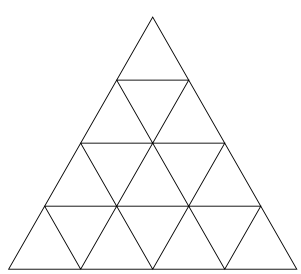


1 2 3

25. Prove that there is no way to pack fifty-four $1 \times 1 \times 4$ bricks into a $6 \times 6 \times 6$ box.

26. Show that if $4 \times 1 \times 1$ bricks and $2 \times 2 \times 2$ cubes fill an $8 \times 8 \times 8$ cube, then the number of $2 \times 2 \times 2$ cubes is even.

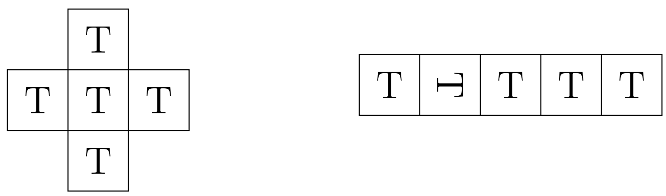
27. Is it possible to write distinct natural numbers from 1 to 16 in the small triangles in the figure below so that the sum of the two numbers in any two triangles that have a common side is prime?



28. (a) A bug sits on each square of a 9×9 board. On a signal each bug crawls onto a neighboring square (so that the old square and the new square share a side). It may happen that several bugs will sit on some squares and none on others. Find the smallest possible number of free squares.

    (b) What if each bug crawls diagonally onto a neighboring square (that is, the old square and the new square share only a vertex)?

29. The figure on the left below shows the top view of five heavy boxes which can be displaced only by rolling them about once of their edges. Their tops are labeled by the letter T. The figure on the right shows the same five boxes rolled into a new position. Which box in this row was originally at the center of the cross?

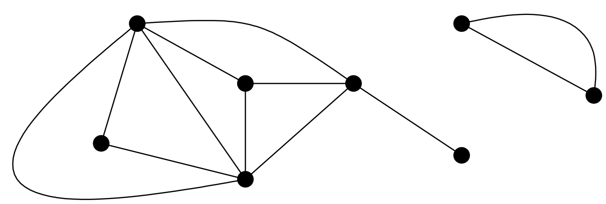


30. One corner of a $(2n+1) \times (2n+1)$ board is cut off. For which values of n is it possible to cover the remaining squares by dominoes so that exactly half of the dominoes are horizontal?

Chapter 10

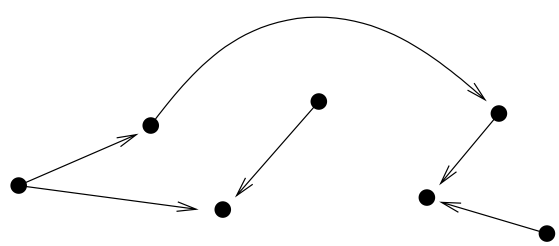
Graphs

Definition 10.1. A **graph** is an object consisting of a set of points called **vertices**, some of which are connected by lines (or arcs) called **edges**.



Definition 10.2. A graph is **simple** if any 2 vertices are connected by at most one edge and there are no loops (edges starting and ending at the same vertex).

Definition 10.3. If the edges are oriented, then we have an **oriented** or **directed** graph. An example of an oriented graph is a one-way road system.

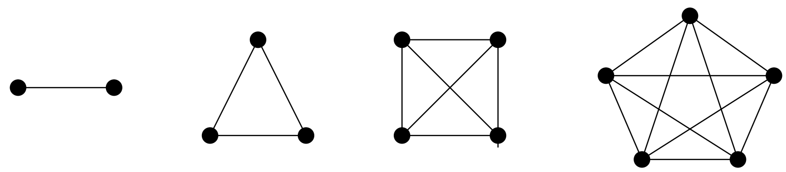


Definition 10.4. If an edge e connects the vertices v_1 and v_2, then we say that v_1 and v_2 are the **endpoints** of e and we write $e = (v_1, v_2)$. Also, we say that v_1 and v_2 are **adjacent vertices**. If two edges e_1 and e_2 share a common vertex, then we say that e_1 and e_2 are **adjacent edges**. A vertex v has **degree** m if m endpoints of edges coincide with v (a loop contributes 2 to the degree of a vertex).

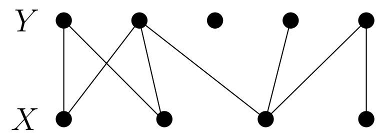
Theorem 10.5. *In any graph, the sum of the degrees of the vertices equals twice the number of the edges.*

Corollary 10.6. *In any graph, the number of vertices with odd degrees is even.*

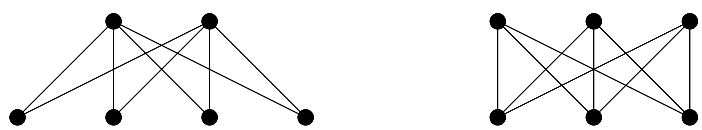
Definition 10.7. An undirected graph in which every two distinct vertices are connected by exactly one edge is called a **complete** graph. K_n denotes the complete graph with n vertices. The graphs K_2, K_3, K_4, and K_5 are shown below.



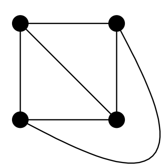
Definition 10.8. If the vertices of a graph can be separated into two parts X and Y so that for every edge in the graph, one of its endpoints belongs to X and the other belongs to Y, then we call such a graph a **bipartite** graph.



Definition 10.9. If every vertex in the set X is connected to every vertex in the set Y by exactly one edge, then the graph is called a **complete bipartite** graph. $K_{m,n}$ denotes the complete bipartite graph with m vertices in the set X and n vertices in the set Y. The graphs $K_{2,4}$ and $K_{3,3}$ are shown below.



Definition 10.10. A graph is called **planar** if it is possible to draw it in such a way that no two edges intersect. For example, the graph K_4 is planar.



Theorem 10.11. *Graphs K_5 and $K_{3,3}$ are not planar.*

Definition 10.12. A **path** is a sequence of edges e_1, e_2, …, e_n such that $e_1 = (x_0, x_1)$, $e_2 = (x_1, x_2)$, …, $e_n = (x_{n-1}, x_n)$. When there are no multiple edges connecting the same vertices in a graph, a path can be denoted by its vertex sequence $x_0,\ x_1,\ \ldots,\ x_n$. A path that begins and ends at the same vertex is called a **cycle**. A path is **simple** if it does not contain the same edge more than once.

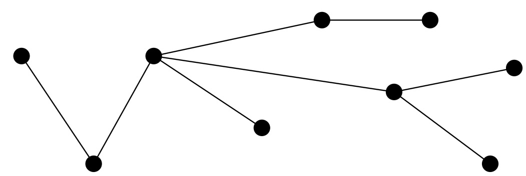
Definition 10.13. An **Euler path (resp. Euler cycle)** is a simple path (resp. cycle) containing every edge of the graph.

Theorem 10.14. *A connected graph has an Euler cycle if and only if each of its vertices has an even degree.*

Definition 10.15. A **Hamilton path (resp. Hamilton cycle)** is a simple path (resp. cycle) visiting every vertex exactly once. (Note: in a cycle, the first and last vertices must coincide, so the start and finish count as one "visit.")

Definition 10.16. If all vertices of a graph can be visited by walking on edges, the graph is **connected**.

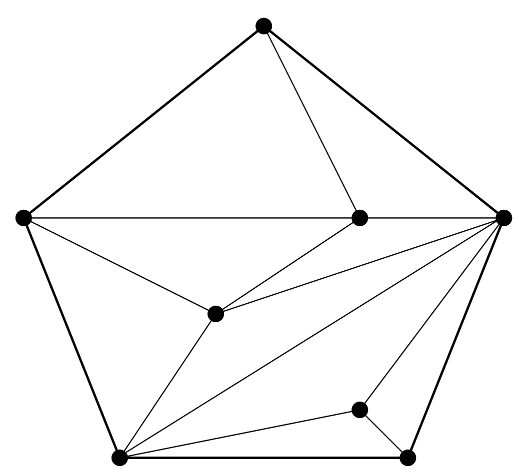
Definition 10.17. A connected graph without simple cycles is called a **tree**. Below is an example of a tree.



Example 10.18. Assume that in a group of people, any two either both know each other or both do not know each other. Prove that in any subset containing six people either there are three people that all know each other or there are three people that all do not know each other.

Solution. *Let's translate this problem into a graph theory problem. Let six vertices a, b, c, d, e, and f represent the six people. If two people know each other, then we use a red edge to join these two vertices. If two people do not know each other, then we use a blue edge to join these two vertices. Since there are edges between every two vertices in the graph, it's a complete graph K_6 with red and/or blue edges. Now the problem has been translated into the following problem.*
Suppose each edge in the complete graph K_6 is colored either red or blue. Prove that there exist either three vertices such that the edges joining them are all red or three vertices such that the edges joining them are all blue.
To prove the above statement, we pick any vertex in K_6, let's call it a. The five edges between this vertex and the other five vertices are each either red or blue. According to Dirichlet's Principle, at least three edges of the five have the same color. Let b, c, and d be the other endpoints of such edges. Without loss of generality, let's assume that edges (a, b), (a, c), (a, d) are red. Now consider the triangle bcd. If one of the edges (b, c), (b, d), (c, d) is red, then we have a red triangle. Otherwise, if (b, c), (b, d), (c, d) are all blue, then the triangle bcd is a blue triangle. This proves that there exists a triangle all of whose edges are colored with the same color.

Example 10.19. Is it possible to draw "a triangular map" inside a pentagon so that the degree of each vertex is even? (See problem 13 in chapter 2 for a definition of a map. A map is called triangular if every region is a triangle. Below is an example of a triangular map inside a pentagon, however, some vertices have an odd degree.)



Solution. *The answer is no. We will prove this by contradiction. Suppose such a map exists. We know (see problem 13 in chapter 2) that every map with all vertices of even degree admits a proper coloring, i.e. its regions can be colored with 2 colors so that no neighboring regions have the same color. Color our map in blue and red so that the (infinite) region outside of the pentagon is blue. All the other regions are triangles. Each edge has a red triangle on one side and a blue region (either a triangle or that infinite outside region) on the other side. Now, count the number of edges (boundaries) in the map in two ways: each red triangle has 3 sides, so the number of edges is a multiple of 3, say,* $3n$. *Each blue triangle has 3 sides, and the infinite region has 5 edges, so the number of edges is a multiple of 3 plus 5, say,* $3m+5$. *Thus we have* $3n = 3m+5$ *where* n *and* m *are integers. But this is impossible.*

Problems

1. Explain why a graph cannot have seven vertices of degrees 2, 2, 3, 3, 3, 4, 4.
2. (a) Does there exist a graph with four vertices of degrees 1, 1, 1, and 5?
   (b) Does there exist a simple graph with four vertices of degrees 1, 1, 1, and 5?
3. (a) Does there exist a graph with eight vertices of degrees 0, 1, 2, 3, 4, 5, 6, 7?
   (b) Does there exist a simple graph with eight vertices of degrees 0, 1, 2, 3, 4, 5, 6, 7?
4. (a) Does there exist a graph with six vertices of degrees 1, 2, 3, 4, 5, 5?
   (b) Does there exist a simple graph with six vertices of degrees 1, 2, 3, 4, 5, 5?
5. A graph has five vertices of degrees 2, 2, 3, 3, 4. How many edges does it have? Draw an example of such a graph.
6. A graph has six vertices of degrees 2, 3, 4, 4, 5, and 6. How many edges does it have? Draw two different examples of such graphs.
7. In a certain kingdom, there are 100 cities, and some pairs of them are connected by roads. Four roads lead out of each city. How many roads are there altogether in the kingdom?
8. In a certain kingdom, some pairs of cities are connected by roads. If exactly three roads lead out of each city, can the kingdom have exactly 100 roads?
9. A simple graph has six vertices. The degrees of five of them are 1, 2, 3, 4, and 5. What is the degree of the sixth vertex?
10. The Math Club at the Smart University consists of seven students who like to write papers together. Every club member has written a joint paper with at least one other club member. Each club member counted the number of other club members with whom they have a joint paper. Six of these numbers are: 1, 2, 3, 4, 5, and 6. What is the seventh number?
11. There are ten people in a chess club. Each counted the number of other members of the club with whom they have played chess. Nine of the numbers are 4, 4, 4, 4, 5, 9, 9, 9, and 9. With how many people did the tenth person play?
12. In a graph with 8 vertices and 15 edges, the degrees of any two vertices either are equal or differ by 1. What are the degrees, and how many vertices of each degree are there?

13. There are eight counties in Sikinia. There are no "four corners" points (like Arizona, Colorado, New Mexico, and Utah). Each county counted the number of neighboring counties. The numbers are 5, 5, 4, 4, 4, 4, 4, 3. Prove that at least one county made a mistake.

14. Can nine line segments be drawn in the plane, each of which intersects exactly 3 others?

15. Prove that in any group of people, the number of people that are mutual friends with an odd number of people is even.

16. Find the number of vertices and edges in K_n and $K_{n,m}$.

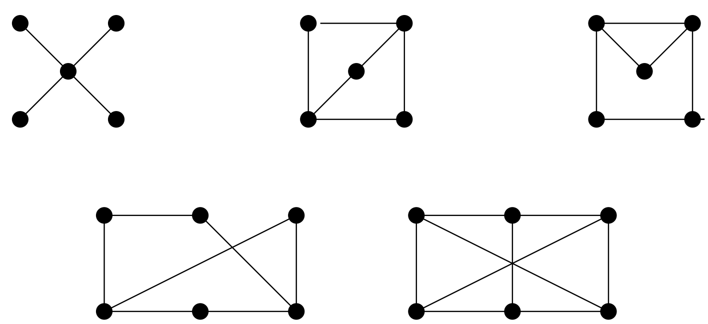
17. In the country of Fifteen there are 15 towns, each of which is connected to at least 7 others by roads. Prove that one can travel (by these roads) from any town to any other town, possibly passing through some towns in between.

18. Prove that a simple graph with n vertices, each of which has degree at least $(n-1)/2$, is connected.

19. In Never-Never-Land there is only one means of transportation: magic carpet. Twenty-one carpet lines serve the capital. A single line flies to Farville, and every other city is served by exactly 20 carpet lines. Prove that it is possible to travel by magic carpet from the capital to Farville (perhaps by transferring from one carpet line to another).

20. In a certain kingdom, some pairs of cities are connected by roads so that 100 roads lead out of each city, and one can travel along those roads from any city to any other. One road is closed for repairs. Prove that one can still get from any city to any other.

21. Determine which of the following graphs are bipartite:



22. A math circle leader assigned 20 problems as homework. At the next meeting, he found out that each student solved exactly 4 problems, and that each problem was solved by exactly 5 students. How many students are in the circle?

23. A connected simple bipartite graph G has eight vertices. Recall that the vertices of a bipartite graph can be divided into two groups A and B so that every edge connects a vertex in group A and a vertex in group B. Both groups for G have four vertices. Three of the vertices in group A have degrees 4, 2, and 2. Three of the vertices in B have degrees 3, 1, and 1. What are the degrees of the remaining vertices?

24. There are ten 9-graders and ten 12-graders interested in a peer mentor program. Each of the 12-graders will be a mentor for one 9-grader, and each 9-grader will have one mentor. It turns out that in this group, every 12-grader knows exactly two 9-graders

and every 9-grader knows exactly two 12-graders (assume that the relationship "knows" is mutual, i.e. either both people know each other or both people do not know each other). Prove that there exists a pairing such that every person would be paired up with one that they know.

25. There are seven kids from school A and seven kids from school B attending a chess night. Each game is between one kid from school A and one kid from school B. The following day, they recall the number of people they have played with. The numbers are as follows: 3, 3, 3, 3, 3, 3, 3, 5, 6, 6, 6, 6, 6, 6. Prove that at least one of them made a mistake.

26. In a connected graph the degrees of four of the vertices equal 3 and the degrees of all other vertices equal 4. Prove that it is impossible to delete one edge in such a way that the graph splits into two isomorphic connected components.

27. Find a necessary and sufficient condition for a graph to have an Euler path but not an Euler cycle.

28. Which of the graphs in problem 21 have

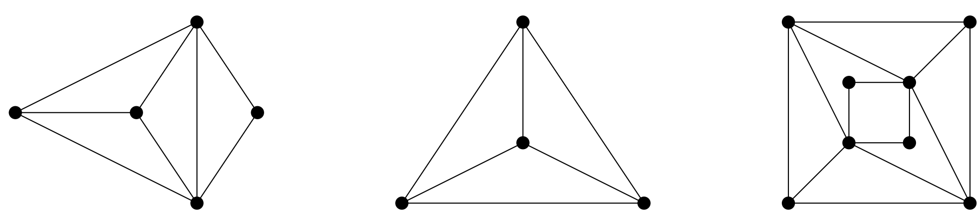
    (a) an Euler path?

    (b) an Euler cycle?

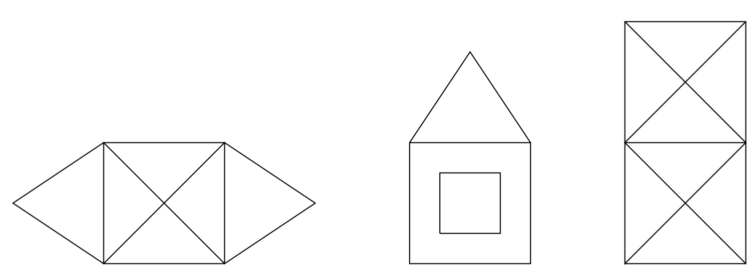
    (c) a Hamilton path?

    (d) a Hamilton cycle?

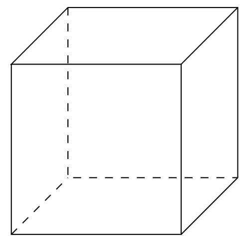
29. Can one draw the following graphs without lifting the pencil from the paper, and tracing over each edge exactly once?



30. Which of the following pictures can be traced without lifting the pencil from the paper so that each line segment is traced exactly once? Does it matter where we start?



31. A bug is crawling along the edges of a cube. It wants to crawl along each edge exactly once. Is this possible?



32. For which of the platonic solids there exists a path along the edges that contains every edge exactly once?

33. (a) Prove that it is impossible to bend a 12 in long wire so that to make a cube whose sides are 1 in long.

    (b) Suppose that we are allowed to cut the wire (still 12 in long) into two pieces, bend each piece, and glue the two pieces together. Prove that it is still impossible to make a 1 in cube.

    (c) What is the least number of pieces into which we need to cut the wire so that to make the cube by bending each piece and then gluing the pieces?

34. A group of islands are connected by bridges in such a way that one can walk from any island to any other. One day, a tourist walked around every island, crossing each bridge exactly once. He visited the island of Thrice three times that day. How many bridges are there to Thrice, if

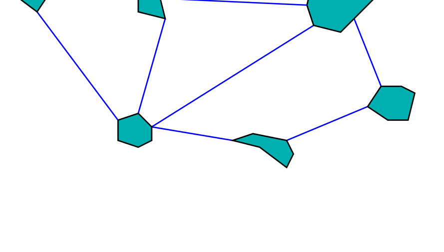
    (a) the tourist neither started nor ended on Thrice?

    (b) the tourist started on Thrice, but didn't end there?

    (c) the tourist started and ended on Thrice?

35. Lake Kingdom has four islands: Castle, Orchard, Rose Garden, and Forest. The islands are connected by six bridges so that there is exactly one bridge between any two islands. Prince's teacher told him to jog every morning as follows: start from the castle, cross every bridge exactly once, then take a break, and walk back to the castle. Prove that the teacher gave the prince an impossible task.

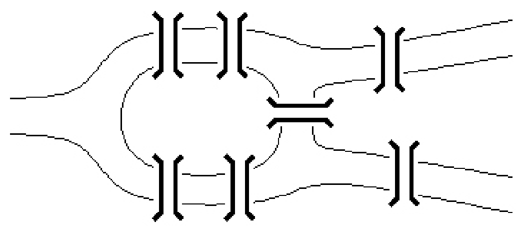
36. A map of major cities and roads of Mathland is shown below.



(a) Is there a tour that starts and ends at the same city and uses each road exactly once?

(b) Is there a tour that starts and ends at the same city and visits each of the other cities exactly once?

37. Below is a map of the river and the bridges in Konigsberg. As we know from problem 7 in chapter 1, it is not possible to design a tour of the town that crosses each bridge exactly once and returns to the starting point. Could the citizens of Konigsberg create such a tour by building a new bridge?



38. For what values of n does K_n have

(a) an Euler path?

(b) a Hamilton path?

39. For what values of n and m does $K_{n,m}$ have

(a) an Euler cycle?

(b) an Euler path?

(c) a Hamilton cycle?

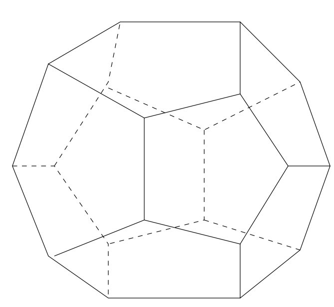
(d) a Hamilton path?

40. What is the smallest number of edges that have to be removed from the graph K_6 in order for the remaining graph to have an Euler cycle?

41. Hamilton's "Round the World" puzzle asks: does [the graph consisting of vertices and edges of] the dodecahedron (shown below) have

(a) a Hamilton path?

(b) a Hamilton cycle?



42. A knight's tour is a sequence of legal moves by a knight starting at some square of a chessboard and visiting each square exactly once. (See problem 8 in chapter 1 for a description and picture of legal knight moves.) A knight's tour is called reentrant if there is a legal move that takes the knight from the last square of the tour back to where the tour began.

    (a) Draw the graph that represents all legal moves of a knight on a 3×4 chessboard.

    (b) Show that there is no reentrant tour on a 3×4 chessboard.

    (c) Find a non-reentrant tour on a 3×4 chessboard.

43. Show that there is no reentrant knight's tour on a 4×4 chessboard.

44. Show that there is no knight's tour at all (reentrant or not) on a 4×4 chessboard.

45. There are 17 scientists who communicate with each other discussing some problems. They discuss only three topics, and each pair discusses at least one of these three. Prove that there are at least 3 scientists who are all pairwise discussing the same topic.

46. Nine mathematicians met at an international conference. They found that among any three of them there are at least two that have a language in common. If every mathematician speaks at most three languages, prove that at least three of the mathematicians can speak the same language.

47. (a) Prove that in a finite simple graph having at least two vertices there are always two vertices with the same degree.

    (b) Does the above hold for graphs with loops but no multiple edges?

    (c) Does the above hold for graphs with multiple edges but no loops?

48. A graph $K_{k,l,m}$ has $k + l + m$ vertices divided into three sets: k vertices in one set, l vertices in another set, and m vertices in the third set. Two vertices are connected if and only if they are in different sets. Prove that there do not exist $k, l, m \geq 1$ such that $K_{k,l,m}$ has exactly 6 edges.

49. Does $K_{1,2,4}$ have

    (a) an Euler cycle?

    (b) an Euler path?

    (c) a Hamilton cycle?

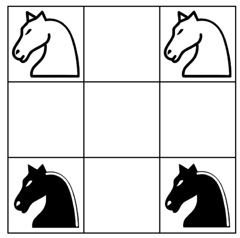
    (d) a Hamilton path?

50. Find a necessary and sufficient condition on k, l, m for which $K_{k,l,m}$ has

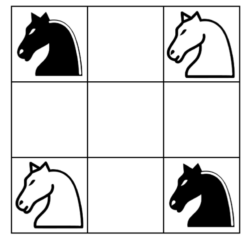
    (a) a Hamilton cycle.

    (b) a Hamilton path.

51. Four knights are placed on a 3×3 board as shown in Figure A below. They can be moved, one at a time, using the usual chess knight's moves. At most one knight can occupy each square at any given time. Is it possible to move the knights to the position shown in Figure B?

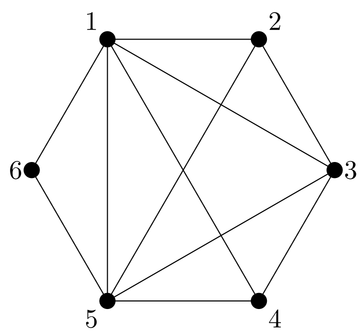


A

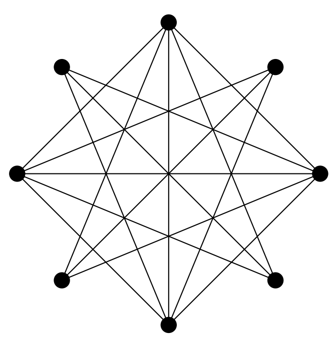


B

52. Given a set of positive integers, a graph is constructed as follows. The vertices represent these integers, and two vertices are connected if and only if the corresponding integers are relatively prime. For example, the set $\{1, 2, 3, 4, 5, 6\}$ produces the following graph.



What is the smallest possible value of the sum of eight positive integers that produce the graph shown below?



Chapter 11

Combinatorics

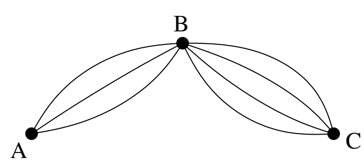
Combinatorics is an area of mathematics primarily concerned with counting. In this chapter, we will study a few techniques for counting the number of certain objects, the number of ways to perform a task, and so on.

Theorem 11.1. *(The Multiplication and Addition Principles.)*

1. *Suppose that a procedure consists of a sequence of two tasks. If there are m ways to perform the first task and for each of these ways, there are n ways to perform the second task, then there are mn ways to perform the procedure.*

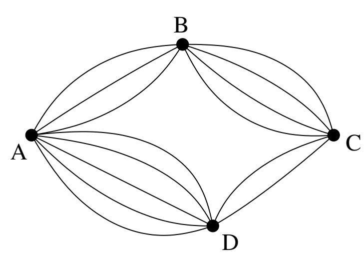
2. *If a task can be performed in one of m ways or in one of n ways, and none of the m ways is the same as one of the n ways, then there is a total of $m + n$ ways to perform the task.*

Example 11.2. There are three towns A, B, and C in Wonderland, and only seven roads. Three roads go from A to B, and four roads go from B to C. In how many ways can we drive from A to C if we want to choose one road from A to B and then one road from B to C? (We do not want to go back and forth.)



Solution. *There are three ways to choose a road from A to B and four ways to choose a road from B to C. So there are $3 \cdot 4 = 12$ ways to drive from A to C.*

Example 11.3. A new town D, five roads from A to D, and two roads from C to D were built. How many ways are there to drive from A to C now?



Solution. *From A to C, we can drive via B or via D. There are $3 \cdot 4 = 12$ ways to drive via B. Similarly, there are $5 \cdot 2 = 10$ ways to drive via D. Thus there are $12 + 10 = 22$ total ways to drive from A to C.*

Definition 11.4. The number of ways to select k objects out of a collection of n objects, if the order does not matter, is called "n choose k" and is denoted $\binom{n}{k}$.

Theorem 11.5. *For $0 \le k \le n$, we have* $\binom{n}{k} = \frac{n!}{k!(n-k)!}$.

Example 11.6. How many ways are there to select two objects out of five?

Solution. *One way to count this is as follows. First, we have five ways to select the first object. After we select the first object, there are four ways to select the second one. The total number of ways is $5 \cdot 4 = 20$. However, each selection was counted twice, as each pair $\{A, B\}$ was also counted as $\{B, A\}$. Therefore, there are actually only 10 distinct selections. Another way is to use the above formula:* $\binom{5}{2} = \frac{5!}{2!3!} = \frac{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5}{1 \cdot 2 \cdot 1 \cdot 2 \cdot 3} = 10$.

Note that in general, $\binom{n}{2} = \frac{n(n-1)}{2}$.

Theorem 11.7. *(Inclusion-Exclusion Principle.) For any finite sets A, B, and C,*

$$\begin{aligned} |A \cup B| &= |A| + |B| - |A \cap B|, \\ |A \cup B \cup C| &= |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|. \end{aligned}$$

Remark. The above principle can be generalized to any number of sets.

Example 11.8. How many of the first 100 natural numbers are divisible by 3 or 5?

Solution. *Let A be the set of all positive integers not exceeding 100 and divisible by 3 and let B be the set of all positive integers not exceeding 100 and divisible by 5. Then $A \cap B$ consists of all positive integers not exceeding 100 and divisible by 15, and we want to find $|A \cup B|$. Using the Inclusion-Exclusion Principle, we have*

$$\begin{aligned} |A \cup B| &= |A| + |B| - |A \cap B|, \\ &= \left\lfloor \frac{100}{3} \right\rfloor + \left\lfloor \frac{100}{5} \right\rfloor - \left\lfloor \frac{100}{15} \right\rfloor \\ &= 33 + 20 - 6 \\ &= 47. \end{aligned}$$

Finally, let us clarify some terminology before we proceed to problems. A few of the problems below deal with multi-digit numbers. The leading digit of a natural number should not be zero. For example, 01234 would not be considered a 5-digit number. It is a 4-digit number and is written as 1234. On the other hand, a string of digits may begin with any digit, including zero. Thus, 01234 would be counted as a 5-digit string.

Problems

1. There are five different teacups, three saucers, and four teaspoons in a cupboard.
   (a) How many ways are there to select a set of three items of different types (i.e. one teacup, one saucer, and one teaspoon)?
   (b) How many ways are there to select two items of different types?
2. (a) If we toss a coin three times, how many different sequences of heads and tails are possible? List all of them.
   (b) What if we toss the coin ten times?
   (c) What if we toss the coin n times?
3. Let's call a natural number "odd-looking" if all of its digits are odd. For example, 1353 and 9175 are odd-looking. How many four-digit odd-looking numbers are there?
4. The old Wonderland language consisted of only three letters: A, B, and C. A word in this language is an arbitrary sequence of no more than four letters.
   (a) How many four-letter words are there in this language?
   (b) How many words are there in this language total?
5. Each of the two novice collectors has 20 stamps and 10 postcards. We call an exchange fair if they exchange a stamp for a stamp or a postcard for a postcard. How many ways are there to carry out one fair exchange between these two collectors?
6. A number (or a word, or any string of characters) is called palindromic if it reads the same backward as forward. For example, the numbers 2002 and 123454321 are palindromic. How many five-digit numbers are palindromic?
7. A die has 1, 2, 3, 4, 5, and 6 dots on its six faces. We roll it three times.
   (a) How many outcomes are possible?
   (b) How many outcomes have at least one occurrence of six?
8. How many license plates with three decimal digits followed by three letters do not contain both the number 0 and the letter O?
9. The modern language of Wonderland is using 6 letters: A, B, C, D, E, F, and allows words of up to six letters. A word is an arbitrary string of letters.
   (a) How many two-letter words are there?
   (b) How many two-letter words are there with two different letters?
   (c) How many three-letter words are there with no letter repeating?
   (d) How about n-letter words with no letter repeating?
10. Suppose we have three different flags and can raise one, two, or all of them up a flagpole as a signal. How many different signals can we send if the order of the flags in the signal matters?
11. In how many ways can a striped rectangular flag be designed if it must consist of 11 horizontal stripes of equal length, each colored either red, blue, or white, and any two adjacent stripes must be of different colors?

12. How many ways are there to choose four cards of different suits and different values from a deck of 52 cards if the order does not matter?

13. The fifth-graders are going to visit Kindergarteners and read books to them. Each fifth-grader will be reading a book to one Kindergartener. There are 20 children in each class. How many ways are there to pair up each fifth-grader with a Kindergartener?

14. Roma has 15 different books that she wants to put on a bookshelf. She wants her two volumes of Roald Dahl's short stories to be next to each other, with volume II to the right of volume I. In how many different orders can she arrange the books?

15. How many ten-digit numbers have at least two equal digits?

16. How many odd five-digit numbers have all digits different?

17. How many five-digit numbers have exactly four equal digits?

18. How many four-digit numbers have the sum of their digits equal to 33?

19. How many positive factors does 4000 have?

20. (a) Find the prime factorization of 30^{10}.
    (b) How many positive factors does 30^{10} have?
    (c) How many of the positive factors of 30^{10} are even?
    (d) How many of the positive factors of 30^{10} are divisible by 12?
    (e) How many of the positive factors of 30^{10} are perfect squares?

21. How many strictly increasing sequences of whole numbers start with 1 and end with 10? For example, $1, 3, 10$ and $1, 2, 4, 5, 6, 8, 10$ are two such sequences.

22. How many diagonals are there in a convex
    (a) octagon?
    (b) n-gon?

23. How many ways are there to put one white and one black rook on an 8×8 chessboard so that they do not attack each other? (The rooks attack each other if they are in the same row or column).

24. How many ways are there to put two identical counters on an 8×8 chessboard so that they are in different rows and in different columns?

25. In how many ways can a black and a white square on a regular chessboard be chosen so that the chosen squares do not lie in the same row or column?

26. How many strings of digits 1 though 5, containing each of these digits exactly once, have 1 to the left of 3? (For example, 14235 is such a string, but 34152 is not.)

27. A "word" is any sequence of letters. How many "words" can be made by rearranging the letters in the following words?
    (a) CIRCLE
    (b) SCIENCE
    (c) SEQUENCE

(d) MATHEMATICS

28. How many 8-digit numbers can be made by rearranging the digits in 10102020?

29. How many seven-digit binary (base 2) numbers have no more than three 1s in them?

30. How many ways are there to put eight identical rooks on a chessboard so that they do not attack each other?

31. How many ways are there to place one white and one black bishop on a chessboard so that they do not attack each other?

32. How many ways are there to place two identical bishops on a chessboard so that they do not attack each other?

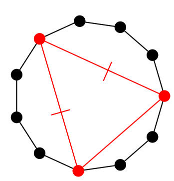
33. Seven girls and seven boys are to be seated in a row on a bench. The girls and boys should alternate. How many ways are there to seat everyone?

34. (a) How many ways are there to split ten people into six pairs?
    (b) Generalize to any even number of people.

35. A fair coin is flipped ten times. How many ways are there to get more heads than tails?

36. If we flip a coin 7 times, what is the probability of getting
    (a) exactly 7 heads?
    (b) exactly 6 heads?
    (c) exactly 5 heads?
    (d) exactly 4 heads?

37. How many ways are there to select three out of 11 vertices of a regular 11-gon so that these three points are vertices of an isosceles triangle? (One example of such three vertices is shown below.)



38. How many even 5-digit numbers have all their digits distinct and not exceeding 5?

39. How many 4-digit positive integers divisible by 5 have four different digits, of which 5 is the largest digit?

40. (a) How many 4-digit strings of digits 0-9 contain exactly two different digits (for example, 1121, 0444, and 5995 are such strings)?
    (b) Generalize to k-digit strings, for $k \geq 2$.

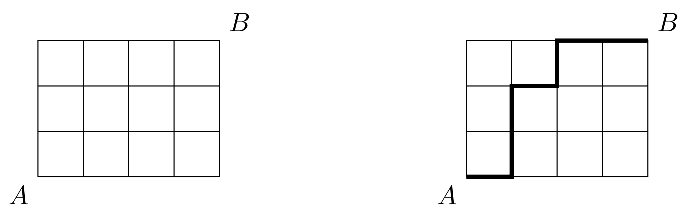
41. Corey and Tony are friends on the same basketball team. There are eight players on the team. How many starting groups of five players include Corey, Tony, or both?

42. There are four married couples in a club. How many ways are there to choose a committee of three members so that no two spouses are members of the committee?

43. There are 31 students in a class, including Pete and John who only talk about computer games whenever they see each other. How many ways are there to choose a soccer team of 11 players so that Pete and John are not on the team together?

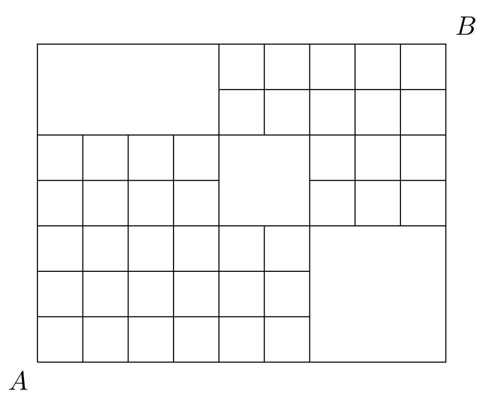
44. There are 4 girls and 6 boys in a chess club. A team of four persons must be chosen for a tournament, and there must be members of both genders on the team. In how many ways can this be done?

45. We want to join points A and B by a path that goes along the lines and has the shortest possible length (which is 7). How many ways are there to do this? For example, one way is shown below.

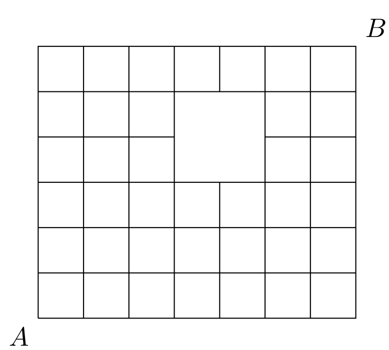


46. How many ways are there to join points A and B by a path that goes along the lines and has the shortest possible length?

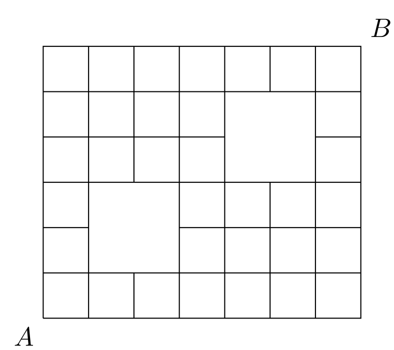
(a)



(b)



(c)



47. A spider has eight different socks and eight different shoes. It wants to put one sock and one shoe on each of its eight legs.

(a) How many different distributions of socks and shoes are there?

(b) Now the spider has decided which sock and which shoe should go on which leg. In how many different orders can it put on its socks and shoes, assuming that, on each leg, the sock must be put on before the shoe?

48. Let $S = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$. How many subsets of S have an odd sum of their elements? (For example, $\{2, 3, 4, 5, 8, 11\}$ is one such subset since 2+3+4+5+8+11 is odd.)

49. Six boxes are numbered 1 through 6. How many ways are there to distribute 20 identical balls among the boxes if some of the boxes can be empty?

50. Six boxes are numbered 1 through 6. How many ways are there to distribute 20 identical balls among the boxes if each box must contain at least one ball?

51. How many triples (a, b, c) of natural numbers are there such that $a + b + c = 10$? In other words, how many ways are there to write 10 as a sum of three natural numbers? (The ways $10 = 2 + 1 + 7$ and $10 = 2 + 7 + 1$ are considered different.)

52. How many ways are there to represent the natural number n as a sum of k natural numbers if the order of the k numbers matters?

53. Susie has twelve identical cookies. She wants to give some to each of her brothers, Sam and Steve, and leave the rest for herself. She wants each person to have at least one cookie. How many ways are there to distribute the cookies?

54. A standard six-faced die is rolled four times. How many ways are there to get a sum of 9?

55. Each of the numbers from 1 to 10 is placed in a bag and drawn at random with replacement. In how many ways can three numbers whose sum is 13 be drawn?

56. You are ordering six doughnuts, and need to choose from among four flavors: glazed, powdered, cream-filled, and jelly-filled. How many different doughnut orders are possible?

57. How many bit strings of length $3n$ consisting of exactly n zeros and $2n$ ones have no consecutive zeros?

58. How many bit strings of length n, where $n \geq 4$, contain exactly two occurrences of 01?

59. How many ways are there to divide 30 students into three groups of 10 students each?

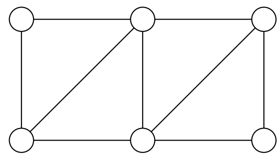
60. (a) Six colors of paint are available to paint all the faces of a cube. Each face of the cube must be painted a distinct color. Two colorings of the cube are considered to be the same if one can be obtained from the other by rotating the cube. How many different colorings are possible?

    (b) Same problem for a dodecahedron and twelve colors of paint.

61. A traveling agent has to visit four cities, each of them five times. In how many different ways can he do this if he does not want to start and finish in the same city?

62. Alice randomly picks three distinct numbers from the set $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ and arranges them in descending order to form a 3-digit number. Bob randomly picks three distinct numbers from the set $\{1, 2, 3, 4, 5, 6, 7, 8\}$ and also arranges them in descending order to form a 3-digit number. What is the probability that Alice's number is larger than Bob's number?

63. Suppose we want to color the six vertices of the graph shown below so that any two adjacent vertices have different colors.



How many ways are there to do this if we have

(a) 3 colors available (say, red, blue, and green)?

(b) 4 colors available (say, red, blue, green, and yellow)?

64. How many 8-bit strings begin with 1010 or end with 01?

65. How many of the first 100 positive integers are not divisible by 5 or 7?

66. How many of the first 1000 positive integers are either a perfect square or a perfect cube?

67. How many strings of five digits (from 0 to 9) contain at least one 4 and at least one 7?

68. How many permutations of the 26 letters of the English alphabet do not contain any of the strings

(a) cat, dog, fish?

(b) dog, rat, bird?

69. How many strings of length 7 over the alphabet $\{a, b, c\}$ contain each letter at least once?

70. How many strings of a's, b's, and c's, of total length 6, have exactly two a's or exactly two b's (or both; also, the two a's or the two b's do not have to be consecutive)? For example, *acaccc*, *acbaab*, and *bcbcaa* are such strings (the first one has exactly two a's, the second has exactly two b's, and the third has both exactly two a's and exactly two b's).

Chapter 13

Calculus

We recall a few important definitions and useful theorems.

Definition 13.1. If a, $x \in \mathbb{R}^+$, $a \neq 1$, and $y \in \mathbb{R}$, then

$$\log_a x = y \qquad \Leftrightarrow \qquad a^y = x.$$

Theorem 13.2. *(Properties of logarithms) If $a, x, y \in \mathbb{R}^+$, $a \neq 1$, and $r \in \mathbb{R}$, then*

1. $\log_a(xy) = \log_a x + \log_a y$,
2. $\log_a\left(\dfrac{x}{y}\right) = \log_a x - \log_a y$,
3. $\log_a(x^r) = r \log_a x$,
4. $\log_a(x) = \dfrac{\ln x}{\ln a}$.

Definition 13.3. A function $f(x)$ is called **even** if $f(-x) = f(x)$ for all x in the domain of f. It is called **odd** if $f(-x) = -f(x)$ for all x in the domain of f.

Definition 13.4. Let $f : A \to B$ be a function. A function $f^{-1} : B \to A$ is called the **inverse** of f if for all $x \in A$ and $y \in B$

$$f^{-1}(y) = x \qquad \Leftrightarrow \qquad f(x) = y.$$

Theorem 13.5. *If f^{-1} is the inverse of $f : A \to B$ where A and B are subsets of $\mathbb{R}$, then the graphs of $y = f(x)$ and $y = f^{-1}(x)$ are symmetric about the line $y = x$.*

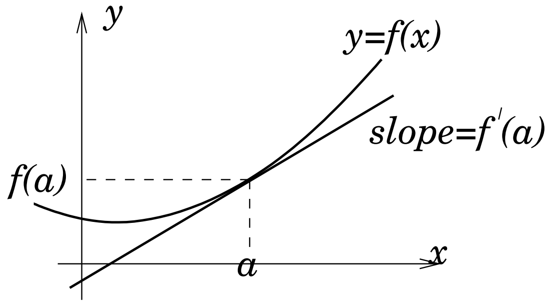
Theorem 13.6. *(Intermediate value theorem) Suppose $f(x)$ is continuous on $[a, b]$. Let N be any number between $f(a)$ and $f(b)$. Then there exists $c \in [a, b]$ such that $f(c) = N$.*

Definition 13.7. The derivative of $f(x)$ at a point a is

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h},$$

if this limit exists.

The derivative $f'(a)$ is the slope of the tangent line to $y = f(x)$ at $(a, f(a))$. Also, $f'(a)$ is the rate of change of $f(x)$ with respect to x at $x = a$.



Theorem 13.8. *(Important derivatives) Whenever the following derivatives are defined,*

$$
\begin{array}{lll}
(x^n)' = nx^{n-1}, & (e^x)' = e^x, & (a^x)' = (\ln a)a^x, \\
(c)' = 0, & (\ln x)' = \dfrac{1}{x}, & (\log_a x)' = \dfrac{1}{(\ln a)x}, \\
(\sin x)' = \cos x, & (\cos x)' = -\sin x, & (\tan x)' = (\sec x)^2, \\
(\csc x)' = -\csc x \cot x, & (\sec x)' = \sec x \tan x, & (\cot x)' = -(\csc x)^2, \\
(\arcsin x)' = \dfrac{1}{\sqrt{1-x^2}}, & (\arccos x)' = -\dfrac{1}{\sqrt{1-x^2}}, & (\arctan x)' = \dfrac{1}{x^2+1}.
\end{array}
$$

Theorem 13.9. *If $f(x)$ is defined on some open interval containing a point c and has a local maximum or minimum c, then c is a critical number of $f(x)$ (i.e. either $f'(c) = 0$ or $f'(c)$ does not exist).*

Definition 13.10. Let $f(x)$ be continuous on an interval $[a, b]$. Divide the interval into n subintervals of equal length: $[x_0, x_1]$, $[x_1, x_2]$, $\ldots$, $[x_{n-1}, x_n]$ where $x_0 = a$ and $x_n = b$. Let $\Delta x = \dfrac{b-a}{n}$ be the length of each subinterval. Then the sum

$$R_n = \sum_{i=1}^{n} f(x_i)\Delta x$$

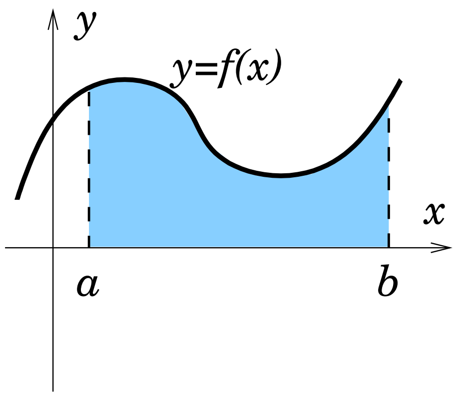
is called the **Riemann sum** of $f(x)$ on $[a, b]$ using n subintervals. It can be proved that the limit of R_n as n approaches infinity exists, and

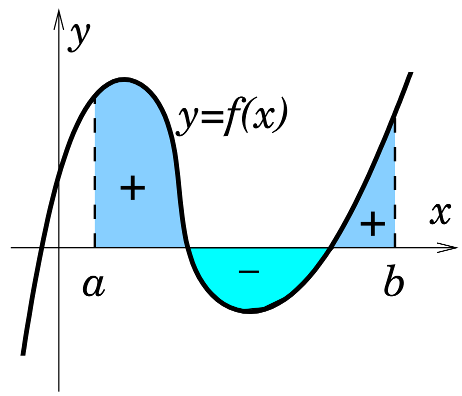
$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$$

is called the integral of $f(x)$ from a to b.

If $f(x) \geq 0$, then $\displaystyle\int_a^b f(x)dx$ is the area of the region under the curve $y = f(x)$ and above the x-axis from a to b.

If $f(x)$ takes on both positive and negative values, then $\displaystyle\int_a^b f(x)dx$ is the sum of the areas under the curve and above the x-axis minus the sum of the areas under the x-axis and above the curve.





Theorem 13.11. *(Fundamental Theorem of Calculus) If $f(x)$ is continuous on $[a,b]$, then*

I. $\dfrac{d}{dx}\left(\displaystyle\int_a^x f(t)dt\right) = f(x)$ *for all* $x \in [a,b]$.

II. *If* $F'(x) = f(x)$ *for all* $x \in [a,b]$, *then* $\displaystyle\int_a^b f(x)dx = F(b) - F(a)$.

Theorem 13.12. *(Substitution Rule) If $g(x)$ is differentiable and $f(x)$ and $g'(x)$ are continuous, then* $\displaystyle\int f(g(x))g'(x)dx = \int f(u)du$, *where* $u = g(x)$, $du = g'(x)dx$.

Theorem 13.13. *(Some important series)*

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots \quad \textit{is divergent.}$$

$$\sum_{n=0}^{\infty} q^n = 1 + q + q^2 + q^3 + \dots = \begin{cases} \dfrac{1}{1-q} & \textit{if} \quad |q| < 1, \\ \textit{divergent} & \textit{if} \quad |q| \geq 1. \end{cases}$$

$$\sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots = e^x \quad \textit{for all } x.$$

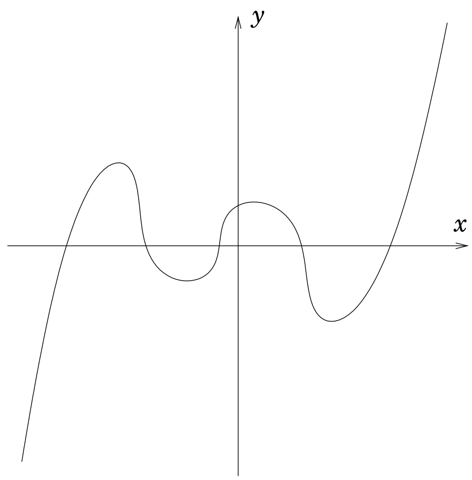
$\Big($*in particular, if* $x = 1$, *then* $\displaystyle\sum_{n=0}^{\infty} \frac{1}{n!} = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \dots = e.\Big)$

$$\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots = \arctan x \textit{ for all } x.$$

$\Big($*in particular, if* $x = 1$, *then* $\displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{1}{2n+1} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots = \arctan 1 = \frac{\pi}{4}.\Big)$

Problems

1. Explain why the curve shown below cannot be the graph of a cubic polynomial.



2. (a) Show that the function $f(x) = \ln(x + \sqrt{x^2 + 1})$ is odd.
   (b) Find the inverse of $f(x)$.

3. Show that any ellipsoid (given by $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$) has a cross-section that is a circle.

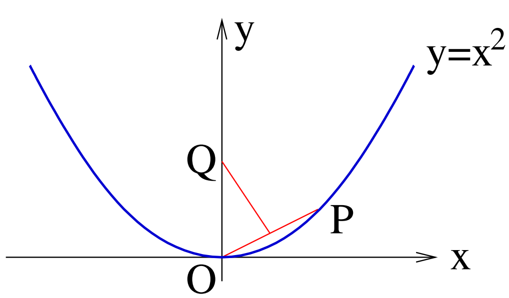
4. Let $T(x)$ denote the temperature at the point x on the surface of Earth at some fixed time. Assuming that T is a continuous function of x, show that at this moment in time there are at least two diametrically opposite points on the equator that have the same temperature.

5. Find all values of a such that $x^2 + ax + 1 \geq \cos x$ for all real x.

6. Find all positive values of a such that that $a^x \geq 1 + x$ for all real x.

7. Show that for $x > 0$,
$$\frac{x}{x^2 + 1} < \arctan x < x.$$

8. The figure below shows a point P on the parabola $y = x^2$ and the point Q where the perpendicular bisector of OP intersects the y-axis. As P approaches the origin along the parabola, what happens to Q? Does it have a limiting position? If so, find it.



9. Find a cubic polynomial $p(x) = ax^3 + bx^2 + cx + d$ that has a local maximum at $(0, 1)$ and a local minimum at $(1, 0)$.

10. Find a polynomial function that has local maxima at $(0, 1)$ and $(2, 1)$ and a local minimum at $(1, 0)$.

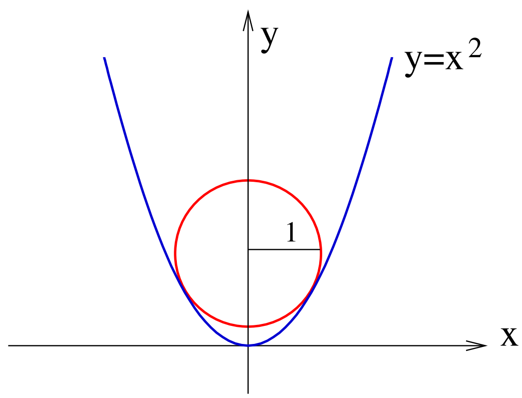
11. Find a number c such that the line $y = x - 1$ is tangent to the parabola $y = cx^2$.

12. Find a quadratic polynomial $y = ax^2 + bx + c$ whose graph passes through the origin and is tangent to the line $y = x - 2$ at the point $(2, 0)$.

13. The parabola $y = x^2 + 2$ has two tangent lines that pass through the origin. Find their equations.

14. Find the number a such that the parabola given by $y = x^2 + a$ is tangent to the lines $y = x$ and $y = -x$.

15. The figure below shows a circle with radius 1 inscribed in the parabola $y = x^2$. Find the center of the circle.



16. Find an equation of a line tangent to both of the following parabolas: $y = (x + 2)^2 - 1$ and $y = 1 - (x - 2)^2$.

17. Find the n-th derivative of $f(x) = \dfrac{1}{x^2 + x}$.

18. Find the n-th derivative of the function $f(x) = \dfrac{x^n}{1 - x}$.

19. The parabola $y = x^2$ and the line $y = mx + 1$ are given. They have two intersection points, A and B. Find the point C on the parabola that lies between A and B and maximizes the area of $\triangle ABC$.

20. Find the interval $[a, b]$ for which the integral $\displaystyle\int_a^b (2 + x - x^2)dx$ has the largest possible value.

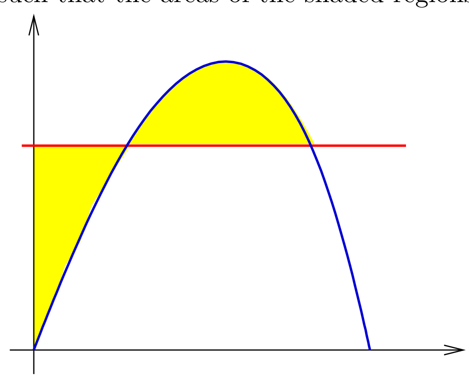
21. Find an equation of the line with a negative slope and passing through the point $(1, 1)$ such that the triangle bounded by this line and the axes is divided by the parabola $y = x^2$ into two regions of equal area.

22. There is a line through the origin that divides the region bounded by the parabola $y = x - x^2$ and the x-axis into two regions with equal area. What is the slope of that line?

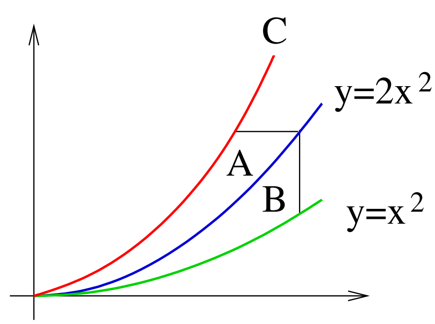
23. Find all values of a for which the area of the region bounded by the line $y = ax$ and the parabola $y = x^2$ is equal to 1.

24. Find all positive numbers a such that the area enclosed by $y = x^2$, $x = a$, and $y = 0$ is equal to the area enclosed by $y = x^3$, $x = a$, and $y = 0$.

25. The figure below shows a horizontal line $y = c$ intersecting the curve $y = 8x - 27x^3$. Find the number c such that the areas of the shaded regions are equal.



26. The figure below shows a curve C with the property that, for every point P on the middle curve $y = 2x^2$, the areas A and B are equal. Find an equation for C.



27. Find a curve that passes through the point $(3, 2)$ and has the property that if the tangent line is drawn at any point P on the curve, then the part of the tangent line that lies in the first quadrant is bisected by P.

28. Find the volume of a 4-dimensional unit ball.

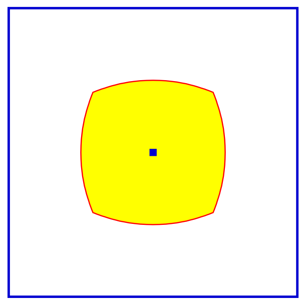
29. Evaluate $\displaystyle\int_0^1 \left(\sqrt[3]{1 - x^7} - \sqrt[7]{1 - x^3}\right) dx$.

30. Evaluate $\displaystyle\int \frac{1}{x^7 - x} dx$.
The straightforward approach would be to start with partial fractions, but that would be too brutal. We could reduce the power of the denominator as follows. Observe that $\displaystyle\int \frac{1}{x^7 - x} dx = \int \frac{x}{x^8 - x^2} dx$. Let $u = x^2$, then $du = 2xdx$, or $\dfrac{du}{2} = xdx$, and we have

$\displaystyle\int \frac{x}{x^8 - x^2}dx = \frac{1}{2}\int \frac{1}{u^4 - u}du$. Now, $u^4 - u$ is better than $x^7 - x$, but can you find an even better substitution?

31. Evaluate the integral: $\displaystyle\int_0^1 \arcsin(x)dx$.

32. The figure below shows a region consisting of all points inside a unit square that are closer to the center than to the sides of the square. Find the area of the region.



33. Let a_1, a_2, ..., a_{30} be real numbers. Show that $a_1 \cos x + a_2 \cos(2x) + \cdots + a_{30} \cos(30x)$ cannot take on only positive values.

34. If a_0, a_1, a_2, ..., a_k are real numbers and $a_0 + a_1 + a_2 + \cdots + a_k = 0$, show that

$$\lim_{n\to\infty}(a_0\sqrt{n} + a_1\sqrt{n+1} + a_2\sqrt{n+2} + \cdots + a_k\sqrt{n+k}) = 0.$$

35. Let $f(x) = a_1 \sin x + a_2 \sin(2x) + a_3 \sin(3x) + \cdots + a_n \sin(nx)$, where $a_1, \ldots, a_n$ are real numbers and n is a positive integer. If it is given that $|f(x)| \le |\sin(x)|$ for all x, show that $|a_1 + 2a_2 + \cdots + na_n| \le 1$.

36. Evaluate $\displaystyle\lim_{n\to\infty}\left(\frac{1}{\sqrt{n}\sqrt{n+1}} + \frac{1}{\sqrt{n}\sqrt{n+2}} + \cdots + \frac{1}{\sqrt{n}\sqrt{n+n}}\right)$.

37. Find the sum of the series $\displaystyle\sum_{n=0}^{\infty}\frac{1}{2^{2n+1}} = \frac{1}{2} + \frac{1}{2^3} + \frac{1}{2^5} + \frac{1}{2^7} + \ldots$.

38. Find the sum of the series

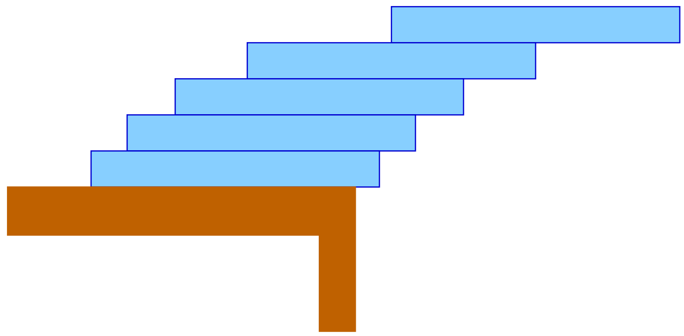
$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{6} + \frac{1}{8} + \frac{1}{9} + \frac{1}{12} + \frac{1}{16} + \frac{1}{18} + \ldots$$

where the terms are the reciprocals of the positive integers whose only prime factors are 2s and 3s.

39. Show that e is irrational.

40. Suppose you have a large supply of books, all the same size, and you stack them at the edge of a table, with each book extending farther beyond the edge of the table than the one beneath it. Show that it is possible to do this so that the top book extends entirely beyond the table. In fact, show that the top book can extend any distance at all beyond the edge of the table if the stack is high enough. Try the following method of stacking: The top book extends half its length beyond the second book. The second

book extends a quarter of its length beyond the third. The third extends one-sixth of its length beyond the fourth, and so on. (You could try it yourself with books, CD cases, or something similar. To make your construction stable, make each book, CD case, etc. extend a little bit less than described in this problem.)
Hint: consider centers of mass.



41. Find a continuous nonnegative function $f(x)$ such that the area of the region under the graph of $f(x)$ (and above the x-axis) from $x = 0$ to $x = t$ is $A(t) = t^3$ for all $t > 0$.

42. Recall that the area of a circle with radius r is $A = \pi r^2$ and the circumference of the circle is $L = 2\pi r$. Notice that $(\pi r^2)' = 2\pi r$. Similarly, the volume of a ball with radius r is $V = \frac{4}{3}\pi r^3$, the surface area is $S = 4\pi r^2$, and $\left(\frac{4}{3}\pi r^3\right)' = 4\pi r^2$. Is this a coincidence? Actually, it isn't. Explain these facts. What is the ratio of the 4-dimensional volume to the usual 3-dimensional volume of its boundary (the analog of the surface area) for a 4-dimensional ball with radius r?

43. What is the ratio of the 5-dimensional volume of a 5-dimensional ball of radius r to the 4-dimensional volume of its boundary (the analog of the surface area)?

Chapter 14

Various problems

Most problems in this section can be solved in a few different ways.

Problems

1. December 25, 2005 was a Sunday. Without using any technology, calculate the day of the week for December 25, 2050.

2. What is the last (units) digit of $1^{1234} + 2^{1234} + 3^{1234} + 4^{1234} + 5^{1234}$?

3. Show that there is no reentrant knight's tour on a 5×5 chessboard (recall that a tour is called reentrant if there is a legal move that takes the knight from the last square of the tour back to where the tour began).

4. Prove that for any integer number n, $n^7 - n$ is divisible by 7.

5. A sequence $\{a_n\}$ is defined recursively by the equations

$$a_0 = a_1 = 1, \qquad n(n-1)a_n = (n-1)(n-2)a_{n-1} - (n-3)a_{n-2}.$$

   Find the sum of the series $\displaystyle\sum_{n=0}^{\infty} a_n$.

6. Evaluate the integral: $\displaystyle\int_{-2}^{3} ||x| - 1|\, dx$

7. Solve the inequality: $|6 - |x| - x| + x \leq 3$.

8. What is the area of the region defined by the inequality

$$|3x - 18| + |2y + 7| \leq 3?$$

9. (a) Find an example of a polygon and a point in its interior, so that no side of the polygon is completely visible from that point.

   (b) Find an example of a polygon and a point in its exterior, so that no side of the polygon is completely visible from that point.

10. The plane is colored with two colors (that is, each point is assigned one of two colors). Prove that there exist three points of the same color which are vertices of a regular triangle.

11. The plane is colored with finitely many colors (that is, each point is assigned one of finitely many colors). Prove that one can find a rectangle with vertices of the same color.

12. Each block of a 25×25 board has either 1 or -1 written on it. Let a_i be the product of all numbers in the ith row and b_j be the product of all numbers in the jth column. Prove that $a_1 + \cdots + a_{25} + b_1 + \cdots + b_{25} \neq 0$.

13. Prove that if 40 coins are distributed among 9 bags so that each bag contains at least one coin, then at least two bags contain the same number of coins.

14. Let twenty distinct positive integers be all less than 70. Prove that among their pairwise differences there are four equal numbers.

15. If a camel (as defined in problem 5 in Chapter 9) starts at the lower-left corner of an 8×8 board, it is easy to check that it can get to the upper-right corner. What is the smallest number of moves necessary to do this?

16. A 6×6 square is tiled by 2×1 dominoes. Prove that it has at least one *fault-line*, that is, a straight line cutting the square without cutting any domino.

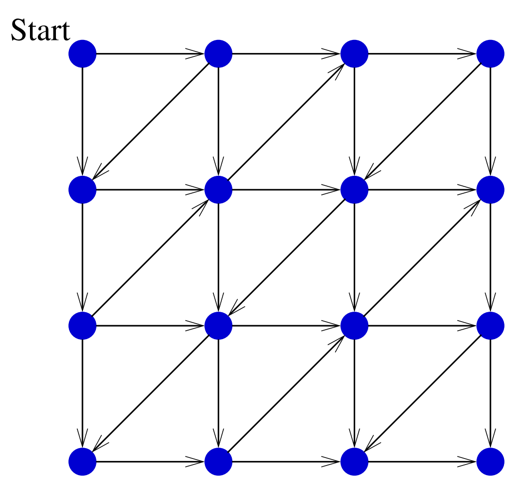
17. Prove that there are no Fibonacci numbers ending with 100.

18. Prove that there are infinitely many Fibonacci numbers divisible by 100.

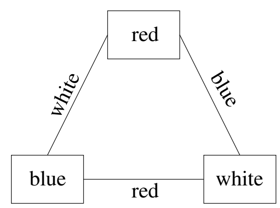
19. How many three-digit numbers satisfy the property that the middle digit is the average of the first and the last digits?

20. How many positive integers not exceeding 1000 are divisible by either 4 or 6?

21. A counter is placed at the top left vertex (marked "Start") of the graph shown below. Two players play the following game. Turns alternate. In one turn, a player may move the counter to any adjacent vertex along an edge, in the direction of the arrow. The player who cannot make a move loses. Find a winning strategy for one of the players.



22. Let n be a positive odd integer. There are n computers. Each pair of computers is connected by one cable. We want to color the computers and the cables in such a way that (1) all computers have different colors, (2) no two cables joined to one computer have the same color, and (3) no computer has the same color as a cable joined to it. Below is an example for $n = 3$:

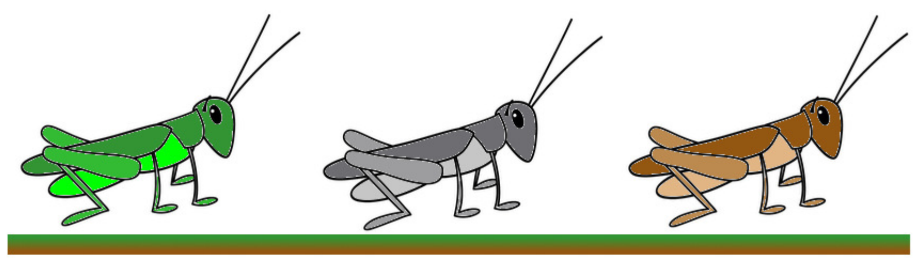


Prove that for any odd n, this can be done using n colors.

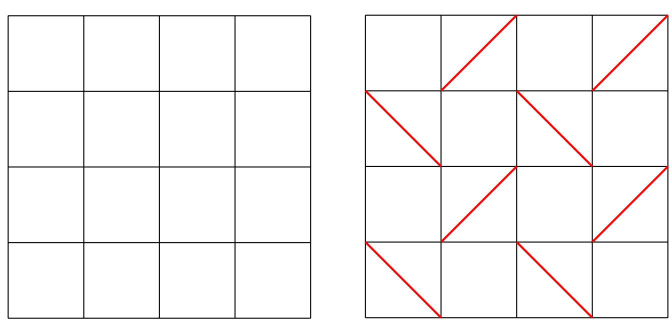
23. Find the units digit of the number $2^{(3^5)}$.

24. Find the units digit of the number $2^{\left(3^{\left(5^7\right)}\right)}$.

25. Three grasshoppers, one green, one gray, and one brown, play leapfrog along a line. Initially, the green one is on the left, the gray one is in the middle, and brown one is on the right, as pictured below. Each second, one grasshopper leaps over another, but not over two others. Can the grasshoppers return to their initial order after 11 leaps?



26. The 4×4 grid is composed of 16 unit squares. What is the largest number of diagonals of the unit squares that can be drawn in such a way that none of them share an endpoint? Below is an example of eight diagonals satisfying this condition. However, it is possible to draw more diagonals.

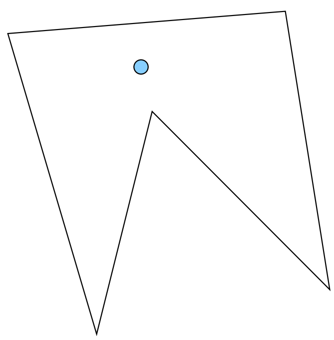


27. Prove that among any 21 distinct positive integers not exceeding 100 there are four different ones, a, b, c, d, such that $a + b = c + d$.

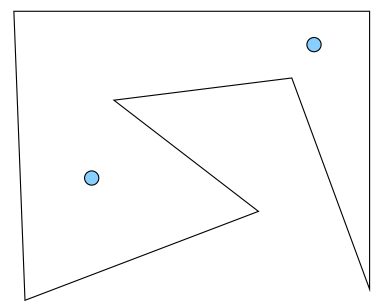
28. Which natural numbers are sums of consecutive smaller natural numbers? For example, $30 = 9 + 10 + 11$ and $31 = 15 + 16$, but 32 has no such representation. Find a simple necessary and sufficient condition.

29. *The Art Gallery Problem.* An art gallery has the shape of an n-gon (not necessarily a convex one). The boundary of the n-gon are the only walls, there are no walls inside it. The gallery is to be guarded by security cameras that can each rotate to obtain a full field of vision. Prove that $\left[\frac{n}{3}\right]$ (the integer part of $\frac{n}{3}$) cameras can survey the building, no matter how complicated its shape.

    Below are two examples.



pentagon, $\left[\frac{5}{3}\right] = 1$
1 camera can survey the building



7-gon, $\left[\frac{7}{3}\right] = 2$
2 cameras can survey the building

Solutions and answers to selected problems

0 Preliminaries

0.1 Review of logic

1. Construct the truth tables.

   (a)

p	q	$p \to q$	$\neg q$	$\neg p$	$\neg q \to \neg p$
T	T	T	F	F	T
T	F	F	T	F	F
F	T	T	F	T	T
F	F	T	T	T	T

   The truth values in the columns for $p \to q$ and $\neg q \to \neg p$ are the same, thus the propositions are logically equivalent.

3. Every student at my university has a computer or has a friend who has a computer.

5. (a) False since $4 \neq -5$.
   (b) True since both $P(4)$ and $Q(4)$ are false.
   (c) True. For example, $x = 4$ makes $P(x)$ false, so $\neg P(x)$ true.
   (d) False. For example, if $x = 4$, then both $P(4)$ and $Q(4)$ are false, so the disjunction is false.
   (e) True. For example, if $x = -5$, then both $P(x)$ and $Q(x)$ are true, so the conjunction is true.
   (f) True. If $P(x)$ holds, then $x = -5$, then $x^2 = 25$, so $Q(x)$ holds, so the implication holds for any x.
   (g) True. For example, if $x = -5$, then both $P(x)$ and $Q(x)$ are true, so the implication is true.
   (h) False. For example, if $x = 5$, then $P(x)$ is false and $Q(x)$ is true, so the biconditional is false.

7. The statement $\forall x \exists y(x + y = 0)$ is true because for any x, let $y = -x$, then $x + y = 0$. The statement $\exists y \forall x(x + y = 0)$ is false because there is no value of y such that for any x the equality $x + y = 0$ holds: for any y, consider $x = -y + 1$, then $x + y = 1$, so $x + y \neq 0$.

9. (a) True. For example, for $x = 1$ and $y = 2$, we have $1 < 2$.
   (b) False. Counterexample: $x = 2$, $y = 1$. We have $2 \not< 1$.
   (c) False. Counterexample: $x = -1$. We have $1 \not< -1$.
   (d) False. There is no such x that for any y, $x < y$, because for any x we can take $y = x$, then $x \not< y$.
   (e) True. For any y, consider $x = y - 1$, then $x < y$.
   (f) False. There is no such y that for any x, $x < y$, because for any y we can take $x = y$, then $x \not< y$.
   (g) True. For any x, consider $y = x + 1$, then $x < y$.

11. (a) False. There is no such x that for any y, $x^2 + y = 5$, because for any x we can take $y = -x^2$, then $x^2 + y = 0 \neq 5$.
   (b) False. For example, for $y = 6$, there is no value of x such that $x^2 + 6 = 5$ because for any real x, we have $x^2 + 6 \geq 0 + 6 = 6 > 5$.
   (c) False. There is no such y that for any x, $x^2 + y = 5$, because if $y \neq 5$, we can take $x = 0$, and we have $x^2 + y = y \neq 5$, while if $y = 5$, we can take $x = 1$, and we have $x^2 + y = 1 + 5 = 6 \neq 5$.
   (d) True. For any x, consider $y = 5 - x^2$, then $x^2 + y = 5$.

13. (a) Not logically equivalent. Consider $P(x) =$"$x = 2$" with domain being the set of real numbers. Then $\forall x(\neg P(x))$ means $\forall x(x \neq 2)$ and is false, while $\neg(\forall x P(x))$ is true since $\forall x P(x)$, or $\forall x(x = 2)$, is false.
   (c) Logically equivalent. The proposition $\forall x(P(x) \wedge Q(x))$ is true if and only if $P(x) \wedge Q(x)$ is always true, which is the case if and only if both $P(x)$ and $Q(x)$ are always true, or $(\forall x P(x)) \wedge (\forall x Q(x))$.
   (e) Not logically equivalent. Consider $P(x) =$"x is an odd integer" and $Q(x) =$"x is an even integer" with domain being the set of integers. Then $\forall x(P(x) \leftrightarrow Q(x))$ is false (e.g. for $x = 1$, $P(x)$ is true and $Q(x)$ is false) while $(\forall x P(x)) \leftrightarrow (\forall x Q(x))$ is true since both $\forall x P(x)$ and $\forall x Q(x)$ are false.
   (g) Logically equivalent. The proposition $\exists x(P(x) \vee Q(x))$ is true if and only if for at least one value of x, $P(x) \vee Q(x)$ is true. i.e. for at least one value of x at least one of $P(x)$ and $Q(x)$ holds, i.e. at least one of $\exists x P(x)$ and $\exists x Q(x)$ holds.
   (i) Not logically equivalent. Consider $P(x) =$"x is an even integer" and $Q(x) =$"$x^2 = -1$" with domain being the set of integers. Then $\exists x(P(x) \to Q(x))$ is true (e.g. for $x = 1$, both $P(x)$ and $Q(x)$ are false, so the implication is true) while $(\exists x P(x)) \to (\exists x Q(x))$ is false (because $\exists x P(x)$ is true and $\exists x Q(x)$ is false, so the implication is false).

15. Let $S = \mathbb{R}$ and $P(x, y) =$"$x < y$." For any x, y, and z, we have that $x < y$ and $y < z$ implies $x < z$. This is the transitivity property, that is, the relation "$<$" is transitive on the set of real numbers. Recall that transitivity is one of the properties of an equivalence relation (you probably have learned about equivalence relations in your previous courses). Note, however, that the above relation is not reflexive or symmetric, so it is not an equivalence relation.

17. The definition is as follows: the sequence $a_1, a_2, a_3, \dots$ converges if there exists a number L such that for any positive ε there exists an index N such that for any $n \geq N$, $|a_n - L| < \varepsilon$. We rewrite this definition using quantifiers:

$$\exists L\ \forall \varepsilon > 0\ \exists N\ \forall n \geq N\ |a_n - L| < \varepsilon$$

(where L and ε are real numbers, and n and N are natural numbers). If we wish to make the domains explicit, we could write it as follows:

$$\exists L \in \mathbb{R}\ \forall \varepsilon \in \mathbb{R}\ \big((\varepsilon > 0) \to \big(\exists N \in \mathbb{N}\ \forall n \in \mathbb{N}\ ((n \geq N) \to (|a_n - L| < \varepsilon))\big)\big).$$

0.2 Types of proofs

1. We will prove the statement by contrapositive: Suppose n is odd. Then $n = 2k+1$ for some integer k. Then $3n+5 = 3(2k+1)+5 = 6k+8 = 2(3k+4)$. Since $3k+4$ is an integer, $3n+5$ is even. Thus we have proved that if n is odd, then $3n+5$ is even. Therefore if $3n+5$ is odd, then n is even.

3. *Solution 1.* Assume $145n-1$ is odd. Then $145n-1 = 2k+1$ for some integer k. Then $n = 2k+2-144n$, and $n+1000 = 2k+1002-144n = 2(k+501-72n)$. Since $k+501-72n$ is an integer, we have that $n+1000$ is even. This is a direct proof.

   Solution 2. Assume to the contrary that for some integer n, $145n-1$ is odd but $n+1000$ is not even, that is, $n+1000$ is odd as well. Then $145n-1 = 2k+1$ and $n+1000 = 2l+1$ for some integers k and l. Adding the above two equations, we have $146n+999 = 2k+2l+2$. It follows that $999 = 2k+2l+2-146n = 2(k+l+1-73n)$. Since 999 is odd but $2(k+l+1-73n)$ is even, we have a contradiction. Thus the original statement is true. This is proof by contradiction.

   Note. Yet another way to prove the given statement is by proving the following two lemmas.

   - If $145n-1$ is odd, then n is even.
   - If n is even, then $n+1000$ is even.

   Perhaps one of the easiest ways to prove the first lemma is by contrapositive similarly to problem 1, and the second lemma can be proved directly.

5. The statement is false; $n = 3$ is a counterexample, since $2^3+1 = 9$ is not prime.

7. By problem 2, an integer is even if and only if its square is even. Therefore an integer is odd if and only if its square is odd. Thus if an odd integer N is a perfect square, then $N = m^2$ for some odd integer m. Thus $m = 2k+1$ for some integer k. Then $N = m^2 = (2k+1)^2 = 4k^2+4k+1 = 4(k^2+k)+1$, so N is of the form $4n+1$. This proof is direct.

   The converse is "if an odd number has the form $4n+1$, then it is a perfect square." This is false because for example $5 = 4\cdot 1+1$ but 5 is not a perfect square. Thus 5 is a counterexample.

9. (a) Let $f(x) = x^{101}+x^{51}+x+1$. Then $f(-1) = -2 < 0$ and $f(1) = 4 > 0$. Since $f(x)$ is a continuous function, by the intermediate value theorem $f(x)$ has a root. This proof is nonconstructive because we did not construct a root, we only proved its existence.

   (b) Suppose $f(x)$ has two distinct roots. By the mean value theorem there is a number c between these roots such that $f'(c) = 0$. However,

   $$f'(x) = 101x^{100} + 51x^{50} + 1 > 0$$

   for all x. We get a contradiction. This is a proof by contradiction.

11. The value $x = \frac{\pi}{6}$ is a root of the equation. This is a constructive proof since we provided an explicit example.

13. The roots of the equation $2x^2 + 8x + 7 = 0$ can be found using the quadratic formula. We have $x = \frac{-8 \pm \sqrt{8}}{4} = -2 \pm \frac{\sqrt{2}}{2}$. It can be proved that both roots are irrational numbers. Since any quadratic equation has exactly two roots (counting complex and with multiplicity), there are no other roots. In particular, there are no rational solutions. This proof is direct.

15. An integer divisible by 8 has the form $8n$ where n is an integer, and

$$8n = (4n^2 + 4n + 1) - (4n^2 - 4n + 1) = (2n + 1)^2 - (2n - 1)^2.$$

This proof is direct and constructive: for any integer divisible by 8, we gave an explicit example of two perfect squares whose difference is equal to that integer.

1 Introduction

1. Assume to the contrary that each of the eleven children contributed at most \$2.72. Then the total amount does not exceed $2.72 \cdot 11 = 29.92$ dollars. However, the total amount is \$30.00. Therefore our assumption is false, thus at least one child contributed at least \$2.73. This kind of proof is called a proof by contradiction (see section 0.2). Such problems can also be solved using the Generalized Dirichlet's Principle (see chapter 3).

2. (a) Any two-digit number N can be written in the form $N = 10a + b$ where b is the units digit of the number and a is its tens digit. (For example, $27 = 10 \cdot 2 + 7$.) Suppose that N is divisible by 3. Then $N = 3k$ for some integer k. Thus

$$\begin{aligned} 10a + b &= 3k, \\ 9a + a + b &= 3k, \\ a + b &= 3k - 9a, \\ a + b &= 3(k - 3a). \end{aligned}$$

Since $k - 3a$ is an integer, $a + b$ is divisible by 3.
Conversely, suppose that $a + b$ is divisible by 3. Then $a + b = 3m$ for some integer m. Thus

$$\begin{aligned} 9a + a + b &= 9a + 3m, \\ 10a + b &= 3(3a + m), \\ N &= 3(3a + m). \end{aligned}$$

Since $3a + m$ is an integer, N is divisible by 3.

(b) Any natural number N can be written in the form

$$N = 10^n a_n + 10^{n-1} a_{n-1} + \cdots + 10a_1 + a_0$$

where a_0 is the units digit of the number, a_1 is the tens digit, and so on (this is called the base 10 expansion of the number N, see chapter 4). Now,

$$\begin{aligned} N &= \underbrace{99\ldots9}_{n} a_n + a_n + \underbrace{99\ldots9}_{n-1} a_{n-1} + a_{n-1} + \cdots + 9a_1 + a_1 + a_0 \\ &= \underbrace{99\ldots9}_{n} a_n + \underbrace{99\ldots9}_{n-1} a_{n-1} + \cdots + 9a_1 + (a_n + a_{n-1} + \cdots + a_1 + a_0). \end{aligned}$$

Since all multiples of 9 are divisible by 3, an argument similar to the one in part (a) shows that N is divisible by 3 if and only if

$$a_n + a_{n-1} + \cdots + a_1 + a_1$$

is (also, see chapter 4 for divisibility properties).

3. False. For example, for $n = 41$, $n^2 + n + 41 = 41^2 + 41 + 41 = 41 \cdot 43$ is not prime.

   Note. You may be tempted to check a few small values of n. You would discover then that for $1 \leq n \leq 39$, the number $n^2 + n + 41$ is indeed prime. However, the above example shows that this is not the case for *all* natural values of n. Thus checking a few examples is not sufficient!

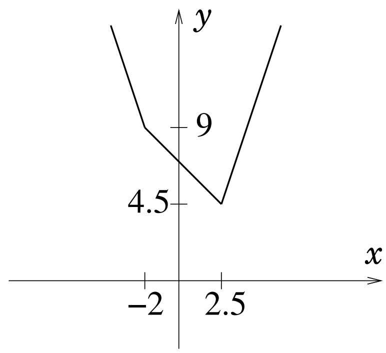
4. Choose a row with the biggest number of stars in it. Note that this row contains at least two stars since if each row contained at most one star then there would be at most four stars total. However, there are six stars. (This argument is the idea behind Dirichlet's box principle, see chapter 3.) So we have the following three cases:
   Case I. This row contains four stars. Then cross it out, and there will only be two stars left. If they are in different columns, then cross out any other row, and the two columns containing the remaining two stars. If the remaining two stars are in one column then cross out one more row, the column containing the stars, and any other column.
   Case II. This row contains three stars. Then cross it out, and there will only be three stars left. We can eliminate three stars by crossing out one more row and two columns.
   Case III. This row contains two stars. Cross it out, and there will only be four stars in three rows left. Therefore at least one of these rows contains two stars. Cross it out, and there will only be two stars left. As above, we can eliminate two stars by crossing out two columns.

5. True. There are several ways to prove this. One is by induction (see chapter 2), another one is considering all possible remainders of n modulo 3 (see chapters 4 and 5). Here is a third way: $n^3 + 2n = n^3 - n + 3n = n(n^2 - 1) + 3n = n(n-1)(n+1) + 3n$. The term $n(n-1)(n+1)$ is the product of three consecutive numbers, and one of them is divisible by 3 (see chapter 4), therefore the product is divisible by 3. The term $3n$ is also divisible by 3. Thus the sum is divisible by 3.

6. Since $|x+2| = \begin{cases} x+2 & \text{if } x+2 \geq 0, \text{ i.e. if } x \geq -2 \\ -x-2 & \text{if } x+2 < 0, \text{ i.e. if } x < -2 \end{cases}$ and
   $|2x-5| = \begin{cases} 2x-5 & \text{if } 2x-5 \geq 0, \text{ i.e. if } x \geq 2.5 \\ -2x+5 & \text{if } 2x-5 < 0, \text{ i.e. if } x < 2.5 \end{cases}$, we have

$$f(x) = |x+2| + |2x-5| = \begin{cases} x+2+2x-5 = 3x-3 & \text{if } x \geq 2.5, \\ x+2-2x+5 = -x+7 & \text{if } -2 \leq x < 2.5, \\ -x-2-2x+5 = -3x+3 & \text{if } x < -2. \end{cases}$$

   So we draw the graph of each linear function for the corresponding interval.



7. No. Consider the four regions of the town, namely, the two river banks and the two islands. Each of them is connected with other regions by either 3 or 5 bridges. Suppose that there is a tour of the town that crosses every bridge exactly once. For each intermediate region on such a tour we must come to the region by a bridge and leave the region by a bridge. So every time the tour visits a region, two bridges are crossed. This means that for every region except the one where we start and the one where we end there must be an even number of bridges connecting that region to others. However, we have 4 regions with an odd number of bridges. Thus we get a contradiction.

   The above solution can be explained in an easier and "smoother" way if we use the graph terminology discussed in chapter 10.

8. The answer to parts (a) and (b) is yes. There are even many such tours. Below are two tours (squares are numbered in the order the knight can visit them). Notice that in the second one, it is possible to go from square number 64 back to square number 1. Such a tour is called reentrant (see chapter 10).

1	40	13	26	3	42	15	28
24	37	2	41	14	27	4	43
39	12	25	60	53	62	29	16
36	23	38	63	56	59	44	5
11	50	57	54	61	52	17	30
22	35	64	51	58	55	6	45
49	10	33	20	47	8	31	18
34	21	48	9	32	19	46	7

1	14	17	42	3	38	19	40
16	43	2	63	18	41	4	37
13	64	15	58	53	56	39	20
44	27	12	55	62	59	36	5
11	30	61	52	57	54	21	50
26	45	28	31	60	51	6	35
29	10	47	24	33	8	49	22
46	25	32	9	48	23	34	7

   Note. You may not be able to find such a tour quickly. However, it is important to recognize the fact that if you tried and were unable to find one, that does not prove that such a tour does not exist.

   The answer to (c) is no. Note that with every move, the knight goes to a square of the opposite color. After two moves it returns to a square of the same color it started on. Thus, after any even number of moves it is going to be on the same color, and after any odd number of moves it is going to be on the opposite color than it started on. To visit every square of the board exactly once, the knight has to make 63 moves. Since 63 is odd, the knight must end on a square of the opposite color. However, the upper-left and lower-right corners of the board are of the same color. Thus it is impossible to end at the lower-right corner.

While in parts (a) and (b) the colors of the chessboard squares are not important, our proof in part (c) constructs a contradiction using the coloring. We will see more examples of such proofs in chapter 9.

2 Principle of Mathematical Induction

1. (a) We will prove this formula by Mathematical Induction.

Basis step. For $n = 1$ we have $1^2 = \frac{1 \cdot 2 \cdot 3}{6}$ which is true.

Inductive step. Suppose the formula holds for $n = k$, i.e.

$$1^2 + 2^2 + 3^2 + \cdots + k^2 = \frac{k(k+1)(2k+1)}{6}.$$

Adding $(k+1)^2$ to both sides gives

$$\begin{aligned} 1^2 + 2^2 + 3^2 + \cdots + k^2 + (k+1)^2 &= \frac{k(k+1)(2k+1)}{6} + (k+1)^2 \\ &= \frac{k(k+1)(2k+1) + 6(k+1)^2}{6} \\ &= \frac{(k+1)(k(2k+1) + 6(k+1))}{6} \\ &= \frac{(k+1)(2k^2 + 7k + 1)}{6} \\ &= \frac{(k+1)(k+2)(2k+3)}{6} \\ &= \frac{(k+1)((k+1)+1)(2(k+1)+1)}{6}. \end{aligned}$$

Thus the formula holds for $n = k + 1$.

(c) Proof by Mathematical Induction.

Basis step. For $n = 1$ we have $1 \cdot 1! = 2! - 1$, or $1 = 2 - 1$ which is true.

Inductive step. Suppose the formula holds for $n = k$, i.e.

$$1 \cdot 1! + 2 \cdot 2! + \cdots + k \cdot k! = (k+1)! - 1.$$

Adding $(k+1) \cdot (k+1)!$ to both sides gives

$$\begin{aligned} 1 \cdot 1! + 2 \cdot 2! + \cdots + k \cdot k! + (k+1) \cdot (k+1)! &= (k+1)! - 1 + (k+1) \cdot (k+1)! \\ &= (k+1)!(1 + k + 1) - 1 \\ &= (k+2)! - 1. \end{aligned}$$

Thus the formula holds for $n = k + 1$.

(e) Proof by Mathematical Induction.

Basis step. For $n = 1$ we have $1 \cdot 2 = \frac{1 \cdot 2 \cdot 3}{3}$ which is true.

Inductive step. Suppose the formula holds for $n = k$, i.e.

$$1 \cdot 2 + 2 \cdot 3 + \cdots + k(k+1) = \frac{k(k+1)(k+2)}{3}.$$

Adding $(k+1)(k+2)$ to both sides gives

$$\begin{aligned}1\cdot 2+2\cdot 3+\cdots+k(k+1)+(k+1)(k+2) &= \frac{k(k+1)(k+2)}{3}+(k+1)(k+2)\\ &= \frac{k(k+1)(k+2)+3(k+1)(k+2)}{3}\\ &= \frac{(k+1)(k+2)(k+3)}{3}.\end{aligned}$$

Thus the formula holds for $n=k+1$.

3. Proof by Mathematical Induction.
Basis step. If $q=1$, then it is true that $3^{2^q}-1=3^2-1=8$ is divisible by $2^{q+2}=2^3=8$.
Inductive step. Assume that the statement holds for $q=k$, i.e. $3^{2^k}-1$ is divisible by 2^{k+2}. We want to prove that the statement holds for $q=k+1$, i.e. $3^{2^{k+1}}-1$ is divisible by 2^{k+3}. Observe that $3^{2^{k+1}}-1=3^{2^k\cdot 2}-1=\left(3^{2^k}\right)^2-1=\left(3^{2^k}-1\right)\left(3^{2^k}+1\right)$. By the induction hypothesis, $3^{2^k}-1$ is divisible by 2^{k+2}. Note that $3^{2^k}+1$ is the sum of two odd numbers, thus it is an even number, that is, divisible by 2. Then the product $\left(3^{2^k}-1\right)\left(3^{2^k}+1\right)$ is divisible by $2^{k+2}\cdot 2=2^{k+3}$.

5. (a) Proof by Mathematical Induction.
Basis step. If $n=1$, the identity says that $F_1F_2=F_2^2$, i.e. $1\cdot 1=1^2$ which is true.
Inductive step. Assume the identity holds for $n=k$, i.e.

$$F_1F_2+F_2F_3+\cdots+F_{2k-1}F_{2k}=F_{2k}^2. \tag{1}$$

We want to prove that it holds for $n=k+1$, i.e.

$$F_1F_2+F_2F_3+\cdots+F_{2(k+1)-1}F_{2(k+1)}=F_{2(k+1)}^2,$$

or, equivalently,

$$F_1F_2+F_2F_3+\cdots+F_{2k+1}F_{2k+2}=F_{2k+2}^2.$$

Using (1) we have:

$$\begin{aligned}F_1F_2+F_2F_3+&\cdots+F_{2k+1}F_{2k+2}\\ &=F_1F_2+F_2F_3+\cdots+F_{2k-1}F_{2k}+F_{2k}F_{2k+1}+F_{2k+1}F_{2k+2}\\ &=F_{2k}^2+F_{2k}F_{2k+1}+F_{2k+1}F_{2k+2}\\ &=F_{2k}(F_{2k}+F_{2k+1})+F_{2k+1}F_{2k+2}\\ &=F_{2k}F_{2k+2}+F_{2k+1}F_{2k+2}\\ &=(F_{2k}+F_{2k+1})F_{2k+2}\\ &=F_{2k+2}^2.\end{aligned}$$

Thus the identity holds for $n=k+1$.

(c) Proof by Mathematical Induction.
Basis step. For $n=1$ the identity is $F_0F_2=F_1^2+(-1)^1$. Since $F_0=0$ and $F_1=F_2=1$, we have $0\cdot 2=1+(-1)$ which is true.
Inductive step. Assume the identity holds for $n=k$, i.e.

$$F_{k-1}F_{k+1}=F_k^2+(-1)^k.$$

We want to show that it holds for $n = k + 1$, i.e.

$$F_{(k+1)-1}F_{(k+1)+1} = F_{k+1}^2 + (-1)^{k+1},$$

or, equivalently,

$$F_k F_{k+2} = F_{k+1}^2 + (-1)^{k+1}.$$

We have

$$\begin{aligned} F_k F_{k+2} &= F_k(F_k + F_{k+1}) \\ &= F_k^2 + F_k F_{k+1} \\ &= F_{k-1}F_{k+1} - (-1)^k + F_k F_{k+1} \\ &= F_{k+1}(F_{k-1} + F_k) + (-1) \cdot (-1)^k \\ &= F_{k+1}^2 + (-1)^{k+1}. \end{aligned}$$

Thus the identity holds for $n = k + 1$.

(d) Hint: recall that multiplication of 2×2 matrices is defined by

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} ae + bg & af + bh \\ ce + dg & cf + dh \end{bmatrix}.$$

(e) Proof by Strong Mathematical Induction.
Basis step. If $n = 1$, then the identity says that $F_0^2 + F_1^2 = F_1^2$, or $0^2 + 1^2 = 1^2$ which is true.

Inductive step. Assume that the identity holds for all $1 \le n \le k$. Namely, we will use that it holds for $n = k$ and $n = k - 1$, i.e.

$$F_{k-1}^2 + F_k^2 = F_{2k-1}$$

and $F_{(k-1)-1}^2 + F_{(k-1)}^2 = F_{2(k-1)-1}$, or equivalently,

$$F_{k-2}^2 + F_{k-1}^2 = F_{2k-3}.$$

We want to prove that it holds for $n = k + 1$, i.e. $F_{(k+1)-1}^2 + F_{k+1}^2 = F_{2(k+1)-1}$, or, equivalently,

$$F_k^2 + F_{k+1}^2 = F_{2k+1}.$$

It may be easier here to work from the right hand side.

$$\begin{aligned} F_{2k+1} &= F_{2k} + F_{2k-1} = F_{2k-1} + F_{2k-2} + F_{2k-1} \\ &= 2F_{2k-1} + F_{2k-2} = 2F_{2k-1} + F_{2k-1} - F_{2k-3} \\ &= 3F_{2k-1} - F_{2k-3} = 3(F_{k-1}^2 + F_k^2) - (F_{k-2}^2 + F_{k-1}^2) \\ &= 3F_{k-1}^2 + 3F_k^2 - F_{k-2}^2 - F_{k-1}^2 = 2F_{k-1}^2 + 3F_k^2 - F_{k-2}^2 \\ &= 2F_{k-1}^2 + 3F_k^2 - (F_k - F_{k-1})^2 \\ &= 2F_{k-1}^2 + 3F_k^2 - F_k^2 + 2F_k F_{k-1} - F_{k-1}^2 \\ &= F_{k-1}^2 + 2F_k^2 + 2F_k F_{k-1} \\ &= F_{k-1}(F_{k-1} + F_k) + F_k(F_k + F_{k-1}) + F_k^2 \\ &= F_{k-1}F_{k+1} + F_k F_{k+1} + F_k^2 \\ &= (F_{k-1} + F_k)F_{k+1} + F_k^2 = F_k^2 + F_{k+1}^2. \end{aligned}$$

Note. The idea of the above inductive step is the following: express F_{2k+1} in terms of F_is with i odd and less than $2k+1$, e.g. in terms of F_{2k-1} and F_{2k-3}, then use the inductive hypothesis to rewrite F_{2k-1} and F_{2k-3} as sums of squares (since we assume that the formula holds for smaller indices), and then rewrite the obtained expression in terms of F_k and F_{k+1} (because the formula we want to prove involves these terms).

7. First we will try to estimate the sum by estimating each term. We see that

$$\frac{1}{3n+1} \leq \text{each term} \leq \frac{1}{n+1},$$

and there are $2n+1$ terms, therefore we have

$$\frac{2n+1}{3n+1} \leq \text{sum} \leq \frac{2n+1}{n+1}.$$

The left inequality doesn't help us, but from the right one we have

$$\text{sum} \leq \frac{2n+1}{n+1} < \frac{2n+2}{n+1} = 2,$$

thus we do not need Mathematical Induction for this part. To show that the sum is bigger than 1, we will use Mathematical Induction.

Basis step. For $n = 1$ we have to check that $1 < \frac{1}{2} + \frac{1}{3} + \frac{1}{4}$. We calculate $\frac{1}{2} + \frac{1}{3} + \frac{1}{4} = \frac{13}{12}$, and we see that this is indeed bigger than 1.

Inductive step. Assume the inequality holds for $n = k$, i.e.

$$1 < \frac{1}{k+1} + \frac{1}{k+2} + \cdots + \frac{1}{3k+1}. \tag{2}$$

We want to prove that it holds for $n = k+1$:

$$1 < \frac{1}{(k+1)+1} + \frac{1}{(k+1)+2} + \cdots + \frac{1}{3(k+1)+1},$$

or

$$1 < \frac{1}{k+2} + \frac{1}{k+3} + \cdots + \frac{1}{3k+1} + \frac{1}{3k+2} + \frac{1}{3k+3} + \frac{1}{3k+4}. \tag{3}$$

Compare (2) and (3), and notice that we "lost" the term $\frac{1}{k+1}$ but "gained" three terms: $\frac{1}{3k+2}$, $\frac{1}{3k+3}$, and $\frac{1}{3k+4}$. If we can show that we gained more than we lost, then the new sum (for $k+1$) is bigger than 1. Thus we want to show that

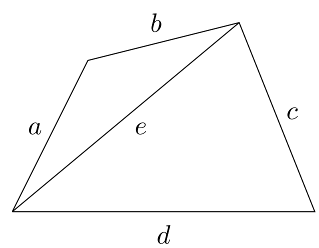
$$\frac{1}{3k+2} + \frac{1}{3k+3} + \frac{1}{3k+4} > \frac{1}{k+1}.$$

The following inequalities are equivalent:

$$\begin{aligned}
\frac{1}{3k+2}+\frac{1}{3k+3}+\frac{1}{3k+4} &> \frac{3}{3k+3}\\
\frac{1}{3k+2}+\frac{1}{3k+4} &> \frac{2}{3k+3}\\
\frac{6k+6}{(3k+2)(3k+4)} &> \frac{2}{3k+3}\\
\frac{3k+3}{(3k+2)(3k+4)} &> \frac{1}{3k+3}\\
(3k+3)^2 &> (3k+2)(3k+4)\\
9k^2+18k+9 &> 9k^2+18k+8\\
9 &> 8
\end{aligned}$$

and the last one is true, which completes our proof.

9. (a) Let a, b, c, and d be the side lengths of a quadrilateral. Draw the diagonal as shown in the picture below and let e be its length.



By the triangle inequality (used twice),

$$d < e + c < a + b + c.$$

11. We will use Mathematical Induction.

Basis step. For $n = 1$, we can color the region inside the circle one color and the region outside of the circle the other color.

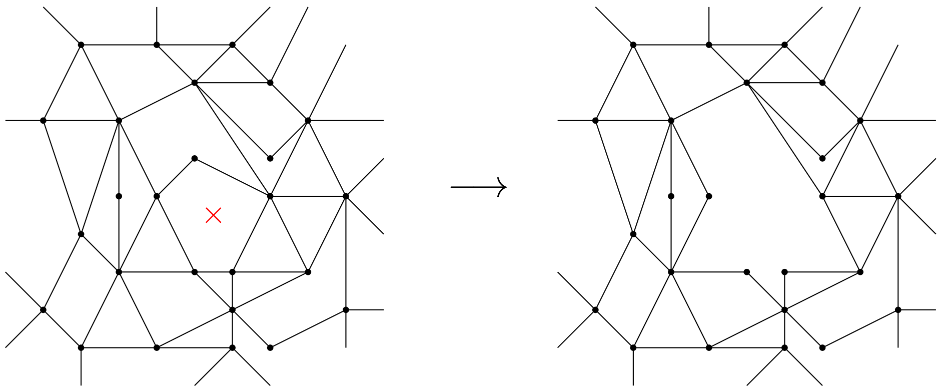
Inductive step. Assume the statement is true for $n = k$ circles. Suppose we have $k + 1$ circles. Let's temporarily remove one circle. Then we are left with k circles, and by our assumption, the regions can be colored using two colors so that no two regions with a common boundary line have the same color. When we put the removed circle back, the condition will be satisfied for all pairs of regions except the ones whose common boundary line is an arc of the new circle. However, if we switch the colors of all regions inside (or outside of, it doesn't matter) the new circle, the condition will be satisfied for all regions again. Thus the statement is true for $n = k + 1$ circles as well.

13. First of all, if at least one point is connected with an odd number of other points (we will call it a vertex of odd degree, see more on the graph terminology in chapter 10), then there is an odd number of regions around it. Since the colors must alternate in a proper coloring, the regions can not be properly colored with two colors.

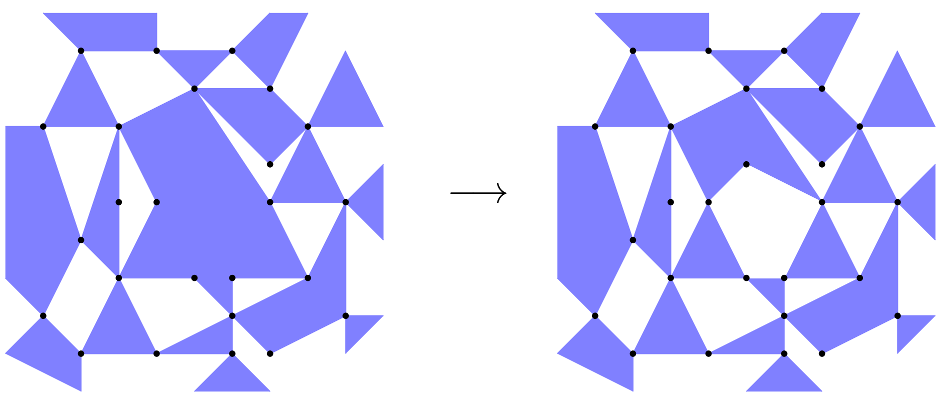
We will show that if the degree of each vertex is even, then the map can be properly colored with two colors. We will use Strong Mathematical Induction, and the induction will be on the number of line segments.

Basis step. For $n = 0$ line segments, the whole plane is one big region. We can color it with any color we like.

Inductive step. Suppose any map with with all vertices of even degree and less than or equal to k line segments can be properly colored with two colors. We wish to show that any map with all vertices of even degree and $k + 1$ line segments can be properly colored. Suppose we are given such a map. Remove temporarily all the boundary lines of any one region. If the boundary contains any vertices of degree 2, then remove these vertices as well.



We get a map with less than k line segments, and the degree of each vertex is still even. By the inductive assumption this new map can be properly colored. Consider a coloring, put the boundary of our region back, and change the color inside it. We get a proper coloring for our original map with $k + 1$ line segments.



15. Proof by Mathematical Induction.
Basis step: for $n = 3$, we have $1 = \frac{1}{2} + \frac{1}{3} + \frac{1}{6}$.
Inductive step: assume that for $k \in \mathbb{N}$, there exist positive integers

$$a_1 < a_2 < \cdots < a_{k-1} < a_k$$

such that

$$1 = \frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_{k-1}} + \frac{1}{a_k}.$$

Observe that $\dfrac{1}{a_k} = \dfrac{1}{a_k+1} + \dfrac{1}{a_k(a_k+1)}$ and $a_k < a_k+1 < a_k(a_k+1)$. Then

$$1 = \frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_{k-1}} + \frac{1}{a_k+1} + \frac{1}{a_k(a_k+1)},$$

where

$$a_1 < a_2 < \cdots < a_{k-1} < a_k + 1 < a_k(a_k+1),$$

thus 1 is written as the sum of $k+1$ fractions of the required form.

17. Proof by Strong Mathematical Induction.
Basis step. For $n = 1$ we have $\det M_1 = \det[5] = 5 = \dfrac{15}{3} = \dfrac{1}{3}(4^2 - 1)$.
Inductive step. Assume the formula $\det M_n = \dfrac{1}{3}(4^{n+1} - 1)$ holds for $1 \leq n \leq k$. We wish to prove it for $n = k+1$.
Case I: if $k = 1$, then $k+1 = 2$, and

$$\det M_2 = \det \begin{bmatrix} 5 & 2 \\ 2 & 5 \end{bmatrix} = 5^2 - 2^2 = 21 = \frac{63}{3} = \frac{1}{3}(4^3 - 1).$$

Case II: if $k \geq 2$, then $k - 1 \geq 1$, and we can assume that the formula holds for $n = k$ and $n = k-1$. Expanding M_{k+1} across the first row and then expanding the second of the two obtained matrices down the first column gives

$$\begin{aligned}
\det M_{k+1} &= \det \begin{bmatrix} 5 & 2 & & & & \\ 2 & 5 & 2 & & 0 & \\ & 2 & 5 & & & \\ & & & \cdots & & \\ & 0 & & & 5 & 2 \\ & & & & 2 & 5 \end{bmatrix}_{(k+1)\times(k+1)} \\
&= 5 \det \begin{bmatrix} 5 & 2 & & & & \\ 2 & 5 & 2 & & 0 & \\ & 2 & 5 & & & \\ & & & \cdots & & \\ & 0 & & & 5 & 2 \\ & & & & 2 & 5 \end{bmatrix}_{k\times k} - 2 \det \begin{bmatrix} 2 & 2 & & & & \\ 0 & 5 & 2 & & 0 & \\ & 2 & 5 & & & \\ & & & \cdots & & \\ & 0 & & & 5 & 2 \\ & & & & 2 & 5 \end{bmatrix}_{k\times k} \\
&= 5 \det M_k - 2^2 \det \begin{bmatrix} 5 & 2 & & & & \\ 2 & 5 & 2 & & 0 & \\ & 2 & 5 & & & \\ & & & \cdots & & \\ & 0 & & & 5 & 2 \\ & & & & 2 & 5 \end{bmatrix}_{(k-1)\times(k-1)} \\
&= 5 \det M_k - 4 \det M_{k-1} = 5 \cdot \frac{1}{3}(4^{k+1} - 1) - 4 \cdot \frac{1}{3}(4^k - 1) \\
&= \frac{1}{3}(5 \cdot 4^{k+1} - 5 - 4^{k+1} + 4) = \frac{1}{3}(4 \cdot 4^{k+1} - 1) = \frac{1}{3}(4^{k+2} - 1).
\end{aligned}$$

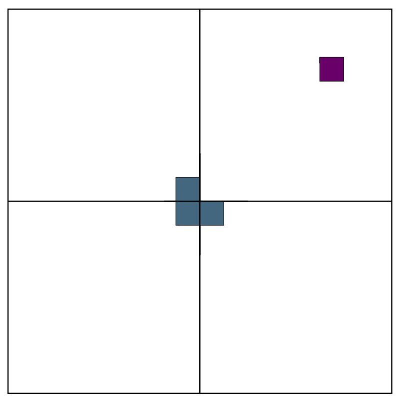
19. Hint: use Strong Mathematical Induction. First show that it is possible to divide a square into 6, 7, or 8 smaller (not necessarily congruent) squares. For the inductive step, assume that it is possible to divide the square into n smaller squares for any n

such that $6 \leq n \leq k$ where $k \geq 8$. To show that it is possible to divide the square into $k+1$ smaller squares, use the fact that it is possible to divide it into $k-2$ smaller squares and divide one of the smaller squares into four (note that this increases the number of the smaller squares by 3).

21. Proof by Mathematical Induction.

Basis step. A 2×2 board with one square removed has the shape of an L-tromino, and thus can be covered by one L-tromino.

Inductive step. Assume that a $2^k \times 2^k$ board with any square removed can be covered by L-trominoes. Now suppose we are given a $2^{k+1} \times 2^{k+1}$ board with one square removed. Divide this board into four $2^k \times 2^k$ boards. One of them has one square removed, and the three others are whole boards. Temporarily remove corner squares from those three whole boards as shown on the picture below.



By the induction assumption, each of these four boards can be covered by L-trominoes. Now place one more L-tromino in the center to cover the 3 squares that we temporarily removed. We are done.

23. (a) First of all, let's write a formula for the given sequence. There are many different ways to do this. One way is to write a recursive formula. If we look at the sequence closely,

$$1007, \quad 10017, \quad 100117, \quad 1001117, \quad \ldots$$

we notice that each term can be obtained from the previous one by shifting its digits to the left (that is, multiplying the number by 10), and then subtracting 53. In other words,

$$a_{n+1} = 10a_n - 53,$$

where a_n is the n-th term of the sequence. We will use Mathematical Induction to prove that for each natural n, the number a_n in this sequence is divisible by 53.

Basis step. For $n = 1$, we have $a_1 = 1007 = 53 \cdot 19$, so it is divisible by 53.

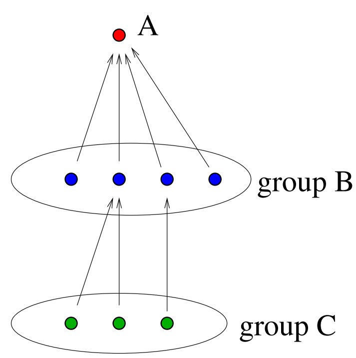
Inductive step. Assume that a_k is divisible by 53, then $a_k = 53x$ for some integer x. Then

$$\begin{aligned} a_{k+1} &= 10a_k - 53 \\ &= 10(53x) - 53 \\ &= 53(10x - 1), \end{aligned}$$

so a_{k+1} is also divisible by 53.

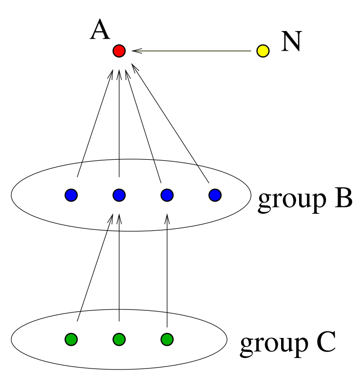
25. Basis step. For $n = 1$ city there is nothing to prove because there is no "any other city". (The step $n = 2$, in which case we have 2 cities and one road between them, so one city can be reached from the other, is also acceptable in this situation.)

Inductive step. Assume the statement is true for $n = k$. We want to prove that the statement is true for $n = k+1$. Suppose we are given $k+1$ cities with roads as described in the problem. Choose one city (let's call this city N) and eliminate it and all roads from and to it for a moment. We are left with k cities. By the inductive hypothesis, there is a city that can be reached from any other city either directly or via at most one other city. Let's call it A, and let's call those cities from which there are direct roads to A group B, and the rest of the cities group C. Then from every city in group C there is a road to at least one city in group B:



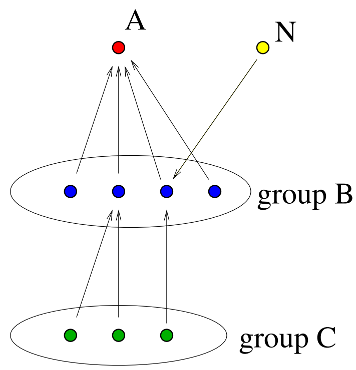
Now we add our $(k+1)$-st city N back. Consider the following three cases.

Case I. The road between A and N goes from N to A.



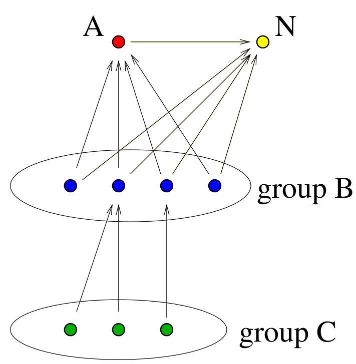
Then we put N into group B, and A is still "a solution city."

Case II. There is at least one road from N to group B.



Then we put N into group C, and A is still a solution city.

Case III. None of the above: the road between A and N goes from A to N, and all the roads between group B and N lead to N.



Then N is a new solution city, and A will join group B.

27. Hint. For the inductive step, temporarily remove two consecutive dots, one red and one blue, in this order, when going clockwise.

3 Dirichlet's Box Principle

1. Think of months as "boxes" and people's birth dates as "objects." Since there are more people (13) than months (12), By Dirichlet's Box Principle, at least two birth dates ("objects") are in the same month ("box").

3. Five. If he pulls our fewer than five socks, it is possible that they are of different colors since there are four colors available. However, if he pulls out five or more socks, then by Dirichlet's box principle at least two socks will be of the same color.

5. There are 100 possible remainders upon division by 100 (namely, 0, 1, 2, . . ., 99). Since we have 120 (more than 100) numbers, by Dirichlet's Box Principle there are at least two numbers with the same remainder. Their difference has remainder 0, and thus is divisible by 100. Since all numbers are distinct, the difference is nonzero, and therefore it ends with two zeros.

7. There are two possible remainders upon division by 2 (namely, 0 and 1). Since we have three numbers, at least two of them have the same remainder. Their difference is even, therefore the product $(x_1 - x_2)(x_1 - x_3)(x_2 - x_3)$ is even.

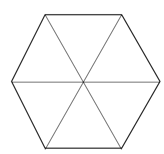
8. Notation clarification:

$$\prod_{1 \le i < j \le 4} (a_i - a_j) = (a_1 - a_2)(a_1 - a_3)(a_1 - a_4)(a_2 - a_3)(a_2 - a_4)(a_3 - a_4).$$

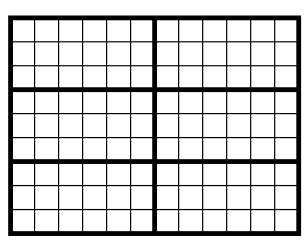
9. There are 11 possible remainders upon division by 11 (namely, 0, 1, 2, . . . , 10). Since we have 12 (more than 11) numbers, by Dirichlet's Box Principle at least two numbers have the same remainder. Their difference is then divisible by 11. Since the given numbers are two-digit, this difference is at most two-digit. Since the numbers are distinct, the difference is non-zero. There are no one-digit numbers divisible by 11. Every two-digit number that is divisible by 11, has the form aa (such numbers are 11, 22, 33, . . . , 99).

11. Each year contains at least 365 days. Since $365 = 52 \cdot 7 + 1$, each year contains 52 whole weeks, and thus at least 52 Fridays. Kevin is paid every other Friday, therefore he is paid at least $\dfrac{52}{2} = 26$ times a year. There are 12 months, and $26 = 2 \cdot 12 + 2$. By the Generalized Dirichlet's Box Principle, at least one month contains three days on which Kevin is paid.

13. Divide the hexagon into 6 regions as shown in the figure below. Since we have 7 (more than 6) points, by Dirichlet's Box Principle there is a region with at least two points in it (or on its boundary). The distance between those two points is at most 1 because each region is an equilateral triangle with all sides of length 1.



15. (a) Divide the rectangle into six 3×6 rectangles as shown below. Since there are 7 points, by Dirichlet's Box Principle at least two of them are in the same 3×6 rectangle. The distance between them is at most the length of the diagonal of a 3×6 rectangle, which is $\sqrt{3^2 + 6^2} = \sqrt{45} < 7$.



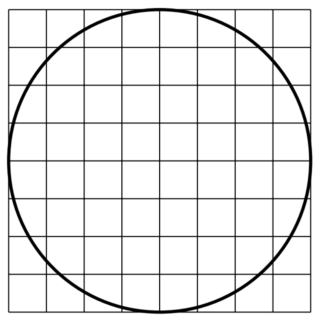
(b) Answer: yes.

Note: Although the proof in part (a) does not work for 6 points, it does not mean that the statement does not hold for 6 points.

Hint: Divide the rectangle into 5 regions with the property that the distance between any two points in one region is less than 7. This is harder than part (a), but is possible!

17. Since $7 \cdot 7 \cdot 7 = 343$, we can divide the cube into 343 small cubes, each with edge 1. Each point is inside at most one small cube (if a point is on the boundary of a small cube, then it is not inside any small cube). Since there are more small cubes than points, there is a small cube (moreover, there are at least 43 of them) that does not contain any points inside it.

19. A circle of radius 4 can be inscribed in an 8×8 square, which, in turn, can be divided into 64 unit squares. At the first glance it may seem that 61 points is not enough since 61 is smaller than 64. Notice, however, that four of the unit squares lie outside of the circle (see the picture below; also, it is easy to check that $4 < 3\sqrt{2}$). In other words, the circle of radius 4 is covered by just 60 unit squares. Thus, by Dirichlet's box principle, at least two of the 61 points are inside one unit square. The distance between them is at most $\sqrt{2}$.



21. There are 10 straight lines (5 horizontal and 5 vertical) that go along the grid lines and divide the square into two pieces. There are 18 dominoes. However, it not possible that a line cuts a single domino, because any line has an even number of squares on each of its sides (note that if a line is horizontal, then both above and below it there is some number of rows of 6, so the number of squares on each side is a multiple of 6; similarly for a vertical line). If a line cuts a single domino, then each of its sides would contain a whole number of dominoes plus a single square, so an odd number of squares total. Thus if a line cuts a domino, it cuts at least two dominoes. If each of the ten lines cuts at least two dominoes, then there must be at least twenty dominoes total, which is a contradiction.

23. If $a_{i+1} - a_i < 15$ for all $1 \leq i \leq 7$, then $a_8 - a_1 = \sum_{i=1}^{7}(a_{i+1} - a_i) \leq 7 \cdot 14 = 98$ (because all a_i's are integers, so all differences are also integers). However, this is impossible since $a_7 - a_1 = 100 - 1 = 99$.

25. Consider the following numbers: 1, 11, 111, $\ldots$, $\underbrace{11\ldots1}_{n+1}$. Since there are n possible remainders upon division by n, at least two of the above numbers, say, $\underbrace{11\ldots1}_{k}$ and $\underbrace{11\ldots1}_{l}$ for $k < l$, have the same remainder. Their difference is $\underbrace{11\ldots1}_{l} - \underbrace{11\ldots1}_{k} = \underbrace{11\ldots1}_{l-k}\underbrace{00\ldots0}_{k} = \underbrace{11\ldots1}_{l-k} \cdot 10^k$ and is divisible by n. Since n is not divisible by 2 or 5, n and 10 are relatively prime. Therefore $\underbrace{11\ldots1}_{l-k}$ is divisible by n.

26. Hint: show that we can choose 2 of the given points whose midpoint is a lattice point.

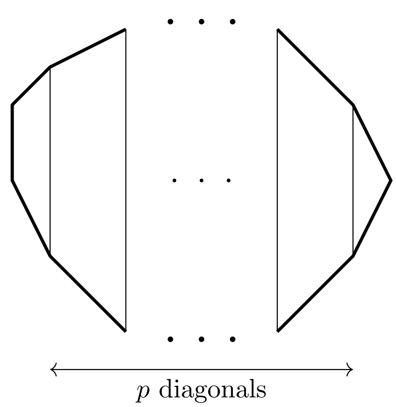
27. Divide the numbers $\{1, 2, \ldots, 2n\}$ into n pairs of consecutive integers: $\{1, 2\}$, $\{3, 4\}$, $\ldots$, $\{2n-1, 2n\}$. Since there are $n+1$ numbers in the set $\{a_1, a_2, \ldots, a_{n+1}\}$, at least two of them are in the same pair, i.e. are consecutive. Their greatest common divisor is 1 because if a natural number p divides both k and $k+1$, then p divides their difference $(k+1) - k = 1$, so $p = 1$. Thus our two consecutive numbers are relatively prime.

29. "Make a box" for each side. There will be $2n$ boxes. We will "put" a diagonal into a "box" if it is parallel to the corresponding side.

box 1: diagonals parallel to side 1	...	box $2n$: diagonals parallel to side $2n$

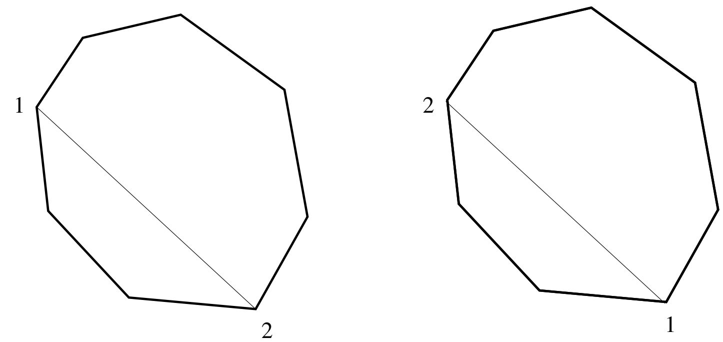
We will figure out the maximal possible number of diagonals that can be parallel to one side (and thus parallel among themselves), i.e. the maximal possible number of diagonals in each box, and we will figure out how many diagonals we have in a $2n$-gon. We will show that $2n$ times the maximal number of diagonals in each box is less than the number of diagonals in a $2n$-gon, thus there is not enough space for all the diagonals in our boxes. Therefore, there is a diagonal that is not in any box, and thus not parallel to any side.

Let p be the maximal possible number of diagonals parallel to the same side. We will find a condition on p. Notice that the vertices of these p diagonals and the 2 vertices of the side they are all parallel to, are distinct (because if 2 line segments have a common vertex, they can not be parallel unless they lie on one line which is impossible in our case). Let us draw the $2n$-gon so that all these p diagonals and the parallel side are vertical, with the side on the left. Then we must also have at least one vertex on the right (because the rightmost line segment must be inside the $2n$-gon):



Thus the number of vertices in this figure is at least $2p+2+1$. So we have $2n \geq 2p+3$. Since $2n$ is an even number and $2p+3$ is odd, we must have $2n \geq 2p+4$. Then $2n-4 \geq 2p$, and $n-2 \geq p$. So there may be at most $n-2$ diagonals in the same box.

Next let's count the number of diagonals in a $2n$-gon. For every diagonal, there are $2n$ ways to choose the first vertex. Once the first vertex has been chosen, there are $2n-3$ ways to choose the second vertex (because the first vertex and its immediate neighbors can not be chosen as the second vertex). Thus there are $2n(2n-3)$ ways to choose an ordered pair of vertices of a diagonal. However, this way we counted each diagonal twice.



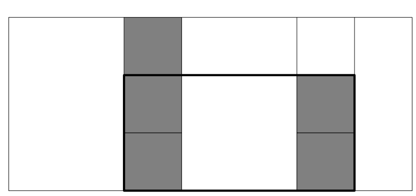
So we must divide by 2. Therefore there are $\dfrac{2n(2n-3)}{2} = n(2n-3)$ diagonals.

Thus we have $2n$ boxes, at most $n-2$ diagonals may be in the same box, therefore at most $2n(n-2) = 2n^2 - 4n$ diagonals may be in the boxes. However, we have $n(2n-3) = 2n^2 - 3n$ diagonals. Since $2n^2 - 3n > 2n^2 - 4n$, there is a diagonal which is not in any box, and thus is not parallel to any side.

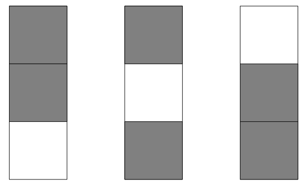
31. Since $f : X \to X$ is onto, it is a 1-1 correspondence from X onto itself, i.e. it permutes the elements of X. Then so does f^k for any integer k. If follows that f^k has an inverse f^{-k}. Since $|X| = n$, there are $n!$ permutations on X. Consider functions f, f^2, f^3, $\ldots$, $f^{n!+1}$. Since there are more functions in this list than distinct permutations, at least two of these functions, say, f^l and f^m for $l < m$, define the same permutation, i.e. $f^l(x) = f^m(x)$ for all $x \in X$. Then $f^{m-l}(x) = f^{-l}(f^m(x)) = f^{-l}(f^l(x) = f^{l-l}(x) = x$ for all $x \in X$.

33. First note that there are $2^{10} - 1 = 1023$ nonempty subsets. Since the smallest two-digit number is 10 and the largest possible sum of ten two-digit numbers is $90+91+\cdots+99 = 945$, there are 936 possible different values of the sum of up to ten two-digit numbers. Since there are more subsets than possible values of the sum, by Dirichlet's Box principle there are two distinct nonempty subsets, say, A and B, with the same sum. Next, note that neither of them can be a subset of the other, as then the sums of their elements could not be equal. It follows that $A - B = A - (A \cap B)$ and $B - A = B - (A \cap B)$ are disjoint, nonempty, and have the same sum of elements.

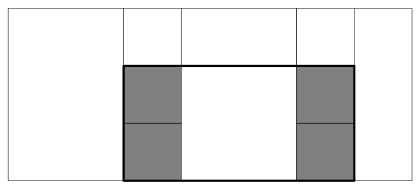
35. Since we have 3 rows and only 2 colors, every column has some color repeated. Since there are 7 columns, there is a color that is repeated (at least twice) in at least 4 columns. So each of these 4 columns contains at least 2 blocks of that color. If there is a column that contains 3 blocks of that color, then choose any other of the 4 columns, and we'll have a rectangle, e.g. as shown in the picture below.



If no column contains 3 blocks of that color, then notice that there are 3 different ways to have 2 out of 3 blocks of some color.



Since we have 4 columns, at least two of them have the same distribution of the colors. Then, again, we have a desired rectangle, e.g. as shown below.



37. The twelve faces of a dodecahedron can be divided into four groups of three so that the three faces in each group have a common vertex, and any two of the three faces in each group have a common edge. Since the sum of all of the numbers is 11 and there are four groups, at least one group has a sum of at most 2. The three faces in this group have either numbers 0, 0, 2 or 0, 1, 1. In either case, two faces having a common edge have equal numbers.

39. Choose any day and track the number of games the player is playing for the next 11 weeks. Consider the sequence $a_1,\ a_2,\ a_3,\ \ldots, a_{77}$, where each a_n represents the total number of games the player has played in the first n days from the chosen start day. Since he is playing at least one game per day, this is a strictly increasing sequence. Define the sequence $b_1,\ b_2,\ b_3,\ \ldots, b_{77}$ by $b_n = a_n + 20$. Note that it is also strictly increasing. Since the player is playing at most 12 games per week, we have $a_{77} \leq 132$ and $b_{77} \leq 152$. Thus there are only 152 possible values for all the numbers in these two sequences. However, there are $77 \cdot 2 = 154$ numbers, so there must be two equal numbers. Since each sequence is strictly increasing, we have $b_i = a_j$ for some i and j. Equivalently, $a_i + 20 = a_j$, which means that exactly 20 games were played on days $i+1$ through j.

4 Number theory

1. (a) The sum of the digits of a number

$$N = \underline{a_n a_{n-1} \ldots a_1 a_0} = a_n \cdot 10^n + a_{n-1} \cdot 10^{n-1} + \cdots + a_1 \cdot 10 + a_0 = \sum_{k=0}^{n} a_k \cdot 10^k$$

is

$$S = a_n + a_{n-1} + \cdots + a_1 + a_0 = \sum_{k=0}^{n} a_k.$$

Since $10 \equiv 1 \pmod 9$, $10^k \equiv 1 \pmod 9$. Then $a_k \cdot 10^k \equiv a_k \pmod 9$, and $\sum_{k=0}^{n} a_k \cdot 10^k \equiv \sum_{k=0}^{n} a_k \pmod 9$. So $N \equiv S \pmod 9$. Thus N is divisible by 9 if and only if S is divisible by 9.

(b) If the sum of the digits of a number is 66, then the number is divisible by 3 (see problem 2(b) in chapter 1) but not divisible by 9. But if a perfect square is divisible by 3 then it must be divisible by 9. Therefore a number with the digital sum 66 cannot be a perfect square.

3. (a) First notice that if k is the last digit of m, then the last digit of m^2 is that of k^2 because $m = 10n+k$ for some integer n, and $m^2 = (10n+k)^2 = 100n^2+20nk+k^2 = (10n^2 + 2nk) \cdot 10 + k^2$. So we consider all possible last digits and compute their squares: $0^2 = 0$, $1^2 = 1$, $2^2 = 4$, $3^2 = 9$, 4^2 ends with 6, 5^2 ends with 5, 6^2 ends with 6, 7^2 ends with 9, 8^2 ends with 4, and 9^2 ends with 1. Thus the last digit of a perfect square can be 0, 1, 4, 5, 6, or 9.

   (b) Since 3 is not listed above, a number ending with 3 cannot be a perfect square.

5. (a) No. Assume $n = a^2$ is a perfect square that ends with 65, then it is divisible by 5. Then a is divisible by 5, and therefore n is divisible by 25. Any number divisible by 25 ends with 00, 25, 50, or 75. Thus it cannot end with 65. We get a contradiction.

7. (a) Let a be a perfect square, then $a = b^2$ for some integer b. If $b \equiv 0 \pmod 7$, then $a \equiv 0 \pmod 7$. If $b \equiv \pm 1 \pmod 7$, then $a \equiv (\pm 1)^2 \equiv 1 \pmod 7$. If $b \equiv \pm 2 \pmod 7$, then $a \equiv (\pm 2)^2 \equiv 4 \pmod 7$. If $b \equiv \pm 3 \pmod 7$, then $a \equiv (\pm 3)^2 \equiv 9 \equiv 2 \pmod 7$. There are no values of b for which $a \equiv 6 \pmod 7$.

9. *Solution 1.* Since $2^4 \equiv 16 \equiv 1 \pmod 5$ and $3^4 \equiv 81 \equiv 1 \pmod 5$, $2^{457} + 3^{457} \equiv 2^{456+1} + 3^{456+1} \equiv 2^{456} \cdot 2 + 3^{456} \cdot 3 \equiv 2^{4 \cdot 114} \cdot 2 + 3^{4 \cdot 114} \cdot 3 \equiv (2^4)^{114} \cdot 2 + (3^4)^{114} \cdot 3 \equiv 1 \cdot 2 + 1 \cdot 3 \equiv 0 \pmod 5$. Therefore $2^{457} + 3^{457}$ is divisible by 5.

   Solution 2. Using the second formula in Theorem 4.22, since 457 is odd, we see that $2^{457} + 3^{457}$ can be factored as $(2+3)$ times another natural number, thus it is divisible by 5.

11. If the units digit of n is 3, then n can be written in the form $n = 10k+3$ for some integer k. Then $n^2+1 = (10k+3)^2+1 = 100k^2+60k+9+1 = 100k^2+60k+10 = 5(20k^2+12k+2)$ is divisible by 5.

13. *Solution 1.* Let us consider all possible cases for n modulo 6.
    If $n \equiv 0 \pmod 6$, then $n^3 + 5n \equiv 0^3 + 5 \cdot 0 \equiv 0 \pmod 6$.
    If $n \equiv 1 \pmod 6$, then $n^3 + 5n \equiv 1^3 + 5 \cdot 1 \equiv 6 \equiv 0 \pmod 6$.
    If $n \equiv 2 \pmod 6$, then $n^3 + 5n \equiv 2^3 + 5 \cdot 2 \equiv 18 \equiv 0 \pmod 6$.
    If $n \equiv 3 \pmod 6$, then $n^3 + 5n \equiv 3^3 + 5 \cdot 3 \equiv 42 \equiv 0 \pmod 6$.
    If $n \equiv 4 \pmod 6$, then $n^3 + 5n \equiv 4^3 + 5 \cdot 4 \equiv 84 \equiv 0 \pmod 6$.
    If $n \equiv 5 \pmod 6$, then $n^3 + 5n \equiv 5^3 + 5 \cdot 5 \equiv 150 \equiv 0 \pmod 6$.

    We see that in each case $n^3 + 5n \equiv 0 \pmod 6$, so $6|(n^3 + 5n)$.

    Solution 2. Observe that $n^3 + 5n = n^3 - n + 6n = (n-1)n(n+1) + 6n$. Since $n-1$, n, and $n+1$ are three consecutive integers, at least one of them is divisible by 2 and at least one of them is divisible by 3. It follows that their product is divisible by 6. Since $6n$ is also divisible by 6, we have that $n^3 + 5n$ is divisible by 6.

15. The number $\dfrac{100}{n+1}$ is natural when $n+1$ is a positive factor of 100. Now, $100 = 2^2 5^2$, so it has nine positive factors (1, 2, 4, 5, 10, 20, 25, 50, and 100). For all values starting with 2, we get a natural value of n. However, $n+1 = 1$ produces $n = 0$ which is not a natural number. Thus there are only eight such values of n.

17. Four consecutive odd integers can be written in the form $2n-3$, $2n-1$, $2n+1$, and $2n+3$ for some integer n. Their sum is $8n$ which is divisible by 8.

19. Let n, $n+1$, $n+2$, and $n+3$ be four consecutive integers whose product is 3024. Then we have

$$\begin{aligned} n(n+1)(n+2)(n+3) &= 3024, \\ \big(n(n+3)\big)\big((n+1)(n+2)\big) &= 3024, \\ (n^2+3n)(n^2+3n+2) &= 3024, \\ (x-1)(x+1) &= 3024, \\ x^2-1 &= 3024, \\ x^2 &= 3025, \end{aligned}$$

where $x = n^2+3n+1$. The above equation has two roots: $x = \pm 55$. Solving $n^2+3n+1 = 55$, we have two cases: $n = 6$ and $n = -9$. In the former, the four consecutive numbers are 6, 7, 8, and 9, and in the latter they are -9, -8, -7, and -6. It is easy to check that the equation $n^2+3n+1 = -55$ does not have real solutions.

21. Since $2^{100} \equiv (2^2)^{50} \equiv 4^{50} \equiv (-1)^{50} \equiv 1 \pmod 5$, the remainder is 1.

Note. There are many other ways to obtain this answer, e.g. $2^{100} \equiv (2^4)^{25} \equiv 16^{25} \equiv 1^{25} \equiv 1 \pmod 5$.

23. If n is composite, then $n = ab$ for some integers $1 < a, b < n$. Then

$$2^n - 1 = (2^a)^b - 1^b = (2^a - 1)((2^a)^{b-1} + \cdots + 2^a + 1).$$

Both factors are bigger than 1: $2^a - 1 > 2^1 - 1 = 1$, and $(2^a)^{b-1} + \cdots + 2^a + 1 > 1$, so $2^n - 1$ is composite.

25. The problem is equivalent to the following question: what is the highest value of n for which $100! = 10^n \cdot a$ where a an integer? Since the prime factorization of 10 is $2 \cdot 5$, let's determine how many factors 2 and 5 appear in the prime factorization of 100!. We will start with 5. First, $\dfrac{100}{5} = 20$ of the numbers 1 through 100 are divisible by 5, so their prime factorizations contain at least one 5 each. However, $\dfrac{100}{25} = 4$ of them are divisible by 25, so their prime factorizations contain a second factor of 5. No numbers from 1 to 100 are divisible by 125, so there is a total of $20 = 4 = 24$ factors of 5 in the prime factorization of 100! It is easy to see that there are many more factors of 2 since there are 50 even numbers, so there are at least 50 factors of 2. Thus, there are 24 zeros at the end of 100!.

27. (a) If there are integers x and y such that $x^2 + 10y = 5xy + 2$, then $x^2 \equiv 2 \pmod 5$. However, we can check all cases for x (namely, we have $x \equiv 0 \pmod 5$, $x \equiv \pm 1 \pmod 5$, or $x \equiv \pm 2 \pmod 5$), and in no case $x^2 \equiv 2 \pmod 5$.

(c) Suppose the equation has an integral solution. Consider the equation modulo 3. The number x is congruent to 0, 1, or 2 modulo 3. Then x^2 is congruent to either 0 or 1 modulo 3 (because $0^2 = 0$, $1^2 = 1$, and $2^2 \equiv 4 \equiv 1 \pmod 3$). The number $3y^2$ is congruent to 0 modulo 3 since $3y^2$ is divisible by 3. Thus the left hand side is congruent to either 0 or 1 modulo 3. However, the right hand side is 17 which is congruent to 2 modulo 3. We get a contradiction.

29. (a) *Solution 1.* Rewrite the equation as $xy - x - y = 0$. Adding 1 to both sides gives $xy - x - y + 1 = 1$, and factoring the left hand side gives $(x-1)(y-1) = 1$. Since $x - 1$ and $y - 1$ are integers and their product is equal to 1, they are either both 1 or both -1. If they are both 1, then $x = 2$ and $y = 2$. If they are both -1, then $x = 0$ and $y = 0$. Thus we have two solution pairs.

    Solution 2. Rewrite the equation as $y(x-1) = x$. Consider the following two cases.

    Case I: $x - 1 = 0$, then $x = 1$. The equation becomes $0 = 1$ which is impossible.

    Case II: $x - 1 \neq 0$. Then $y = \dfrac{x}{x-1}$. If y is an integer, then $(x-1) \mid x$. Since $(x-1) \mid (x-1)$, $(x-1) \mid (x-(x-1))$, i.e. $(x-1) \mid 1$. Then either $x - 1 = 1$ or $x - 1 = -1$. Therefore either $x = 2$ (and then $y = 2$) or $x = 0$ (and then $y = 0$).

    (c) The given equation can be rewritten as follows:

$$\begin{aligned} x^2 - 2x + y^2 - 2y + z^2 - 2z + 2 &= 0 \\ x^2 - 2x + 1 + y^2 - 2y + 1 + z^2 - 2z + 1 &= 1 \\ (x-1)^2 + (y-1)^2 + (z-1)^2 &= 1. \end{aligned}$$

    It follows that one of $x - 1$, $y - 1$, and $z - 1$ is ± 1 and the other ones are 0, thus we have the following solutions: $(2, 1, 1)$, $(0, 1, 1)$, $(1, 2, 1)$, $(1, 0, 1)$, $(1, 1, 2)$, and $(1, 1, 0)$.

31. (a) Notice first of all that y must be even because $3y = 100 - 2x$ and the right hand side is even. Second, y must be positive. Third, y cannot exceed 32 because if $y \geq 34$, then $3y \geq 102$ and then x would have to be negative. If y satisfies all the above conditions, namely, y is even and $2 \leq y \leq 32$, then $3y$ is even and $6 \leq 3y \leq 96$, so $100 - 3y$ is even and $4 \leq 100 - 3y \leq 94$, so there exists a positive integer x such that $2x = 100 - 3y$. Thus for any even y such that $2 \leq y \leq 32$ we have a unique solution pair (x, y). There are 16 even numbers satisfying $2 \leq y \leq 32$, thus 16 pairs are solutions to the given equation.

33. (a) Yes. Let's find the remainder of 1239999 upon division by 4567 and subtract the remainder from 1239999. We'll get a number that satisfies the required conditions. Since $1239999 = 4567 \cdot 271 + 2342$, the number $1237657 = 1239999 - 2342 = 4567 \cdot 271$ is divisible by 4567.

    Note. This, of course, is not the only such number. Another one is $1237657 - 4567 = 1233090$. Also, we could have started with e.g. 12399999, etc.

35. Appending 3 to a three-digit number on the left is equivalent to adding 3000. Let the original number be x. Then we have $3000 + x = 9x$. It follows that $8x = 3000$, so $x = 375$.

37. Appending 36 to a number on the right is equivalent to adding 36 to 100 times the number. Let the original number be x. Then we have $100x + 36 = 103x$. It follows that $3x = 36$, so $x = 12$.

39. The last digit in the product is 4 which means the last digit in $\square$12 multiplied by the last digit in 30$\square$ has to end with 4. So the last digit in 30$\square$ has to be either 2 or 7. Now, the last two digits in the product are determined by the last two digits in each number, thus the product is 33,824 in the first case (because $12 \cdot 2 = 24$) and 33,884 in the second case (because $12 \cdot 7 = 84$). Since $33,824/302 = 112$ and $33,884/307$ is not an integer, only the first case works, thus the whole equation is $112 \times 302 = 33,824$. The covered digits are 1, 2, 2, and their product is 4.

41. Since $792 = 8 \cdot 9 \cdot 11$, the number $13xy45z$ must be divisible by 8, 9, and 11. An integer is divisible by 8 if and only if the three-digit number formed by its last three digits is divisible by 8, so $45z$ must be divisible by 8. This gives us $z = 6$ (it is easy to see that 456 is the only number of the above form that is divisible by 8). Now we have the number $13xy456$. It is divisible by 9 if and only if the sum of its digits is divisible by 9. Since $1 + 3 + 4 + 5 + 6 = 19$, we see that $x + y$ must be 8 or 17. Finally, the number is divisible by 11 if and only if the alternating sum of its digits is divisible by 11, that is, $1 - 3 + x - y + 4 - 5 + 6 = x - y + 3$ must be 0 or 11. This implies that $x - y$ is -3 or 8. Combining with the previous condition, we have four cases:
If $x + y = 8$ and $x - y = -3$, then there are no integer solutions.
If $x + y = 8$ and $x - y = 8$, then $x = 8$ and $y = 0$.
If $x + y = 17$ and $x - y = -3$, then $x = 7$ and $y = 10$. However, 10 is not a single digit, so again we have no solutions.
If $x + y = 17$ and $x - y = 8$, then there are no integer solutions.
Thus the number is 1380456.

43. (a) Suppose that the number $\sqrt[3]{25}$ is rational. Then it can be written as an irreducible quotient: $\sqrt[3]{25} = \dfrac{m}{n}$, $m, n \in \mathbb{Z}$, $\gcd(m, n) = 1$. Then $25 = \dfrac{m^3}{n^3}$, so $25n^3 = m^3$. Now there are several ways to get a contradiction.

Way 1. From the last equation, $5 \mid m$, so $m = 5a$ for some integer a. Then $25n^3 = (5a)^3$, so $25n^3 = 125a^3$, thus $n^3 = 5a^3$. Now we see that $5 \mid n$. Thus both m and n are divisible by 5, which contradicts the condition $\gcd(m, n) = 1$.

Way 2. If $n = 1$, then $25 = m^3$ which is impossible. If $n > 1$, then $n \mid m$ which contradicts $\gcd(m, n) = 1$.

Way 3. We have $5 \cdot 5 \cdot n^3 = m^3$. Both n and m can be written as products of primes. Since n and m are cubed, the number of 5s on the left is 2 plus a multiple of 3, and the number of 5s on the right is a multiple of 3. This contradicts the fundamental theorem of arithmetic.

5 Case analysis

1. (a) Consider the following two cases.

Case I: $2x - 2 \geq 0$, or, equivalently, $x \geq 1$. Then $|2x - 2| = 2x - 2$, and the equation becomes $x^2 + 2x - 2 = 1$, or $x^2 + 2x - 3 = 0$. We have two roots: $x = 1$ and $x = -3$. However, only the first root satisfies the condition $x \geq 1$, so we disregard the second root.

Case II: $2x - 2 < 0$, or, equivalently, $x < 1$. Then $|2x - 2| = -(2x - 2)$, and the equation becomes $x^2 - (2x - 2) = 1$, or $x^2 - 2x + 1 = 0$. We have one root: $x = 1$. It does not satisfy the condition $x < 1$, so we disregard it. (Note, however, that this was a root in the first case, so we still keep it in the end.)

So the equation has only one root: $x = 1$.

3. (a) Consider the following two cases.

Case I: $5x - 6 \geq 0$, or, equivalently, $x \geq \dfrac{6}{5}$. Then $|5x - 6| = 5x - 6$, and the inequality becomes $x^2 - 5x + 6 \leq 0$, or $(x - 2)(x - 3) \leq 0$. The solution interval is $[2, 3]$, all of whose values satisfy the condition $x \geq \dfrac{6}{5}$.

Case II: $5x-6<0$, or, equivalently, $x<\frac{6}{5}$. Then $|5x-6|=-(5x-6)$, and the inequality becomes $x^2+5x-6\le 0$, or $(x+6)(x-1)\le 0$. The solution interval is $[-6,1]$, all of whose values satisfy the condition $x<\frac{6}{5}$.

The solution set to the original inequality is the union of the above intervals, that is, $[-6,1]\cup[2,3]$.

(c) Consider the following two cases.

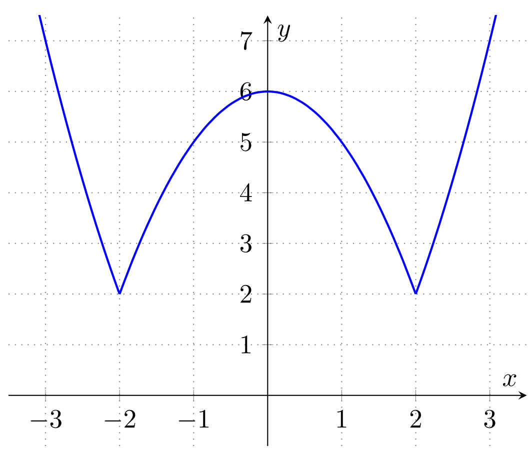
Case I: $7x+15\ge 0$, or, equivalently, $x\ge -\frac{15}{7}$. Then $|7x+15|=7x+15$, and the inequality becomes $x^2-7x-15\ge 3$, or $x^2-7x-18\ge 0$, or $(x+2)(x-9)\ge 0$. The solution set is $(-\infty,-2]\cup[9,\infty)$. However, we should restrict it to the values satisfying the condition $x\ge -\frac{15}{7}$, so we obtain $\left[-\frac{15}{7},-2\right]\cup[9,\infty)$.

Case II: $7x+15<0$, or, equivalently, $x<-\frac{15}{7}$. Then $|7x+15|=-(7x+15)$, and the inequality becomes $x^2+7x+15\ge 3$, or $x^2+7x+12\ge 0$, or $(x+4)(x+3)\ge 0$. The solution set is $(-\infty,-4]\cup[-3,\infty)$. However, we should restrict it to the values satisfying the condition $x<-\frac{15}{7}$, so we obtain $(-\infty,-4]\cup\left[-3,-\frac{15}{7}\right)$.

The solution set to the original inequality is the union of the above solution sets, that is, $(-\infty,-4]\cup[-3,-2]\cup[9,\infty)$.

(e) Answer: the inequality does not have any real solutions.

5. (a) The graph of $y=|x^2-4|$ is obtained from the graph of $y=x^2-4$ by reflecting the portion below the x-axis (that is, between $x=-2$ and $x=2$) about the x-axis. Then we shift the graph 2 units up to obtain the following curve.

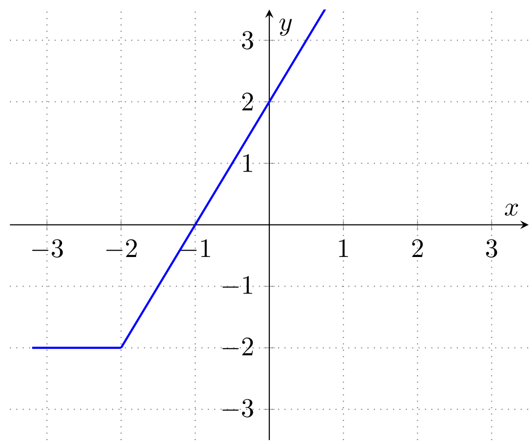


(c) First let's sketch the graph of $g(x)=x+|x+2|$.

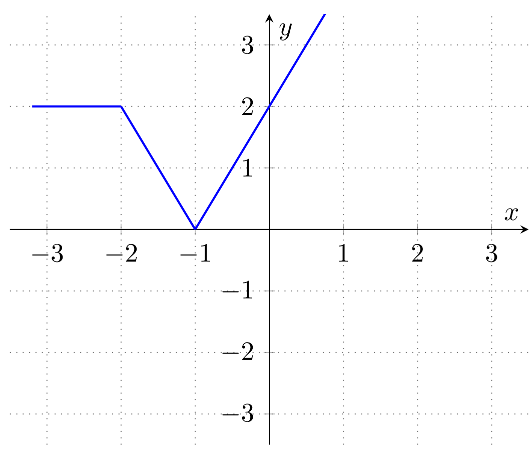
Case I: $x+2\ge 0$, or $x\ge -2$. Then $|x+2|=x+2$, so $g(x)=x+(x+2)=2x+2$.

Case II: $x+2<0$, or $x<-2$. Then $|x+2|=-(x+2)$, so $g(x)=x-(x+2)=-2$.

Thus $g(x)=\begin{cases} 2x+2 & \text{if } x\ge -2 \\ -2 & \text{if } x<-2 \end{cases}$. Its graph is shown below.



Since $f(x) = |x + |x + 2|| = |g(x)|$, the graph of $f(x)$ is obtained from the graph of $g(x)$ by reflecting the piece below the x-axis about the x-axis:



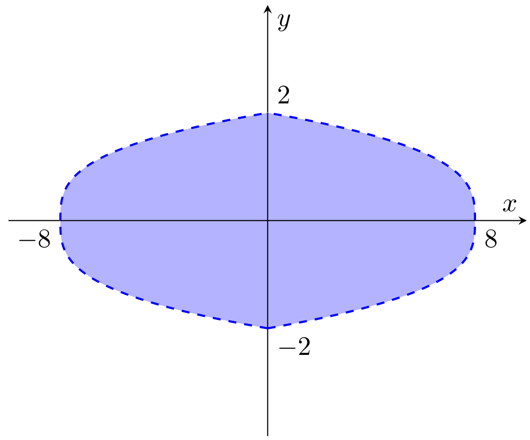
7. (a) Case I: $x \geq 0$, $y \geq 0$. The inequality becomes $x + y^3 < 8$, or $x < 8 - y^3$.

Case II: $x \geq 0$, $y < 0$. The inequality becomes $x - y^3 < 8$, or $x < 8 + y^3$.

Case III: $x < 0$, $y \geq 0$. The inequality becomes $-x + y^3 < 8$, or $x > y^3 - 8$.

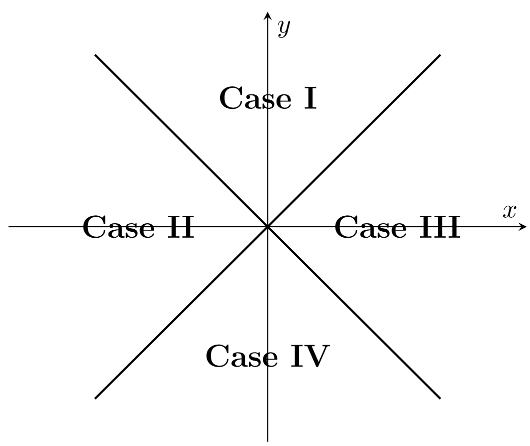
Case IV: $x < 0$, $y < 0$. The inequality becomes $-x - y^3 < 8$, or $x > -8 - y^3$.

Now we draw the corresponding region in each quadrant, and we get the following figure.



Note: since the inequality is strict, the boundary of the region is excluded.

(c) Since $y - x$ can be nonnegative or negative and $x + y$ can be nonnegative or negative, we have four cases.



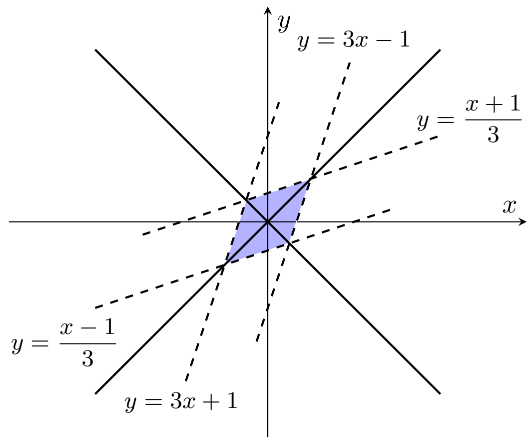
Case I: $y - x \geq 0$, $x + y \geq 0$. The inequality becomes $2y - 2x + y + x \leq 1$, or $3y \leq x + 1$, or $y \leq \dfrac{x+1}{3}$.

Case II: $y - x \geq 0$, $x + y < 0$. The inequality becomes $2y - 2x - y - x \leq 1$, or $y \leq 3x + 1$.

Case III: $y - x < 0$, $x + y \geq 0$. The inequality becomes $-2y + 2x + y + x \leq 1$, or $-y \leq 1 - 3x$, or $y \geq 3x - 1$.

Case IV: $y - x < 0$, $x + y < 0$. The inequality becomes $-2y + 2x - y - x \leq 1$, or $-3y \leq 1 - x$, or $y \geq \dfrac{x-1}{3}$.

Now we sketch the region in each case.



9. (a) The given equation is in the form $a^b = 1$ where $a = x$ and $b = x^2 - 7x + 12$. Thus we consider the following three cases.

   Case I: $a = 1$. Then $x = 1$.

   Case II: $a \neq 0$ and $b = 0$. The second equation gives $x = 3$ or $x = 4$, which are both nonzero, so they are roots of the original equation as well.

   Case III: $a = -1$ and b is even. Then $x = -1$, and $b = (-1)^2 - 7(-1) + 12$ is indeed even.

   Thus there are four roots: -1, 1, 3, and 4.

   (c) Answer: -1, 1.

11. (a) Let us consider the following three cases for the second equation.

   Case I: $x = 1$. Then the first equation implies $y = 0$.

   Case II: $x \neq 0$ and $y = 0$. Then the first equation becomes $x^{2x} = 1$, thus we have three cases again.

   Case IIA: $x = 1$. This gives the same solution as in Case I.

   Case IIB: $x \neq 0$ and $2x = 0$. This system does not have any solutions.

   Case IIC: $x = -1$ and $2x$ is even. Indeed, $2x$ is even for $x = -1$.

   Case III: $x = -1$ and y is even. This will give the same solution as in Case IIC.

   Thus there are only two solutions: $(1, 0)$ and $(-1, 0)$.

   (b) Answer: $(1, -1)$, $(1, 1)$, $(4, -2)$.

13. *Solution 1.* Let's consider all four possible cases for the false statement.

   Case I: Statement A is false. Then Statement D is false. Since only one statements is false, this case is not possible.

   Case II: Statement B is false. Then A is true, which implies that D is true, which in turn implies that C is true. So this case works out. Let's still consider the other cases in order to exclude any other possibilities.

   Case III: Statement C is false. Then B is true, so A is false, which is not possible.

   Case IV: Statement D is false. Then C is false as well, which is not possible.

   Thus the false statement must be Statement B.

   Solution 2. Consider two cases for Statement A.

   Case I: Statement A is true. Then D is false, C is true, and B is false. This case works.

Case II: Statement A is false. Then Statement D is false. Since only one statements is false, this case is not possible.

Therefore the false statement is Statement B.

15. First, consider two cases for the green octopus.
Case I: the green octopus says the truth. Then the blue octopus has six legs, thus must say the truth. However, the blue octopus says it has eight legs, so that cannot be the case.
Case II: the green octopus lies. Then consider two cases for the blue octopus.
Case IIA: the blue octopus says the truth. Then it has eight legs, and the first statement of the purple octopus is true. So it says the truth, but then it can't have nine legs. So this case is impossible.
Case IIB: the blue octopus lies also. Therefore the purple octopus lies as well. Then, the first statement of the striped octopus is correct, therefore its second statement must be correct as well.
Thus, the striped octopus is the only one with eight legs.

17. Consider all possible factorizations of 36 into a product of three factors, and compute the sum of these factors:

- $1 + 1 + 36 = 38$
- $1 + 2 + 18 = 21$
- $1 + 3 + 12 = 16$
- $1 + 4 + 9 = 14$
- $1 + 6 + 6 = 13$
- $2 + 2 + 9 = 13$
- $2 + 3 + 6 = 11$
- $3 + 3 + 4 = 10$

Only the sum 13 is repeated. So the friend's birthday must be on the 13th, and that's why the information given first was not sufficient. Once she was told that there was an "oldest" sibling, she was able to determine that the ages must be 2, 2, and 9.

6 Finding a pattern

1. (a) $a_n = n^2$.

   (c) $a_n = 2n + 3$.

   (e) $a_n = 5 - 2n$.

   (g) $a_n = \left\lfloor \frac{3n}{2} \right\rfloor = \begin{cases} \frac{3n-1}{2} & \text{if } n \text{ is odd} \\ \frac{3n}{2} & \text{if } n \text{ is even} \end{cases}$.

   (i) $a_n = \frac{n}{2^n}$.

3. Let $S_n = \dfrac{1}{1\cdot 3} + \dfrac{1}{3\cdot 5} + \cdots + \dfrac{1}{(2n-1)(2n+1)}$. Then

$$\begin{aligned}
S_1 &= \frac{1}{1\cdot 3} = \frac{1}{3},\\
S_2 &= \frac{1}{1\cdot 3} + \frac{1}{3\cdot 5} = \frac{6}{15} = \frac{2}{5},\\
S_3 &= \frac{2}{5} + \frac{1}{5\cdot 7} = \frac{15}{35} = \frac{3}{7},\\
S_4 &= \frac{3}{7} + \frac{1}{7\cdot 9} = \frac{28}{63} = \frac{4}{9}.
\end{aligned}$$

We guess from the above calculations that $S_n = \dfrac{n}{2n+1}$. We will prove this formula by Mathematical Induction.

Basis step. If $n = 1$, the formula $S_1 = \dfrac{1}{2\cdot 1+1}$ is correct.

Inductive step. Suppose the formula $S_n = \dfrac{n}{2n+1}$ holds for $n = k$, i.e. $S_k = \dfrac{k}{2k+1}$. We want to prove that it holds for $n = k+1$, i.e. $S_{k+1} = \dfrac{k+1}{2(k+1)+1}$.

Indeed, $S_{k+1} = \dfrac{1}{1\cdot 3} + \dfrac{1}{3\cdot 5} + \cdots + \dfrac{1}{(2k-1)(2k+1)} + \dfrac{1}{(2(k+1)-1)(2(k+1)+1)} = S_k + \dfrac{1}{(2k+1)(2k+3)} = \dfrac{k}{2k+1} + \dfrac{1}{(2k+1)(2k+3)} = \dfrac{k(2k+3)+1}{(2k+1)(2k+3)} = \dfrac{2k^2+3k+1}{(2k+1)(2k+3)} = \dfrac{(2k+1)(k+1)}{(2k+1)(2k+3)} = \dfrac{k+1}{2k+3} = \dfrac{k+1}{2(k+1)+1}$.

5. Let $P(n) = \displaystyle\prod_{i=1}^{2n-1}\left(1 - \frac{(-1)^i}{i}\right) = \left(1 - \frac{-1}{1}\right)\left(1 - \frac{1}{2}\right)\left(1 - \frac{-1}{3}\right)\cdots\left(1 - \frac{-1}{2n-1}\right)$.

First we compute the first few values of $P(n)$: $P(1) = 2$, $P(2) = \dfrac{4}{3}$, $P(3) = \dfrac{6}{5}$, $P(4) = \dfrac{8}{7}$. It appears that $P(n) = \dfrac{2n}{2n-1}$. This formula can be computed using Mathematical Induction.

7. Computing the first few terms of the sequence, we have $a_1 = 1$, $a_2 = 2$, $a_3 = 4$, $a_4 = 8$, $a_5 = 16$. It appears that $a_n = 2^{n-1}$. We will prove this formula by Strong Mathematical Induction. The basis step consists of $a_1 = 1$ and $a_2 = 2$ which are given. For the inductive step, assume $a_n = 2^{n-1}$ for all n from 1 to k, inclusive. Then $a_{k+1} = a_k + 2a_{k-1} = 2^{k-1} + 2\cdot 2^{k-2} = 2^{k-1} + 2^{k-1} = 2^k$ as desired.

9. First we compute $f_n(x)$ for the first few values of n:

$$\begin{aligned}
f_1(x) &= 2x+1,\\
f_2(x) &= 2(2x+1)+1 = 4x+3,\\
f_3(x) &= 2(4x+3)+1 = 8x+7,\\
f_4(x) &= 2(8x+7)+1 = 16x+15.
\end{aligned}$$

It appears that $f_n(x) = 2^n x + 2^n - 1$, so we will try to prove this formula by Mathematical Induction.

Basis step. If $n = 1$, then our formula gives $f_1(x) = 2x + 1$ which is true.

Inductive step. Suppose the formula holds for $n = k$, i.e. $f_k(x) = 2^k x + 2^k - 1$. Then $f_{k+1}(x) = f_1 \circ f_k(x) = 2(2^k x + 2^k - 1) + 1 = 2^{k+1}x + 2^{k+1} - 2 + 1 = 2^{k+1}x + 2^{k+1} - 1$. Thus the formula holds for $n = k + 1$.

11. (a) As discussed in chapter 4, the units digit of a positive number is its remainder upon division by 10.

Solution 1. First we find the units digit of 107^n for some small values of n:

$$\begin{aligned}
107^1 &\equiv 7 \pmod{10},\\
107^2 &\equiv 107 \cdot 107 \equiv 7 \cdot 7 \equiv 49 \equiv 9 \pmod{10},\\
107^3 &\equiv 107^2 \cdot 107 \equiv 9 \cdot 7 \equiv 63 \equiv 3 \pmod{10},\\
107^4 &\equiv 107^3 \cdot 107 \equiv 3 \cdot 7 \equiv 21 \equiv 1 \pmod{10},\\
107^5 &\equiv 107^4 \cdot 107 \equiv 1 \cdot 7 \equiv 7 \pmod{10},\\
107^6 &\equiv 107^5 \cdot 107 \equiv 7 \cdot 7 \equiv 49 \equiv 9 \pmod{10}.
\end{aligned}$$

We see that the units digits start repeating. Namely, as we keep multiplying our number by 107, the 4-tuple of units digits 7, 9, 3, and 1 keep repeating. More precisely,

$$107^n \equiv \begin{cases} 7 \pmod{10} & \text{if } n \equiv 1 \pmod 4\\ 9 \pmod{10} & \text{if } n \equiv 2 \pmod 4\\ 3 \pmod{10} & \text{if } n \equiv 3 \pmod 4\\ 1 \pmod{10} & \text{if } n \equiv 0 \pmod 4 \end{cases}.$$

This formula can be proved by Strong Mathematical Induction.
Basis step. If $n = 1$, then $107^n \equiv 7 \pmod{10}$ is true.
Inductive step. Suppose the formula holds for all $1 \le n \le k$. We want to prove that it holds for $n = k + 1$.
Case I: $k + 1 = 2$. Then $k + 1 \equiv 2 \pmod 4$, and $107^2 \equiv 9 \pmod{10}$ is true.
Case II: $k + 1 = 3$. Then $k + 1 \equiv 3 \pmod 4$, and $107^3 \equiv 3 \pmod{10}$ is true.
Case III: $k + 1 = 4$. Then $k + 1 \equiv 0 \pmod 4$, and $107^4 \equiv 1 \pmod{10}$ is true.
Case IV: $k + 1 \ge 5$. Then $k - 3 = (k + 1) - 4 \ge 1$, and we assumed that the formula above was true for $n = k - 3$.
We consider all possible remainders of $k + 1$ modulo 4.
Case IVA: $k+1 \equiv 1 \pmod 4$. Then $k-3 \equiv 1 \pmod 4$, so $107^{k+1} \equiv 107^{k-3} \cdot 107^4 \equiv 107^{k-3} \cdot 1 \equiv 7 \pmod{10}$.
Case IVB: $k+1 \equiv 2 \pmod 4$. Then $k-3 \equiv 2 \pmod 4$, so $107^{k+1} \equiv 107^{k-3} \cdot 107^4 \equiv 107^{k-3} \cdot 1 \equiv 9 \pmod{10}$.
Case IVC: $k+1 \equiv 3 \pmod 4$. Then $k-3 \equiv 3 \pmod 4$, so $107^{k+1} \equiv 107^{k-3} \cdot 107^4 \equiv 107^{k-3} \cdot 1 \equiv 3 \pmod{10}$.
Case IVD: $k+1 \equiv 0 \pmod 4$. Then $k-3 \equiv 0 \pmod 4$, so $107^{k+1} \equiv 107^{k-3} \cdot 107^4 \equiv 107^{k-3} \cdot 1 \equiv 1 \pmod{10}$.
Thus the formula holds for $n = k + 1$.
Now, since $107 \equiv 3 \pmod 4$, $107^{107} \equiv 3 \pmod{10}$, so the units digit of 107^{107} is 3.

Solution 2. As in solution 1, we compute the units digits of powers of 107 for a few small exponents and discover the pattern. However, we will prove only part of this pattern rather than the whole thing, namely, we will prove that if the exponent is divisible by 4, then the power of 107 ends with 1. That is, we will prove that for any nonnegative integer n, the number 107^{4n} ends with 1. The base case is $107^0 = 1$ which ends with 1. For the inductive step, assume 107^{4k} ends with 1.

Then $107^{4(k+1)} \equiv 107^{4k} \cdot 107^4 \equiv 107^{4k} \cdot 107^4 \equiv 1 \cdot 7^4 \equiv 7^4 \equiv 49^2 \equiv 9^2 \equiv 1 \pmod{10}$, so $107^{4(k+1)}$ ends with 1 as well. Now, $107^{107} \equiv 107^{104} \cdot 107^3 \equiv 107^{4\cdot 26} \cdot 107^3 \equiv 1 \cdot 7^3 \equiv 343 \equiv 3 \pmod{10}$, so 107^{107} ends with 3.

Solution 3. Since $107^4 \equiv 1 \pmod{10}$ and $107^3 \equiv 3 \pmod{10}$ (as we saw above), $107^{107} \equiv 107^{104} \cdot 107^3 \equiv (107^4)^{26} \cdot 107^3 \equiv 1^{26} \cdot 107^3 \equiv 1 \cdot 3 \equiv 3 \pmod{10}$, so 107^{107} ends with 3.

13. First we find the remainder of 2^n upon division by 12 for some small n:

$$\begin{aligned} 2^1 &\equiv 2 \equiv 2 \pmod{12}, \\ 2^2 &\equiv 4 \equiv 4 \pmod{12}, \\ 2^3 &\equiv 8 \equiv 8 \pmod{12}, \\ 2^4 &\equiv 16 \equiv 4 \pmod{12}, \\ 2^5 &\equiv 32 \equiv 8 \pmod{12}. \end{aligned}$$

We see that the remainders 4 and 8 start repeating. Namely,

$$2^n \equiv \begin{cases} 4 \pmod{12} & \text{if } n \text{ is even} \\ 8 \pmod{12} & \text{if } n \geq 3 \text{ is odd} \end{cases}.$$

As in problem 11, this can be proved by Strong Mathematical Induction. Since 100 is even, the remainder of 2^{100} upon division by 12 is 4.

15. (a) For $f(x) = \sin(x)$, the first few derivatives are:

$$\begin{aligned} f'(x) &= \cos(x), \\ f''(x) &= -\sin(x), \\ f'''(x) &= -\cos(x), \\ f^{(4)}(x) &= \sin(x), \\ f^{(5)}(x) &= \cos(x). \end{aligned}$$

We got $\cos(x)$ again, so the derivatives $\cos(x)$, $-\sin(x)$, $-\cos(x)$, $\sin(x)$ will repeat. Therefore

$$f^{(n)}(x) = \begin{cases} \cos(x) & \text{if } n \equiv 1 \pmod 4 \\ -\sin(x) & \text{if } n \equiv 2 \pmod 4 \\ -\cos(x) & \text{if } n \equiv 3 \pmod 4 \\ \sin(x) & \text{if } n \equiv 0 \pmod 4 \end{cases}.$$

This formula can be proved by Strong Mathematical Induction (the proof is similar to that in problem 11).

(c) For $h(x) = 2e^{5x}$, the first few derivatives are:

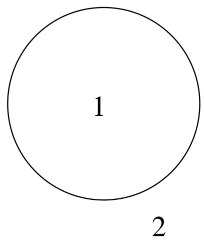
$$\begin{aligned} h'(x) &= 2 \cdot 5e^{5x}, \\ h''(x) &= 2 \cdot 5 \cdot 5e^{5x}, \\ h'''(x) &= 2 \cdot 5 \cdot 5 \cdot 5e^{5x}. \end{aligned}$$

We guess that $h^{(n)}(x) = 2 \cdot 5^n e^{5x}$, and prove this formula by Mathematical Induction.

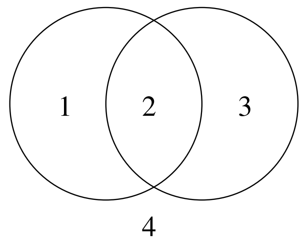
Basis step. We have $h'(x) = 2 \cdot 5e^{5x}$, so the formula holds for $n = 1$.

Inductive step. Suppose $h^{(k)}(x) = 2 \cdot 5^k e^{5x}$, then $h^{(k+1)}(x) = \left(2 \cdot 5^k e^{5x}\right)' = 2 \cdot 5^k \cdot 5e^{5x} = 2 \cdot 5^{k+1} e^{5x}$.

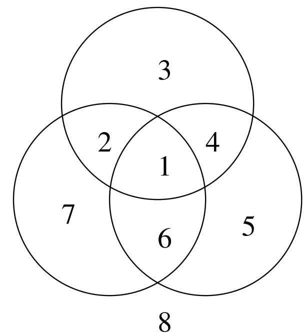
17. First we find the number of regions for a few small values of n.



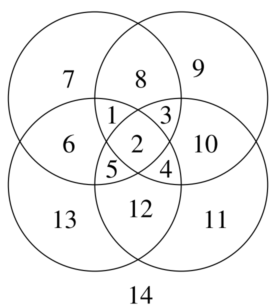
$n = 1$
2 regions



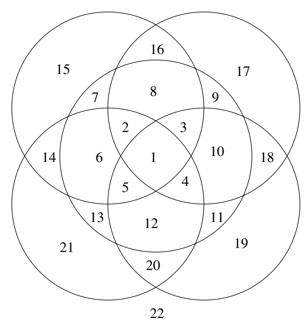
$n = 2$
4 regions



$n = 3$
8 regions



$n = 4$
14 regions



$n = 5$
22 regions

The sequence is 2, 4, 8, 14, 22, ... The differences between consecutive terms are 2, 4, 6, 8, ... We guess that the differences are increasing consecutive even numbers, so the number of regions into which n circles divide the plane is $2+2+4+6+\cdots+2(n-1) = 2+2(1+2+3+\cdots+(n-1)) = 2+2\frac{(n-1)n}{2} = 2+(n-1)n = n^2-n+2$.

Now we will prove this formula by Mathematical Induction.

Basis step. If $n = 1$, the formula gives 2, and it is true that there are 2 regions.

Inductive step. Suppose the formula is true for $n = k$ circles. Given $k + 1$ circles, temporarily remove one circle. The remaining k circles divide the plane into $k^2 - k + 2$ regions. Now add the $(k+1)$-th circle back. This new circle intersects the old k circles in $2k$ points. Thus the intersection points divide the new circle into $2k$ arcs. Therefore, the number of regions increases by $2k$ (each arc divides an old region into 2). Then, since k circles divided the plane into $k^2 - k + 2$ regions, $k+1$ circles will divide it into $k^2 - k + 2 + 2k = k^2 + 2k + 1 - k - 1 + 2 = (k+1)^2 - (k+1) + 2$ regions, and thus the formula holds for $k+1$.

19. (a) We compute the first few Fibonacci numbers: 0, 1, 1, 2, 3, 5, 8, 13, 21, 34, and notice that every third Fibonacci number is even. More precisely, F_n is even if and only if $n \equiv 0 \pmod 3$. Therefore exactly one third of $F_1, F_2, \ldots, F_{99}$ are even which gives 33 numbers, and F_0 is even, thus we have 34 even numbers total.

The pattern described above can be proved by Strong Mathematical Induction as follows.

Basis step. If $n = 0$, $F_n = F_0 = 0$ is even.

Inductive step. Suppose the statement "F_n is even if and only if $n \equiv 0 \pmod 3$" holds for $0 \le n \le k$. We will prove that the statement holds for $n = k+1$.

Case I: $k+1 = 1$. Then $k+1 \not\equiv 0 \pmod 3$, and $F_1 = 1$ is indeed odd.

Case II: $k+1 = 2$. Then $k+1 \not\equiv 0 \pmod 3$, and $F_2 = 1$ is indeed odd.

Case III: $k+1 \ge 3$. Then we consider all possible remainders of $k+1$ modulo 3.

Case IIIA: $k+1 \equiv 0 \pmod 3$. Then $k \equiv 2 \pmod 3$ and $k-1 \equiv 1 \pmod 3$. By the inductive hypothesis F_k and F_{k-1} are both odd, so $F_{k+1} = F_k + F_{k-1}$ is even.

Case IIIB: $k+1 \equiv 1 \pmod 3$. Then $k \equiv 0 \pmod 3$ and $k-1 \equiv 2 \pmod 3$. By the inductive hypothesis F_k is even and F_{k-1} is odd, so $F_{k+1} = F_k + F_{k-1}$ is odd.

Case IIIC: $k+1 \equiv 2 \pmod 3$. Then $k \equiv 1 \pmod 3$ and $k-1 \equiv 0 \pmod 3$. By the inductive hypothesis F_k is odd and F_{k-1} is even, so $F_{k+1} = F_k + F_{k-1}$ is odd.

(c) Answer: 20.

20. Hint: do not try to list all the ways. There are too many of them! Instead, replace 10 by small numbers, and guess the pattern. Prove your guess using Mathematical Induction.

7 Working backwards

1. (a) Since

$$\begin{aligned} 46 &= 1 \cdot 32 + 14, \\ 32 &= 2 \cdot 14 + 4, \\ 14 &= 3 \cdot 4 + 2, \\ 4 &= 2 \cdot 2, \end{aligned}$$

we have $d = \gcd(46, 32) = 2$. Then

$$\begin{aligned} 2 &= 14 - 3 \cdot 4 \\ &= 14 - 3(32 - 2 \cdot 14) = 7 \cdot 14 - 3 \cdot 32 \\ &= 7(46 - 1 \cdot 32) - 3 \cdot 32 = 7 \cdot 46 - 10 \cdot 32, \end{aligned}$$

so $x = 7$ and $y = -10$.

(c) Since

$$\begin{aligned} 96 &= 1 \cdot 54 + 42, \\ 54 &= 1 \cdot 42 + 12, \\ 42 &= 3 \cdot 12 + 6, \\ 12 &= 2 \cdot 6, \end{aligned}$$

we have $d = \gcd(96, 54) = 6$. Then

$$\begin{aligned} 6 &= 42 - 3 \cdot 12 \\ &= 42 - 3(54 - 1 \cdot 42) = 4 \cdot 42 - 3 \cdot 54 \\ &= 4(96 - 1 \cdot 54) - 3 \cdot 54 = 4 \cdot 96 - 7 \cdot 54, \end{aligned}$$

so $x = 4$ and $y = -7$.

3. Choose any remainder r_7 and quotients q_1 through q_8. For example, if we let all of these be equal to 2, then

$$\begin{aligned}
r_6 &= q_8 \cdot r_7 = 2 \cdot 2 = 4,\\
r_5 &= q_7 \cdot r_6 + r_7 = 2 \cdot 4 + 2 = 10,\\
r_4 &= q_6 \cdot r_5 + r_6 = 2 \cdot 10 + 4 = 24,\\
r_3 &= q_5 \cdot r_4 + r_5 = 2 \cdot 24 + 10 = 58,\\
r_2 &= q_4 \cdot r_3 + r_4 = 2 \cdot 58 + 24 = 140,\\
r_1 &= q_3 \cdot r_2 + r_3 = 2 \cdot 140 + 58 = 338,\\
b &= q_2 \cdot r_1 + r_2 = 2 \cdot 338 + 140 = 816,\\
a &= q_1 \cdot b + r_1 = 2 \cdot 816 + 338 = 1970.
\end{aligned}$$

Then reversing the order of the above equations gives divisions in Euclid's algorithm.

5. To obtain the smallest possible value of a (and $b < a$), we need to choose the remainders and quotients in

$$\begin{aligned}
r_4 &= q_6 \cdot r_5,\\
r_3 &= q_5 \cdot r_4 + r_5,\\
r_2 &= q_4 \cdot r_3 + r_4,\\
r_1 &= q_3 \cdot r_2 + r_3,\\
b &= q_2 \cdot r_1 + r_2,\\
a &= q_1 \cdot b + r_1
\end{aligned}$$

as small as possible. These remainders and quotients have to be positive, and also we must have $r_4 > r_5$, so r_4 must be at least 2. To get the smallest possible value of r_3, we let $r_5 = 1$, $r_4 = 2$, and $q_5 = 1$ (so $q_6 = 2$). After that, we let all quotients be equal to 1, so

$$\begin{aligned}
r_4 &= q_6 \cdot r_5 = 2 \cdot 1 = 2,\\
r_3 &= q_5 \cdot r_4 + r_5 = 1 \cdot 2 + 1 = 3,\\
r_2 &= q_4 \cdot r_3 + r_4 = 1 \cdot 3 + 2 = 5,\\
r_1 &= q_3 \cdot r_2 + r_3 = 1 \cdot 5 + 3 = 8,\\
b &= q_2 \cdot r_1 + r_2 = 1 \cdot 8 + 5 = 13,\\
a &= q_1 \cdot b + r_1 = 1 \cdot 13 + 8 = 21.
\end{aligned}$$

7. Let $f^5(N) = 1$. First, $f(x) = 1$ has only one solution, namely, $x = 2$ (to find it, we solve $\dfrac{x}{2} = 1$ to obtain $x = 2$ and $3x + 1 = 1$ to obtain $x = 0$; however, 0 is not odd, so we disregard it). So $f^4(N) = 2$. Since 2 is not in the form $3x + 1$, $f(x) = 2$ also has only one solution, namely, $x = 4$ (as $\dfrac{4}{2} = 2$). Thus $f^3(N) = 4$. However, $f(x) = 4$ has two solutions: $x = 8$ and $x = 1$ (which are obtained by solving $\dfrac{x}{2} = 4$ and $3x + 1 = 4$, respectively). So $f^2(N)$ is either 8 or 1. For the first case, we solve $f(x) = 8$ to find the unique solution $x = 16$ (as 8 is not in the form $3x+1$), so $f(N) = 16$. Then N is 32 or 5 (again, we solve two equations: $\dfrac{N}{2} = 16$ and $3N + 1 = 16$). For the case $f^2(N) = 1$, we already know from the above work that there is only one solution: $N = 4$. Thus the given equation has three roots total: 4, 5, and 32.

9. Let's look at such list of numbers from the end. The last value in the list has two options: it must be either the smallest or the largest, that is, either 1 or n. The next to last value also has two options: it must be either the smallest or the largest from the first $n-1$ values. (That is, if the last value is 1, then the next to last one must be either 2 or n, and if the last value is n, then the next to last one must be either 1 or $n-1$). The value before the next to last one again has two options, and so on, until the second value in the list. The very first value then will have just one option: whichever value remains at that point. So, there are 2^{n-1} ways to place the numbers 1 through n so that to satisfy the given condition.

11. *Solution 1.* Suppose the 4-tuple 0, 5, 0, 5 does occur. Then before it we must have the digit 0, and before that 5, and before that another 0, and so on—in fact, we will prove that all the digits in our sequence must be 0s and 5s. Indeed, solving

$$a_n \equiv a_{n-4} + a_{n-3} + a_{n-2} + a_{n-1} \pmod{10}$$

for a_{n-4} gives

$$a_{n-4} \equiv a_n - a_{n-3} - a_{n-2} - a_{n-1} \pmod{10}.$$

This implies

$$a_{n-4} \equiv a_n - a_{n-3} - a_{n-2} - a_{n-1} \pmod{5}.$$

Thus if four consecutive digits are divisible by 5, then the previous digit is divisible by 5. It follows that all the digits in the sequence are divisible by 5. However, the starting sequence 1, 2, 3, 4 contains numbers not divisible by 5. We have a contradiction.

Solution 2. If the 4-tuple 0, 5, 0, 5 does occur, look at *the very first time* it occurs. Then working backwards, determine a few digits before these four:

$$\begin{array}{ll}
1,2,3,4,\ldots & \qquad\qquad\qquad\qquad \ldots,\underline{0,5,0,5},\ldots \\
1,2,3,4,\ldots & \qquad\qquad\qquad\quad \ldots,0,\underline{0,5,0,5},\ldots \\
1,2,3,4,\ldots & \qquad\qquad\qquad \ldots,5,0,\underline{0,5,0,5},\ldots \\
1,2,3,4,\ldots & \qquad\qquad \ldots,0,5,0,\underline{0,5,0,5},\ldots \\
1,2,3,4,\ldots & \qquad \ldots,5,0,5,0,\underline{0,5,0,5},\ldots \\
1,2,3,4,\ldots & \ldots,\underline{0,5,0,5},0,\underline{0,5,0,5},\ldots
\end{array}$$

We see that we have this 4-tuple in the sequence again, hence the one we started with was not the first occurrence. We have a contradiction.

13. We want to force our opponent to take the last counter. Thus we have to leave 1 counter on our last turn. To ensure that we will be able to do that, we will leave 6 counters on our next to last turn (then if our opponent takes 1, we take 4 and leave 1; if our opponent takes 2, we take 3; if they take 3, we take 2; if they take 4, we take 1). On the turn before the next to last we'll leave 11, and so on. Continue in this manner and notice the pattern: we want to leave a number congruent to 1 modulo 5. Thus we want to go first, take 1 counter and leave 26. Then no matter how our opponent plays we will be able to leave 21, 16, 11, 6, 1. (This is a winning strategy for the first player.)

15. In order to win we want to take the last counter. We do not want to leave 1 or 2 counters as our opponent would be able to take them and win. So leaving 1 or 2 counters is a bad position. However, leaving 3 counters is good: our opponent will be able to take 1 or 2 counters leaving us with 2 or 1, respectively. We do not want to leave 4 or 5 counters as our opponent would be able to take 4 or 2, respectively, and either win right away or get into a good position. Leaving 6 is good: our opponent will be able to

take 1, 2, or 4, leaving us with 5, 4, or 2 respectively, and all of these are bad positions. Now we see that we want to go first and take 4 counters on our first turn. This will leave 6. Then, as described above, we will win. (This is a winning strategy for the first player.)

17. This problem is similar to problem 15, just involves bigger numbers. Working in a way similar to the one given above for problem 15, we find that leaving 3, 6, or 9 counters is good while leaving 1, 2, 4, 5, 7, or 8 counters is bad. We notice that the numbers 3, 6, 9 are multiples of 3 and guess that this pattern continues. We will prove that leaving multiples of 3 is indeed a winning strategy. Suppose we leave a multiple of 3. Our opponent will take a power of 2. Since a power of 2 is not divisible by 3, they will leave a number not divisible by 3. Then we can take its remainder (1 or 2) upon division by 3 and leave a multiple of 3 again. Thus we want to go first, take 2 counters, and leave 48 (or take 8 and leave 42, or take 32 and leave 18). Then, each time we will be able to leave a multiple of 3. Thus sooner or later we'll leave 0, and we will win.

19. In order to win, we want to leave 1 match on our last turn so that our opponent cannot take it and loses. We do not want to leave 2 matches on our turn as our opponent would be able to take 1 and leave 1. It is good to leave 3 matches: our opponent can only remove 1 leaving us with 2. However, it is bad to leave 4, 5, or 6 as our opponent would be able to take 1, 2, or 3, respectively, and leave 3. Continuing in this manner, we see that leaving 7 is good, leaving any number from 8 to 14 inclusive is bad, and leaving 15 is good again. Notice that the numbers of matches we should leave on our turn (going backwards) are 1, 3, 7, 15. These all are one less than powers of 2. So it seems (and we will confirm this below) that the strategy is to always leave one less than a power of 2. Thus we get a winning strategy for the first player. If we play first, we should remove 45 matches leaving 255; the opponent will remove some number from 1 to 127 (inclusive), leaving some number between 254 and 128 (inclusive). We will then be able to remove the some number between 127 and 1 leaving 127 (which means that we removed no more than half); the opponent will remove some number between 1 and 63 leaving some number between 126 and 64. We will remove the required number between 63 and 1 to leave 63; the opponent will remove 1-31 leaving 62-32. We will remove the required number between 31 and 1 to leave 31; the opponent will remove 1-15 leaving 30-16. We will remove the required number between 15 and 1 to leave 15. Then, as described above, we will leave 7, 3, and, finally, 1, and we win.

21. Writing 0 is a losing move. Writing 1 is good since then the only opponent's choice is to subtract 1 and write 0. Writing 2 is bad since the opponent can subtract 1 and write 1. Writing 3 is good since the opponent can only subtract 1 or 3 and write 2 or 0, respectively, both of which are bad positions. Then, 4 is a bad position since the opponent can subtract 1 and write 3 which is good. Continuing in this manner, we notice that every odd number is a good position and every even number is bad. If we don't want to go all the way to 55 and check this manually, we need to prove that this pattern will continue. To do so, we can prove that from any odd number one can only go to an even number, and from any even number it is possible to go to an odd one. Indeed, any divisor of an odd number is odd, and subtracting an odd number from an odd number always results in an even number. Thus, if we write an odd number on our turn, our opponent will be forced to write an even number. Next, 1 is a divisor of any number, thus from any even number we can subtract 1 and obtain an odd one. That is, when our opponent writes an even number, we can get to an odd one. Therefore, we will always force our opponent to a bad position and then be able to get to a good

one on our turn. Since 55 is odd, the first player will be forced to go to a bad position (even number), and the second player has a winning strategy.

23. Again, working backwards, we will determine good and bad positions. First, 1000 is good. Then, 999 is bad as it is possible to add 1 and get to 1000. The number 998 is bad as well as it is possible to add 2 and get to 1000. So is 997, and 996, and, in fact, all numbers from 501 to 999 are bad as from each one it is possible to get to 1000. However, 500 is good as our opponent can only add a number from 1 to 499 and get to a range from 501 to 999, and all numbers in that range are bad. Continuing in similar manner, we find that 251-499 are bad, 250 is good, 126-249 are bad, 125 is good, 63-124 are bad, 62 is good, 32-61 are bad, 31 is good, 16-30 are bad, 15 is good, 8-14 are bad, 7 is good, 4-6 are bad, 3 is good, and 2 is bad. Since, starting from 2, the first player can get to 3, which is a good position, the first player has a winning strategy in this game.

25. First we mark (with a plus sign) the bottom left corner as a good position, and those squares to which it is possible to get to it we mark (with a minus sign) as a bad position (because if we go there, our opponent will be able to move to the winning position). Then those squares from which it is possible to get to only bad positions we mark as good (if we go there, we know our opponent will only be able to go to a bad position, from which we'll move to a good one).

−	−					
+	−					

+						
−	−					
+	−	+				

We mark those squares from which it is possible to get to a good one as bad again, and repeat this process until we have filled the whole board.

−	−					
+	−					
−	−	−	−			
+	−	+	−			

+	−	+	−	+	−	+
−	−	−	−	−	−	−
+	−	+	−	+	−	+
−	−	−	−	−	−	−
+	−	+	−	+	−	+
−	−	−	−	−	−	−
+	−	+	−	+	−	+

Note that at each step, for a square to be marked as bad it is sufficient to have just one move from it to a good one; while for a square to be marked as good all moves from this square must lead to bad ones. Since the top right corner is a good position, the second player has a winning strategy. The first player will have to move to one of the bad positions, and then the second player can move to a good one at each step.

27. The process is similar to that of the previous two problems. The first step and the whole board are shown below.

							−
						−	
					−		
				−			
			−				
		−					
−	−						
+	−						

−	−	+	−	−	−	−	−
+	−	−	−	−	−	−	−
−	−	+	−	−	−	−	−
+	−	−	−	−	−	−	−
−	−	+	−	−	−	−	−
+	−	−	+	−	+	−	+
−	−	−	−	−	−	−	−
+	−	+	−	+	−	+	−

If the counter is initially placed in one of the bad positions, the first player has a winning strategy (namely, the first player should move the counter to any of the good positions on each move). If the counter initially placed in one of the good positions, the second player has a winning strategy.

29. Possible positions in this game are of the form (a, b) where $0 \leq a \leq 6$ and $0 \leq b \leq 6$ represent the numbers of counters in the first and second pile, respectively. These can be organized in a square grid.

second pile							
6							
5							
4							
3							
2							
1							
0							
	0	1	2	3	4	5	6

first pile

Then this game is equivalent to that in problem 25. We have determined before that the second player has a winning strategy, namely, on each move they should put the counter in an (odd, odd) position.

+	−	+	−	+	−	+
−	−	−	−	−	−	−
+	−	+	−	+	−	+
−	−	−	−	−	−	−
+	−	+	−	+	−	+
−	−	−	−	−	−	−
+	−	+	−	+	−	+

However, notice that an odd position, horizontally or vertically, in the board game corresponds to an even number of counters left in the corresponding pile. (In particular, the bottom left corner corresponds to zero counters in each pile.) Thus, in this game the second player has a winning strategy: they should always leave an even number of counters in each pile.

31. Hint. This game is equivalent to the following one.

    A counter is placed in the upper right corner of an 8×8 board. Two players take turns moving a counter. It can be moved one space to the left, one space down, diagonally one space to the left and one down, or diagonally one space to the left and one up. The player who cannot make a move loses.

    Answer. The winning positions in this game are the same as those in problem 25.

33. The derivative of a cubic polynomial is a quadratic polynomial. We want this quadratic polynomial to have integer roots. Instead of trying random coefficients a, b, c, and d, let's choose the roots of the quadratic polynomial (the derivative of f), and then find f.

    Choose the roots, e.g. $r_1 = 3$ and $r_2 = 5$. Then $(x-3)(x-5) = x^2 - 8x + 15$. Now $f(x)$ can be any antiderivative of this polynomial, say, $\frac{1}{3}x^3 - 4x^2 + 15x - 3$. However, we want it to have integer coefficients, so let's multiply this function by 3: $f(x) = x^3 - 12x^2 + 45x - 9$. Then $f'(x) = 3x^2 - 24x + 45 = 3(x^2 - 8x + 15) = 3(x-3)(x-5)$ has integer roots.

    Here is another choice of roots and the constant d: $r_1 = -3$, $r_2 = 4$, $(x+3)(x-4) = x^2 - x - 12$, an antiderivative is $\frac{1}{3}x^3 - \frac{1}{2}x^2 - 12x - \frac{1}{6}$, multiply by 6: $f(x) = 2x^3 - 3x^2 - 72x - 1$. Then $f'(x) = 6x^2 - 6x - 72 = 6(x^2 - x - 12) = 6(x+3)(x-4)$ has integer roots.

35. Start with a matrix in reduced echelon form with integer entries, and perform a few operations (i.e. work backwards in the reducing algorithm) to modify some (or all) coefficients. Here is an example:

$$\begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix} \leftarrow \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & -2 \\ 0 & -2 & 4 \end{bmatrix} \leftarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & 3 & 6 \\ 0 & -2 & 4 \end{bmatrix} \leftarrow \begin{bmatrix} 1 & 1 & 1 \\ -2 & 1 & 4 \\ 0 & -2 & 4 \end{bmatrix} \leftarrow$$
$$\begin{bmatrix} 1 & 1 & 1 \\ -2 & 1 & 4 \\ 3 & 1 & 7 \end{bmatrix} \leftarrow \begin{bmatrix} 4 & 4 & 4 \\ -2 & 1 & 4 \\ 3 & 1 & 7 \end{bmatrix}.$$

8 Invariants

1. *Proof 1.* We consider all possible cases of signs of the two numbers changed:

    - positive, positive $\to$ negative, negative,
    - positive, negative $\to$ negative, positive,
    - negative, negative $\to$ positive, positive.

    We see that the number of positive numbers either does not change or changes by ± 2. Thus the parity of the number of positive numbers is an invariant. We start with the set containing 3 positive numbers. It is not possible to reach 6 positive numbers because 3 is odd but 6 is even.

    Proof 2. When two numbers are multiplied by -1, the product of all the numbers does not change. Initially the product is -36, so it will always be -36. It is not possible to make it 36.

3. The product of an odd number and 3 is an odd number, and the product of an even number and 3 is an even number. So multiplication by 3 does not change the parity of the number. Also, the difference of an odd number and 2 is an odd number, and the difference of an even number and 2 is an even number. So neither of the permitted operations changes the parity of the number. The initial set consists of four odd numbers. Thus the four numbers will always be odd. Therefore it is not possible to reach 2 odd and 2 even numbers.

5. When we change the signs of two numbers, the product of all numbers does not change. Initially the product is 1. Since the product of twenty-five -1s is -1, it is not possible to change all numbers into -1.

7. When we replace a by $a+2b$ or $a-2b$ (where a and b are two numbers in the set), we do not change its parity (if a is even, then $a \pm 2b$ is even, and if a is odd, then $a \pm 2b$ is odd). Thus the parity of each number will always be the same. Initially we have two even and two odd numbers. It is not possible to make all of the numbers even.

9. When we add one coin to each of two stacks, the total number of coins increases by 2, therefore, it doesn't change its parity. Initially the total number of coins is 21, which is odd, so it will always remain odd. The sum of six equal numbers would be divisible by 6, and thus even. Therefore it is not possible to reach six equal stacks.

11. The parity of the number of $-$ signs does not change:

    - if two $+$s are replaced by a $+$, then the number of $-$s does not change,
    - if two $-$s are replaced by a $+$, then the number of $-$s is decreased by 2,
    - if a $+$ and a $-$ are replaced by a $-$, then the number of $-$s does not change.

    Therefore if we had an even number of $-$ signs, then a $+$ will remain in the end, and if we had an odd number of $-$ signs, then a $-$ sign will remain in the end.

13. We have seen in problem 1 in chapter 4 that any number is congruent to the sum of its digits modulo 9. Thus the question is equivalent to whether there are more numbers among 1, 2, ..., 10^6 congruent to 1 or congruent to 2 modulo 9. Remainders of consecutive natural numbers modulo 9 are 1, 2, 3, ..., 8, 0, and this 9-tuple repeats. The last number in our sequence is $10^6 \equiv 1 \pmod 9$, thus there will be more 1s.

15. Hint. Prove that the described operation does not change the number modulo 9. Prove that the starting number and any number with ten distinct digits are not congruent modulo 9.

17. When we replace a and b (let $a \geq b$) by $a-b$, the sum of all the numbers changes by

    $$-a-b+(a-b) \equiv -2b \equiv 0 \pmod 2.$$

    So the parity of the sum does not change. Initially the sum is

    $$1+2+\cdots+(4n-1) \equiv \frac{(4n-1)4n}{2} = (4n-1)2n,$$

    which is even. Thus the sum of the numbers is always even. Therefore an even number will remain in the end.

19. *Proof 1.* The sum of the numbers does not change since a, b, c, d, $\dots$ are replaced by $2b-a$, $2c-b$, $2d-c$, $\dots$. The sum of the original numbers is 45. However, the sum of ten 5s is 50. Therefore it is not possible to reach ten 5s.

    Proof 2. Since $2b - a \equiv a \pmod 2$, $2c - b \equiv b \pmod 2$, etc., and we start with 5 even and 5 odd numbers, we will always have 5 even and 5 odd numbers. Therefore it is not possible to reach ten 5s.

21. Let the integers in the order they are written be a_1, a_2, a_3, a_4, a_5, and a_6. The sets $\{a_1, a_2, a_3, a_4, a_5, a_6\}$ and $\{1, 2, 3, 4, 5, 6\}$ are equal. Thus the sum of all the a_is is

    $$a_1 + a_2 + a_3 + a_4 + a_5 + a_6 = 1 + 2 + \cdots + 6 = 21.$$

    When we add its place number to each integer, we get

    $$a_1 + 1,\ a_2 + 2,\ a_3 + 3,\ a_4 + 4,\ a_5 + 5,\ a_6 + 6.$$

    The sum of these is

    $$\begin{aligned}&(a_1 + 1) + (a_2 + 2) + (a_3 + 3) + (a_4 + 4) + (a_5 + 5) + (a_6 + 6) =\\ &(a_1 + a_2 + a_3 + a_4 + a_5 + a_6) + (1 + 2 + 3 + 4 + 5 + 6) = 21 + 21 = 42.\end{aligned}$$

    Suppose that all the sums $a_1 + 1$, $a_2 + 2$, $a_3 + 3$, $a_4 + 4$, $a_5 + 5$, and $a_6 + 6$ have different remainders upon division by 6. Then the remainders are a permutation of the set $\{0, 1, 2, 3, 4, 5\}$ whose sum is

    $$0 + 1 + 2 + 3 + 4 + 5 \equiv 15 \equiv 3 \pmod 6.$$

    Since $42 \not\equiv 3 \pmod 6$, we get a contradiction.

23. (a) Answer: no. Hint: consider, for example, the parity of the sum of all six numbers.

    (b) Answer: no. Remark: the parity of the sum of all six numbers does not produce a contradiction in this case. We need to find another invariant that does.

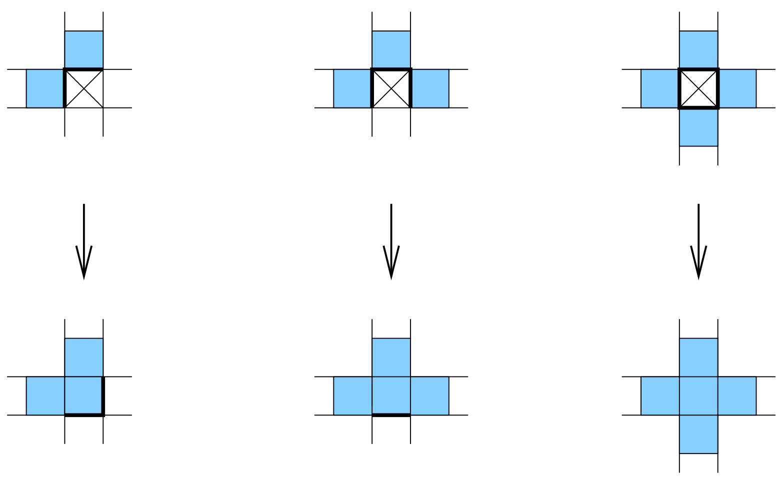
25. Note that when we replace two chips of different colors by a single chip of a third color, all three numbers of white, black, and red chips change parity. So if any of them are of the same parity, they will always be of the same parity, and if two numbers are of different parity, they will always be of different parity. If only one chip remains in the end, that means that the original number of chips of that color was of different parity than the other tho numbers. In other words, the color of the chip remaining in the end corresponds to the number – a, b, or c – that is of different parity than the other two. (However, if all of a, b, and c are of the same parity, it is not possible to reach just a single chip.)

27. Answer: zero. Hint: prove that after the first minute, the number of zombies is always divisible by 3.

29. No. If 5 heads are cut off and 17 new heads grow, the number of heads increases by 12; if 15 are cut off and 24 grow, the number increases by 9; if 17 are cut off and 2 grow, the number decreases by 15; if 20 are cut off and 14 grow, the number decreases by 6. Observe that all of these changes are divisible by 3. Initially the dragon had 100 heads, which is congruent to 1 modulo 3. Since every change is divisible by 3, the number of heads of the dragon will be congruent to 1 modulo 3 as long as the dragon is still alive. To die, it needs to loose all its heads in a single blow of the sword, but for that it would need to have 5, 15, 17, or 20 heads, and none of those numbers is congruent to 1 modulo 3. Therefore the dragon will never die.

31. (a) No. Consider the five vertices only (not the five intersection points of the diagonals). Initially all five of their labels are 1s. Observe that when all signs on one side or one diagonal are changed, exactly two of the five vertex labels are changed. So the number of 1s among them doesn't change parity (as it either increases by 2 or decreases by 2 or doesn't change). Therefore the number of 1s will always be odd, and thus it is not possible to eliminate all of them.

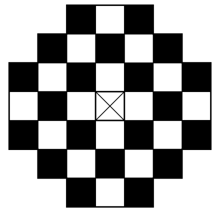
33. Consider cells with two, three, or four infected neighbors. Notice that when the infection spreads to such a cell, the perimeter of the contaminated area cannot increase (but it may decrease). Namely, when a cell with two infected neighbors becomes infected, the perimeter of the contaminated area does not change. When a cell with three infected neighbors becomes infected, the perimeter decreases by 2. When a square with four infected neighbors becomes infected, the perimeter decreases by 4. (Some examples are shown below.) Initially the perimeter is at most $4 \cdot 9 = 36$. Since it cannot increase, it cannot become 40.



35. First we divide the Parliament into two houses randomly. We will say that a Parliament member is unsatisfied with their placement if they have two or more enemies in their house. If there are unsatisfied members, we will choose any one of them and move them to the other house. Now there is at most one enemy in their house. By this move we reduced the number of hostile pairs (because the member moved was in at least two hostile pairs and now they are in at most one hostile pair, and no pairs not containing that member were affected by their move). If any unsatisfied members remains, then again we will choose one of them and move them, thus reducing the number of hostile pairs again. And so on. Since it is not possible for the number of hostile pairs to become negative, sooner or later there will be no unsatisfied members.

9 Coloring

1. There are 36 squares, and each domino covers 2, so we need 18 dominoes. Color the figure as a chessboard (see picture below). Since each domino covers one black square and one white square, 18 dominoes must cover 18 black and 18 white squares. However, the figure has 20 black and 16 white squares. Therefore it is not possible to cover it with dominoes.



3. The task is not possible because in each move, the chess knight goes from a white square to a black one or from a black square to a white one. Therefore after eleven (or any odd number of) moves the knight will be on the opposite color from the one it started on. Thus it cannot be at the original position after 11 moves.

5. No, because a camel will always stay on one color, while the top-left and top-right corners of the chessboard have opposite colors.

7. *Proof 1.* Suppose such a covering is possible. A 10×10 board has 100 squares. Color the board using the standard chessboard coloring. Then it has 50 black squares and 50 white squares. Each T-tetromino covers either 3 black and 1 white or 1 black and 3 white squares. Suppose there are n T-tetrominoes covering 3 black squares. Then there are $25 - n$ T-tetrominoes covering 1 black square. Then all 25 tetrominoes cover $3n+(25-n) = 2n+25$ black squares. They must cover 50, so $2n+25 = 50$, or $2n = 25$. However, this equation has no integer solutions since 25 is not divisible by 2. We get a contradiction.

   Proof 2. Suppose such a covering is possible. A 10×10 board has 100 squares. Color the board using the standard chessboard coloring. Then it has 50 black squares and 50 white squares. Each T-tetromino covers either 3 black and 1 white or 1 black and 3 white squares. Either way, it covers an odd number of black squares. Thus 25 (and odd number of) T-tetrominoes must cover an odd number of black squares. However, the board has 50, that is, an even number of black squares. We get a contradiction.

9. *Proof 1.* Suppose such a covering is possible. A 10×10 board has 100 squares. Consider the standard chessboard coloring. Then the board has 50 black squares and 50 white squares. Each T-tetromino covers either 3 black and 1 white or 1 black and 3 white squares. Suppose there are n T-tetrominoes covering 3 black squares. Then there are $15 - n$ T-tetrominoes covering 1 black square. Each L-tetromino covers 2 black and 2 white squares. Therefore 10 L-tetrominoes cover 20 black squares. Thus all 25 tetrominoes cover $3n + (15 - n) + 20 = 2n + 35$ black squares. They must cover 50, so $2n + 35 = 50$, or $2n = 15$. However, this equation has no integer solutions since 15 is not divisible by 2. We get a contradiction.

   Proof 2. Suppose such a covering is possible. A 10×10 board has 100 squares. Consider the standard chessboard coloring. Then the board has 50 black squares and 50 white squares. Each T-tertromino covers either 1 or 3 black squares, that is, an odd number of black squares. Each L-tetromino covers 2 black squares, that is, an even number of black squares. Then 15 (an odd number of) T-tetrominoes and any number of L-tetrominoes together must cover an odd number of black squares. However, the board has an even number of black squares. We get a contradiction.

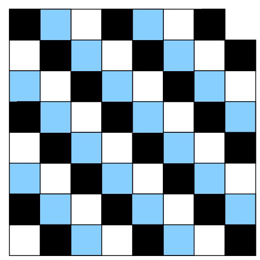
11. *Proof 1.* Suppose such a covering is possible. Color the board using the stripe pattern with alternating black and white stripes (see the first of the four colorings on page 51). There are 50 black squares. The rest of the argument is the same as that in proof 1 for problem 7. Each L-tetromino covers either 1 or 3 black squares. Let n

tetrominoes cover 3 black squares, then $25 - n$ tetrominoes cover 1 black square, so all 25 tetrominoes together cover $3n + (25 - n) = 2n + 25$ black squares. Therefore $2n + 25 = 50$, or $2n = 25$. However, this equation has no integer solutions. We get a contradiction.

Another approach is to use the coloring in proof 1 but to write a proof similar to proof 2 for problem 7.

13. Suppose the upper right corner has been removed.

Proof 1. Color the board diagonally using three colors as shown below.

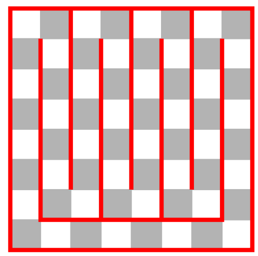


The board contains 21 white, 22 black, and 20 blue squares. Since each straight tromino must cover 1 white, 1 black, and 1 blue square, the board cannot be covered by straight trominoes.

Proof 2. Suppose such a covering is possible. Color the board using horizontal stripes of three colors, say, from top to bottom: white, blue, black. Then there are 23 white squares (three rows minus one removed square), 24 blue squares (three full rows), and 16 black squares (two full rows). Each tromino covers either three squares of the same color or one square of each color. Let a be the number of trominoes covering three white squares, b the number of trominoes covering three blue squares, c the number of trominoes covering three black squares, and d the number of trominoes covering one square of each color. Then, for the total number of squares of each color, we have: $3a + d = 23$, $3b + d = 24$, and $3c + d = 16$. Subtracting the first equation from the second we have $3b - 3a = 1$, or $3(b - a) = 1$. The left-hand side is divisible by 3 and the right-hand side is not. We get a contradiction.

15. Suppose such a covering is possible. Color the board using the stripe coloring using two colors, say, black and white, starting with black (see the first of the four colorings on page 51). Then there are 12 black columns and 11 white columns, so there are 23 more black squares than white ones. Each 2×2 tile covers 2 black and 2 white squares, so if there are n of such tiles they cover $2n$ black and $2n$ white squares. Each 3×3 tile covers either 6 or 3 black squares and, respectively, either 3 or 6 white squares. Let m be the number of 3×3 tiles that cover 6 black and 3 white squares, and let k be the number of 3×3 tiles that cover 3 black and 6 white squares. Then the total number of black squares covered by 3×3 tiles is $6m + 3k$, and the total number of white squares covered by 3×3 tiles is $3m + 6k$. Thus the total number of black squares covered by all tiles is $2n + 6m + 3k$, and the total number of white squares covered by all tiles is $2n + 3m + 6k$. Since there are 23 more black squares than white squares, we have $(2n+6m+3k)-(2n+3m+6k) = 23$, or $3m-3k = 23$, where n, m, and k are integers. However, we see that the left-hand side is divisible by 3, but the right-hand side is not. We get a contradiction.

17. Yes. Consider the following picture (we will show the black squares in gray to make it easier to see the red borders).



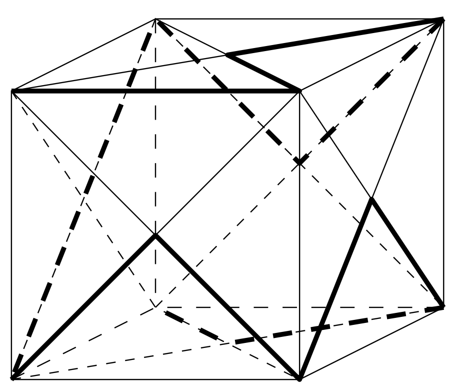
Note that the squares of the whole board are organized into one continuous "loop." If one black square and one white square are removed, then the loop breaks into one or two pieces. Namely, if the removed black and white squares are consecutive in this loop, we get one piece consisting of 62 squares which can be paired up and covered by 31 dominoes. If the removed black and white squares are not consecutive, we get two pieces. Note that each piece will have ends of opposite colors and hence contain an even number of squares. Therefore each piece can be covered by dominoes.

19. (a) Notice that every piece of a face diagonal connects a vertex and a face midpoint. Thus if we only use face diagonals, vertices and midpoints must alternate. However, there are 8 vertices and 6 midpoints, so there is no way to make them alternate (there are too many vertices).

    Note. We could color all the marked points, e.g. let vertices be black, and let midpoints be white. Then black and white points must alternate, but there are 8 black points and 6 white points, so that's impossible. In this problem coloring points was not very useful because it was easy to refer to the points as vertices and face midpoints. In fact, coloring made our solution longer. However, in many problems there is no "natural" division of the points and such a coloring could provide a way to explain a similar argument.

    (b) If one edge is allowed, then we could have two vertices in the beginning, after which we would be left with 6 midpoints and 6 vertices, and we can make them alternate. Again, let vertices be black and midpoints white, then a path could be e.g. *bbwbwbwbwbwbwb*.

    Here is an example. (But there are many other such paths.)

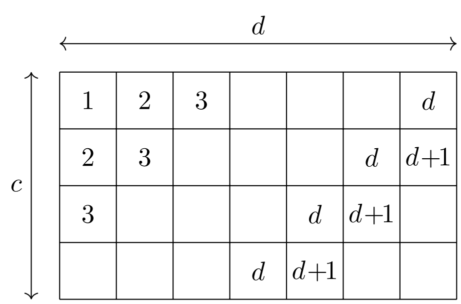


21. Let a be the number of rows and let b be the number of columns. If $n|a$, then $a = nk$ for some integer k, and each column contains nk squares. Thus we can cover each column

by k "vertical" $1 \times n$ tiles. Similarly, if $n|b$, then $b = nk$ for some integer k, and each row contains nk squares. Thus we can cover each row by k "horizontal" $1 \times n$ tiles.

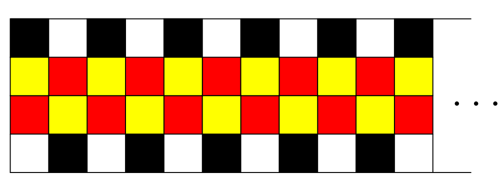
Now suppose that $n \nmid a$ and $n \nmid b$ but an $a \times b$ board can be covered by $1 \times n$ tiles. Color the board diagonally using n colors. Each tile must cover exactly one square of each color. Therefore each color must appear the same number of times. We will show below that this is not possible, thus obtaining a contradiction.

If $a > n$, then in the first n rows each color appears exactly b times (because each color appears exactly once in each column of length n). Therefore if we throw these first n rows away, each color must still appear the same number of times. Similarly, we can throw away the next set of n consecutive rows, and so on, until less than n rows remain. Similarly for the columns. So now we reduced our board to, say, a $c \times d$ board where $c < n$ and $d < n$, and each color must appear the same number of times. Without loss of generality we can assume that $c \leq d$. This $c \times d$ piece is colored diagonally, and we can assign numbers 1 through n to our colors so that they appear in the increasing order as shown in the picture below.



Since $d < n$, the number of colors is at least $d+1$, so the first $d+1$ "diagonals" are of different colors. Since only $c-2 \leq d-2 < d$ "diagonals" remain, colors d and $d+1$ will not repeat. Therefore in this piece there are c squares of color d but only $c-1$ squares of color $d+1$. Thus the colors are not distributed evenly. We get a contradiction.

23. No. Assume that such a reentrant knight tour exists. Color the board as shown on the picture below.



Notice that from any black square a knight can only get to a yellow square; from any white square a knight can only get to a red square. Since there are 123 squares of each color, the tour must contain 123 pairs "black, yellow" and 123 pairs "white, red." However, there is no way to get from a yellow square to a white one or from a red square to a black one. We get a contradiction.

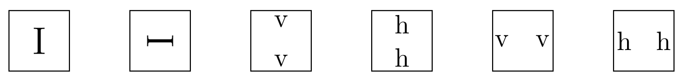
25. Suppose such a filling is possible. "Color" the small (i.e. $1 \times 1 \times 1$) cubes of the $6 \times 6 \times 6$ cube as follows. Color each other "level" as in Example 9.2, and the other "levels" all white.

Then the $6 \times 6 \times 6$ cube contains 27 black cubes. Each $4 \times 1 \times 1$ brick fills either none or two, so, an even number of black cubes. Therefore all bricks together must fill an even number of black cubes. We get a contradiction.

27. Hint 1. Use a chessboard-like coloring (namely, two regions that share an edge have different colors).

    Hint 2. Consider even and odd numbers.

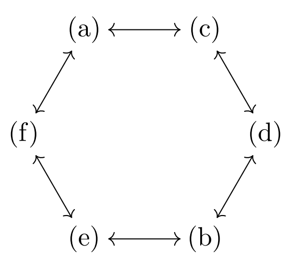
29. The fourth box from the left. Consider the following six cases for the position of T on the box.



(a) (b) (c) (d) (e) (f)

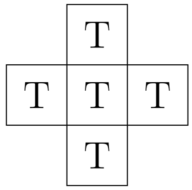
(a) the T is on the top or bottom face (so when it's on the top, we see it when we look from above) and positioned vertically (correctly or upside down) when we look in the direction discussed in the problem,
(b) the T is on the top or bottom face and is positioned sideways (to the right or to the left),
(c) the T is on a side face that is represented by a horizontal segment in our diagram and positioned vertically (correctly or upside down) if we look at that face from the side, standing on the floor,
(d) the T is on a side face that is represented by a horizontal segment in our diagram and positioned sideways,
(e) the T is on a side face that is represented by a vertical segment in our diagram and positioned vertically,
(f) the T is on a side face that is represented by a vertical segment in our diagram and positioned sideways.

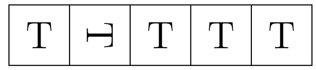
Observe that when a box is rolled about one of its edges, only the following changes between these six positions are possible:



No matter how the box is rolled, an even number of steps is needed to get to the same position, e.g. from (a) to (a), and an odd number of steps is needed to get from (a) to (b).

Now, color the floor like a chessboard with whole squares directly under each box in the original position. Note that initially four of the squares are on one color, and one (the one in the center of the cross) is on the opposite color. Every time a box is rolled, it changes the color of the square it occupies. In the final position, three boxes are on one color and two are on the opposite color.



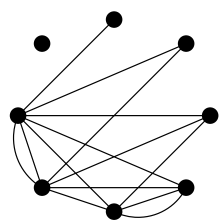


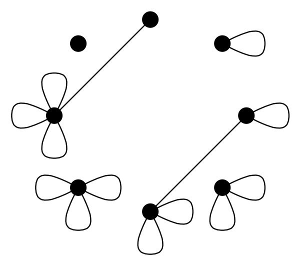
Since these three boxes are in the same position (a), it follows that they came from some three of the four that were on the same color initially, so they took an even number of rolls. The fourth one is on the opposite color in the end, so it took an odd number of rolls, thus it is the second box from the left that is in position (b). Therefore the center box, after an even number of rolls, is the fourth one from the left in the final position.

10 Graphs

1. By corollary 10.6, in any graph, the number of vertices of odd degree is even. Here there are 3 vertices of degree 3, so there is no such graph.

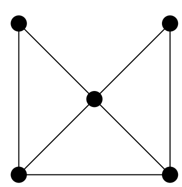
3. (a) Yes. Here are two examples.





   (b) No. If one vertex has degree 7, then it is connected with every other vertex. Hence there cannot be a vertex of degree 0.

5. Since the sum of the degrees is $2 + 2 + 3 + 3 + 4 = 14$, by Theorem 10.5 the graph has seven edges. Below is an example of such a graph.



7. Consider the graph in which vertices represent cities and edges represent roads. Since four roads lead out of each city, the degree of each vertex is 4. There are 100 cities, so there are 100 vertices in the graph, and thus the sum of all their degrees is 400. By Theorem 10.5, there are 200 roads.

9. *Solution 1.* Let vertices A, B, C, D, E have degrees 1, 2, 3, 4, and 5, respectively, and let F be the sixth vertex. Since E has degree 5, it is connected to every other vertex. Since A has degree 1, it is not connected to any vertex but E. Then, since D has degree 4 but is not connected to A, it is connected to B, C, E, and F. Now, B has degree 2, and we already know that it is connected to D and E, so it is not connected to C or F. Finally, C has degree 3 and is connected to D and E but not A or B, so it must be

connected to F. Thus F is connected to C, D, and E, but not A or B, so F also has degree 3.

Solution 2. Since the graph is simple, the degree of the sixth vertex is at most 5. The sum of the degrees of all six vertices must be even, so it follows that the degree of the sixth vertex is odd. Therefore it is 1, 3, or 5. We will show that it cannot be 1 or 5. Indeed, if it were 1, there would be two vertices of degree 1. The vertex of degree 5 must be connected to every other vertex. Then the vertices of degree 1 are not connected to any vertex other than the one of degree 5. Then no vertex can have degree 4 as there are not four vertices to which it can be connected. Now, if the degree of the sixth vertex were 5, then there would be two vertices of degree 5. They must connect to every other vertex in the graph. Then there cannot be a vertex of degree 1. Therefore the degree of the sixth vertex is 3.

11. Answer: 5. Hint: consider the graph in which vertices represent the club members, and two vertices are connected if and only if the corresponding members have played chess. Then use reasoning similar to that in solution 1 to problem 9.

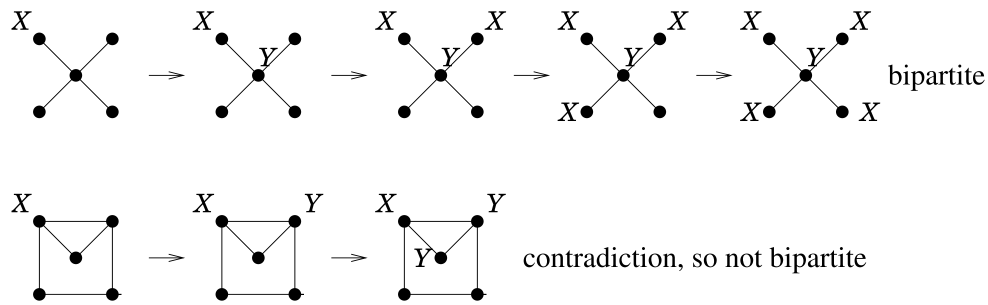
13. If this were possible, consider the following graph with 8 vertices: each vertex represents a county, and two vertices are connected if and only if the corresponding counties are neighbors. Then the degree of each vertex is the number of the neighbors of that county. Thus we would have a graph with 8 vertices of degrees 5, 5, 4, 4, 4, 4, 4, 3. But in any graph, the sum of the degrees of all the vertices is even. The sum $5+5+4+4+4+4+4+3 = 33$ is odd. We have a contradiction.

15. Let vertices represent people, and edges represent friendship (two vertices are connected if and only if the corresponding people are friends). Then the degree of each vertex is the number of friends of the corresponding person. Since in any graph the number of vertices of odd degree is even, we have that the number of people with an odd number of friends is even.

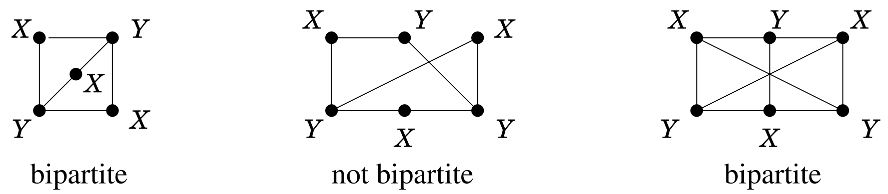
17. Suppose that there is no way to get from town A to town B. Then they are in two different connected components. However, since each town is connected to at least 7 others, each connected component contains at least eight towns. This is impossible.

19. Let us represent this transportation system by a graph. Each city is represented by a vertex, and each carpet line is an edge. Then the degree of the capital is 21, the degree of Farville is 1, and the degree of each other vertex is 20. If the capital and Farville were in different connected components, then the connected component of Farville would contain itself and a few other cities that have degree 20 each. Then the sum of the degrees of all vertices in this connected component would be odd, which is impossible.

21. We can start with any vertex and assume it's in set X. Then consider any vertex connected with the first one, and if the graph is bipartite, this vertex must be in the other set, say, Y. Then consider any vertex connected with the first or second one, and so on. If we ever run into a situation when two vertices in one set are connected, the graph is not bipartite. If not, we'll have a division of the set of vertices into two sets X and Y such that there are no edges within one set, and hence the graph is bipartite.



Similarly:

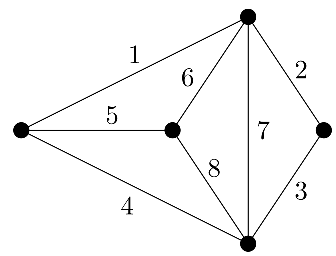


23. Let the degrees of the remaining vertices be a (in group A) and b (in group B). The sum of degrees of vertices in the first group must be equal to the sum of degrees of vertices in the second group. Thus $4 + 2 + 2 + a = 3 + 1 + 1 + b$, or $3 + a = b$. Since the graph is connected, the degree of each vertex is at least 1, thus $a \geq 1$. This implies that $b \geq 4$. However, the graph is simple, so $b \leq 4$, and thus $b = 4$ and $a = 1$.

25. If nobody made a mistake, we would be able to draw a bipartite graph with 14 vertices, seven vertices representing the kids from school A and seven vertices representing the kids from school B, with two vertices connected if and only if the corresponding students have played a game. Then the sum of the degrees of the seven vertices representing the kids from school A should be equal to the sum of the degrees of the seven vertices representing the kids from school B (both sums being equal to the number of edges). However, it is not possible to divide the given 14 numbers into two groups such that the sums are equal because one group must contain the 5, and the other group must consist of 3s and 6s only. The sum of the numbers in the first group is congruent to 2 modulo 3, and the sum of the numbers in the second group is congruent to 0 modulo 3, so the sums cannot be equal.

27. A graph has an Euler path but not an Euler cycle if and only if it is connected and has exactly two vertices of odd degree. The fact that it does not have an Euler cycle follows from Theorem 10.14. Here is a proof that it contains an Euler path. Connect the two vertices of odd degrees. Then all degrees become even and thus there exists an Euler cycle. Now remove this added edge and obtain an Euler path of the original graph.

29. The first graph can be drawn. A possible order of edges is shown below.



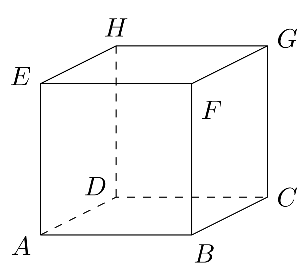
The second graph cannot be drawn since there are four vertices and the degree of each vertex is 3. The third graph cannot be drawn either because there are four vertices of odd degree.

31. No. The degree of each vertex is 3. Since there are more than two (eight, to be precise) vertices of an odd degree, a path that uses all edges exactly once is not possible.

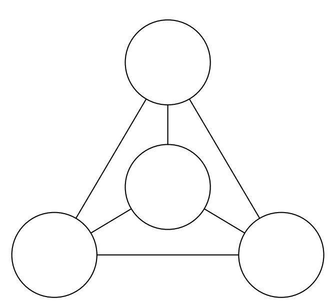
33. (a) Note that the possibility to bend a wire into a cube is equivalent to the existence of an Euler path. The cube does not have an Euler path because there are eight vertices of odd degree.

    (b) Each vertex of an odd degree must be an endpoint of a piece of wire. Since two pieces have only four endpoints but there are more than four vertices in a cube and each of them has an odd degree, two pieces cannot be bent into a cube.

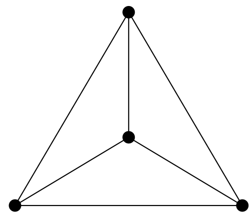
    (c) Each vertex of a cube must be an endpoint of a piece of wire. So at least four pieces are needed. Four pieces are possible, for example $ABCDAEFGHE$, BF, CG, and DH. (There are many different solutions.)



35. If we draw a plan of the islands and bridges, we obtain the following picture.

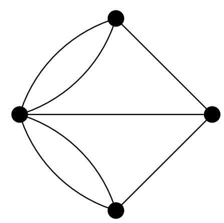


Thus the problem is equivalent to existence of an Euler path for the following graph.



There four vertices of odd degree, therefore there is no Euler cycle.

37. We will use a graph to represent the city as follows. Let each of the four pieces of land be represented by a vertex, and let each bridge be represented by an edge connecting the corresponding vertices. Then we get the following graph:



Existence of a tour described in the problem is equivalent to existence of an Euler cycle in this graph. We know that an Euler cycle exists if and only if the degree of each vertex is even. However, the degrees of the vertices in our graph are 5, 3, 3, and 3, i.e. all odd, and this is how we know that no such tour exists. Now, if we add one more bridge we will change the degrees of two vertices (corresponding to the pieces of land that this bridge connects), thus the degrees of two vertices will become even, but the degrees of the other two vertices will remain odd. Therefore no such tour will exist even if one bridge is built.

39. First of all, recall that $K_{n,m}$ has 2 groups of vertices, n vertices in group A, m vertices in group B, and every vertex in group A is connected to every vertex in group B.

    (a) We know that an Euler cycle exists if and only if the degree of each vertex is even. The graph $K_{n,m}$ has n vertices of degree m and m vertices of degree n. Thus an Euler cycle exists iff both m and n are even.

    (b) By problem 27, an Euler path exists if and only if the graph has at most 2 vertices of odd degree. We have the following cases:

    (1) No vertices of odd degree, i.e. all the degrees are even. Then both m and n are even.

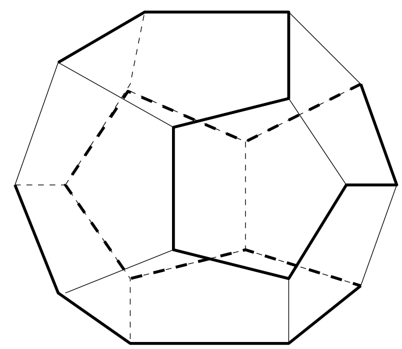
    (2) There are two vertices of odd degree and they are in the same group, say, in group A of n vertices. Since all the vertices in this group have the same (odd) degree, and we can have at most 2 vertices of odd degree, there are only 2 vertices in this group, thus $n = 2$. Since their degree is odd, m is odd. Thus we have $n = 2$ and m is odd. Similarly, we could have $m = 2$ and n odd.

    (3) There are two vertices of odd degree, one in group A and the other in group B. Then both m and n are odd, thus all the degrees are odd, but we can have at most 2 odd degrees, so $n = m = 1$.

    (c) A Hamilton cycle is a cycle that visits every vertex exactly once. If a Hamilton cycle starts at a vertex in group A, then its second vertex belongs to group B, the next one belongs to group A, the fourth one belongs to group B, and so on, i.e. A and B will alternate. It must eventually come back to the original vertex, therefore the number of vertices in group A must be equal to the number of vertices in group B. Thus $m = n$. Conversely, if $m = n$, then a Hamilton cycle can be found by alternating groups A and B.

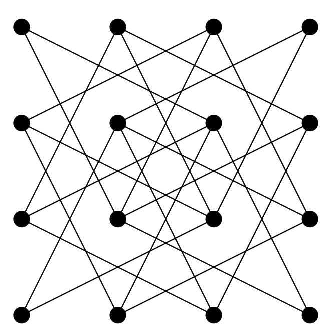
(d) A path does not return to the starting point, thus in addition to the case $m = n$ (in this case a path has the form ABAB...AB), we have $m = n - 1$ (then we can find a path of the form ABAB...ABA), and $m = n + 1$ (then we can find a path of the form BABA...BAB).

41. (a) Yes. See the picture below.

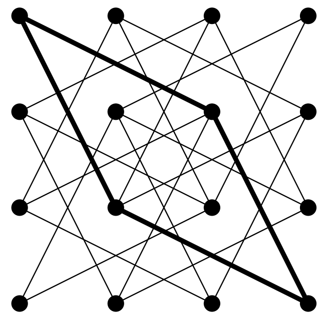


(b) Yes. In the above picture, we can connect the two ends of the path.

43. First draw the graph representing all possible moves of a knight:



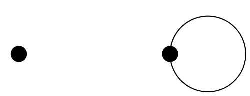
A reentrant tour is a Hamilton cycle. Thus we have to show that this graph has no Hamilton cycle. Notice that there are 4 vertices of degree 2, and in order to visit a vertex of degree 2 we have to use both its edges. Consider the upper left corner vertex and the lower right corner vertex. We must use both edges at each of them. Then we get a cycle. There is no way of adding anything to this cycle (because if we add more edges, we'll have to go through some vertex more than once). However, this cycle misses many vertices. Thus there is no Hamilton cycle.



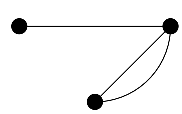
45. Let scientists be represented by vertices. Connect all vertices. We get the complete graph K_{17}. Color the edges using three colors, say, blue, red, and green, according to the topic discussed by the scientists these vertices represent. We have to prove that there are three vertices connected (pairwise) by 3 edges of the same color. Choose any vertex, say, vertex A. It is connected with 16 other vertices. Among the 16 edges connecting vertex A with other vertices at least 6 are of the same color, say, blue. Look at those 6 vertices. If at least two of them are connected by a blue edge then we have a blue triangle. If not, look at the K_6 graph for those 6 vertices. All its edges are red and green. By example 10.18, it contains at least one triangle with all 3 sides of the same color, either red or green.

47. (a) Suppose a simple graph has n vertices. The degree of each vertex is an integer that is at least 0 and at most $n-1$, so there are n possible values for degrees. If all vertices have different degrees, then for each integer from 0 to n, inclusive, there must be exactly one vertex of that degree. In particular, there is a vertex of degree 0 and a vertex of degree n. However, this is impossible because the vertex of degree 0 is not connected to any other vertices, while the vertex of degree n must be connected to every other vertex.

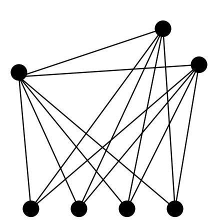
    (b) No. For example, consider the graph with two vertices which are not connected, but one of them has a loop. The degrees are 0 and 2.



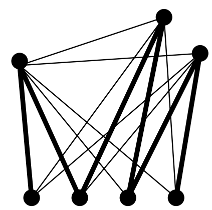
    (c) No. For example, consider the graph with three vertices, with one edge between two vertices and two edges between a different set of two vertices. The degrees are then 1, 2, and 3.



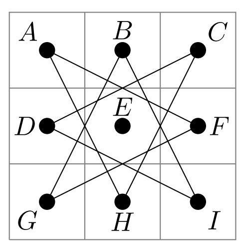
49. The graph $K_{1,2,4}$ is shown below.



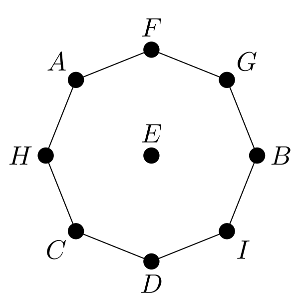
(a) The degrees of the vertices are 6, 5, 5, 3, 3, 3, 3. Since there are vertices of odd degree, there is no Euler cycle.
(b) Since there are more than two vertices of odd degree, by problem 27 there is no Euler path.
(c) No. In any cycle containing four vertices, some two vertices of the group of 4 have to be consecutive. However, there cannot be an edge between them.
(d) Yes. See the picture below.



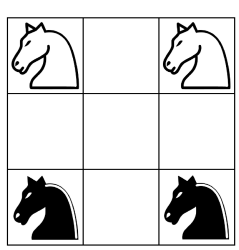
51. We will show that this is impossible. Let us draw the graph representing all legal moves of a knight on such a board:



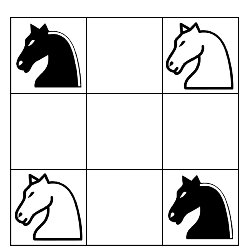
This graph can be drawn as follows.



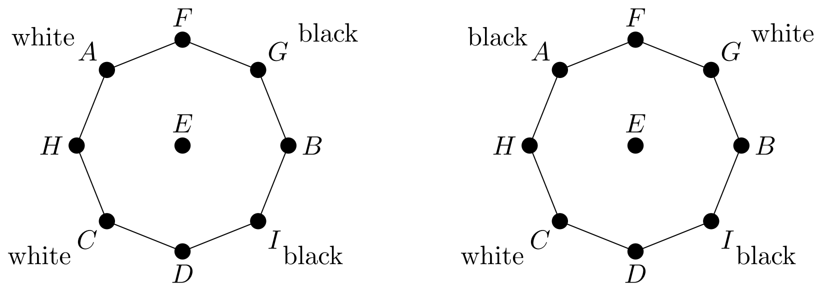
Now, mark the positions of the white and black knights given in the figures A and B in the problem:



A



B



Since in the first graph the white knights are next to each other and the black knights are next to each other, while in the second graph the white and black knights alternate, and the knights cannot "jump over" each other, there is no way to go from one position to the other.

52. Hint: draw the complement of the given graph, that is, the graph that has the same vertices, but two vertices are connected if and only if they are not connected in the given graph. The edges in this graph represent pairs of integers that have a common factor. In this case, the graph has fewer edges and might be easier to work with.

11 Combinatorics

1. (a) There are 5 ways to select a teacup. For each of those selections, there are 3 ways to select a saucer. Thus, there are $5 \cdot 3$ ways to select a teacup and a saucer. For each of these, there are 4 ways to select a spoon, thus there are $5 \cdot 3 \cdot 4 = 60$ selections total.

   (b) There are $5 \cdot 3 = 15$ ways to select a teacup and a saucer, $5 \cdot 4 = 20$ ways to select a teacup and a spoon, and $3 \cdot 4 = 12$ ways to select a saucer and a teaspoon. So there are $15 + 20 + 12 = 47$ ways to select two items with different names.

3. There are 5 odd digits, so each of the four digits in the number can be chosen in 5 different ways. Thus there are $5 \cdot 5 \cdot 5 \cdot 5 = 5^4 = 625$ such numbers.

5. There $20 \cdot 20 = 400$ ways to exchange a stamp for a stamp, and $10 \cdot 10 = 100$ ways to exchange a postcard for a postcard. So there are $400 + 100 = 500$ ways to carry out a fair exchange between the collectors.

7. (a) There are 6 possible results for each of the three rolls, thus there are $6^3 = 216$ possible outcomes total.

   (b) Of the 216 possible outcomes, $5^3 = 125$ do not have any 6s. Thus, $216 - 125 = 91$ outcomes have at least one 6.

9. (a) There are 6 ways to choose each letter, so there are $6^2 = 36$ two-letter words.

   (b) There are 6 ways to choose the first letter, and, after the first letter has been chosen, there are 5 ways to choose the second letter. So there are $6 \cdot 5 = 30$ two-letter words.

   (c) If all letters have to be different, each next letter starting with the second one has one less choices than the previous one. So there are $6 \cdot 5 \cdot 4 = 120$ three-letter words.

(d) Answer: $\dfrac{6!}{(6-n)!}$ if $n \leq 6$, and 0 if $n > 6$.

11. There are three ways to choose the color of the first strip and two ways to choose the color of each subsequent strip (as it has to be different from the one chosen at the previous step). So there are $3 \cdot 2^{10} = 3072$ ways to choose all the colors.

13. Let's order the fifth graders. There are 20 ways to choose a Kindergartener for the first fifth grader, 19 ways to choose a Kindergartener for the second fifth grader, and so on, so $20 \cdot 19 \cdot \dots \cdot 2 \cdot 1 = 20!$ ways to pair up all fifth-graders with Kindergarteners.

15. Since the first digit cannot be 0, there are $9 \cdot 10^9$ ten-digit numbers total, and $9 \cdot 9!$ of them have all digits different, so $9 \cdot 10^9 - 9 \cdot 9!$ numbers have at least two equal digits.

17. There are five ways to chose the digit that is different from the rest. Then, since the leading digit cannot be 0, there are 9 ways to choose the leftmost digit, and there are 9 ways to choose the other digit that appears in the number. Thus there are $5 \cdot 9 \cdot 9 = 405$ such numbers total.

19. Since the prime factorization of 4000 is $2^5 5^3$, a positive integer is a factor of 4000 if and only if it has prime factorization of the form $2^a 5^b$ where $0 \leq a \leq 5$ and $0 \leq b \leq 3$. So, there are 6 choices for a and 4 choices for b. Thus there are $6 \cdot 4 = 24$ positive factors.

21. Each of the numbers 2 through 9 has two choices: be present or not be present in the sequence. Thus there are $2^8 = 256$ such sequences.

23. There are 64 ways to choose a square for the white rook. The black rook must be in a different row and in a different column, so there are 49 ways to choose a square for it. Thus there are $64 \cdot 49 = 3136$ ways total.

25. Let us choose a black square first, and then a white square. There are 32 ways to choose a black square. Once it is chosen, there are 24 ways to choose a white square since out of the 32 white squares 4 are in the same row and 4 are in the same column as the black one. So there are $32 \cdot 24 = 768$ ways total. Alternatively, there are 64 ways to choose the first square, 24 ways to choose the second square (since it must be of the opposite color and in a different row and in a different column), however, we must divide by 2 since every choice of two squares was counted twice.

27. (a) Notice that the letter C appears twice, and all other letters are different. Make, for a moment, the two Cs different, for example, color one of them red: CIRCLE. Then there are 6! ways to rearrange the letters. However, since in the original word the two Cs were indistinguishable, we now counted each rearrangement twice: once with the black C before the red one, and once with the black C after the red one. For example, we counted the rearrangements ERCLCI and ERCLCI as different, while they are actually the same rearrangement if the two Cs are identical. So, we divide by 2 to account for double counting, and get $\dfrac{6!}{2} = 360$ different "words."

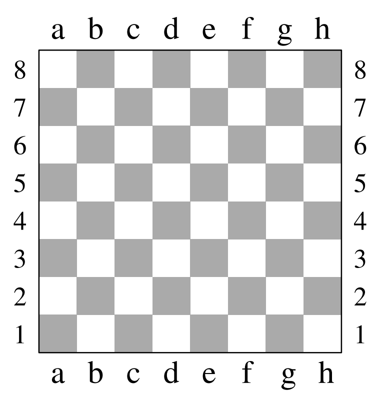
    (b) Here both C and E appear twice, so using the same idea as in part (a), we have to divide by $2 \cdot 2 = 4$ (for example, the rearrangement NEICECS is counted 4 times when the two Cs and the two Es are made different: NEICECS, NEICECS, NEICECS, and NEICECS). Thus, there are $\dfrac{7!}{4} = 1260$ different "words."

(c) In the word "SEQUENCE" the letter E appears three times. If we make these three Es distinguishable, there will be 8! arrangements. However, since there are 3! ways to order the three Es, we divide by 3! and get $\frac{8!}{3!} = 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 = 6720$ different "words."

(d) Since three letters (M, A, and T) appear twice each, we now have to divide 11! by $2^3 = 8$, so the number of different "words" is $\frac{11!}{8}$.

29. The first digit in a binary number must be a 1, and there should be no more than two 1s among the remaining six digits. There is only one number that has all 0s in those positions, six that have exactly one 1 among those six digits, and $\binom{6}{2} = 15$ numbers that have two 1s among those six digits, so there are $1 + 6 + 15 = 22$ such numbers total.

31. *Solution 1.* Observe that if the white bishop is on any border square (a1-a8, h1-h8, b1-g1, or b8-g8 – there are 28 of these), then it occupies or attacks eight squares, so the black bishop has 56 choices. If the white bishop is on any square that is not on the border but is adjacent to a border square (b2-b7, g2-g7, c2-f2, or c7-f7 – there are 20 of these), then it occupies or attacks ten squares, so the black bishop has 54 choices. If the white bishop is on a square in the next "layer" (c3-c6, f3-f6, d3, e3, d6, or e6 – there are 12 of these), then it occupies or attacks twelve squares, so the black bishop has 52 choices. Finally, if the white bishop is on one of the four squares in the center (d4, e4, d5, or e5), then it occupies or attacks 14 squares, so the black bishop has 50 choices. So there are $28 \cdot 56 + 20 \cdot 54 + 12 \cdot 52 + 4 \cdot 50 = 3472$ ways total.



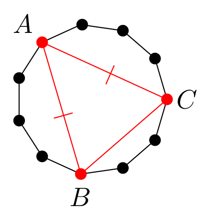
Solution 2. If the two bishops are on squares of opposite color, they do not attack each other. There are $64 \cdot 32 = 2048$ ways to place them since there are 64 ways to choose a square for the white bishop and then 32 ways to choose a square for the black bishop. If they are on squares of the same color, we need to make sure they are not on a line parallel to a diagonal. This requires considering a few cases. If the white bishop is on any border square (a1-a8, h1-h8, b1-g1, or b8-g8 – there are 28 of these total), the black bishop has 24 possibilities. If the white bishop is on any square that is not on the border but is adjacent to a border square (b2-b7, g2-g7, c2-f2, or c7-f7 – there are 20 of these), the black bishop has 22 choices. If the white bishop is on a square in the next "layer" (c3-c6, f3-f6, d3, e3, d6, or e6 – there are 12 of these), the black bishop has 20 choices. Finally, if the white bishop is on one of the four

squares in the center (d4, e4, d5, or e5), the black bishop has 18 choices. So there are $64 \cdot 32 + 28 \cdot 24 + 20 \cdot 22 + 12 \cdot 20 + 4 \cdot 18 = 3472$ ways total.

33. The arrangement can be GBGBGBGBGBGBGB or BGBGBGBGBGBGBG. There are $7! \cdot 7!$ ways to seat the children in each of these arrangements, so there are $2(7!)^2$ ways to seat them total.

35. There are $2^{10} = 1024$ possible outcomes total, and $\binom{10}{5} = \frac{10 \cdot 9 \cdot 8 \cdot 7 \cdot 6}{5!} = 9 \cdot 2 \cdot 7 \cdot 2 = 252$ of them have five heads and five tails. Of the remaining $1024 - 252 = 772$ outcomes, half have more heads than tails, and the other half have more tails than heads. Therefore there are $\frac{772}{2} = 386$ ways to get more heads than tails.

37. Note that since the number of vertices is not divisible by 3, the triangle cannot be equilateral. There are 11 ways to select the vertex at which the angle is different from the other two (this vertex is labeled A in the picture below). For each of these selections, there are 5 ways to select the other two vertices (labeled B and C) so that they are equidistant from the first vertex. Thus the number of such triangles is $11 \cdot 5 = 55$.



39. The integer is a multiple of 5 when its units digit is 0 or 5. Let us consider these cases separately. If the units digit is 0, then 5 must be in one of the other 3 positions. Once we chose the position for 5, the other two digits have $4 \cdot 3 = 12$ choices (since the digits cannot repeat). So we have $3 \cdot 12 = 36$ numbers in this case. If the units digit is 5, then the first digit has 4 choices (since it cannot be 0 or 5), and the other two digits have $4 \cdot 3 = 12$ choices (again, since the digits cannot repeat). So we have $4 \cdot 12 = 48$ numbers in this case. Thus there are $36 + 48 = 84$ such numbers total.

41. *Solution 1.* There are $\binom{8}{5} = \binom{8}{3} = \frac{8 \cdot 7 \cdot 6}{6} = 56$ ways to choose any five players from the team, but $\binom{6}{5} = \binom{6}{1} = 6$ of them do not include Corey or Tony. Therefore $56 - 6 = 50$ starting groups of five include at least one of Corey and Tony.

    Solution 2. There are $\binom{6}{4} = \binom{6}{2} = \frac{6 \cdot 5}{2} = 15$ ways to choose Corey, not Tony, and 4 more players out of the remaining 6. There are also 15 ways to choose Tony, not Corey, and 4 more players out of the remaining 6. Finally, there are $\binom{6}{3} = \frac{6 \cdot 5 \cdot 4}{6} = 20$ ways to choose Corey, Tony, and 3 more players out of the remaining 6. Therefore there are $15 + 15 + 20 = 50$ ways total.

43. There are $\binom{31}{11} = \frac{31!}{11!20!}$ ways to choose a team of 11 players, and $\binom{29}{9} = \frac{29!}{9!20!}$ of them include both Pete and John (since if both Pete an John are on the team, we still have to select 9 players out of 29 remaining students). Therefore $\frac{31!}{11!20!} - \frac{29!}{9!20!} =$

$\dfrac{29!}{9!20!}\left(\dfrac{31\cdot 30}{11\cdot 10}-1\right)=\dfrac{29!}{9!20!}\left(\dfrac{93}{11}-1\right)=\dfrac{82\cdot 29!}{11\cdot 9!20!}$ ways do not have both Pete and John on the same team.

45. Note that there is a total of seven pieces of the path, of which four go to the right and the other three go up. To construct a path, we have to choose which three of the seven moves go up, thus the number of paths is $\dbinom{7}{3}=35$.

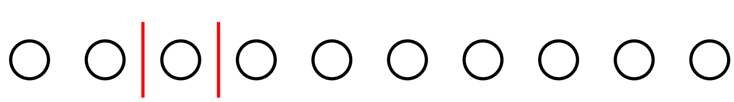
47. (a) There are 8! ways to distribute the socks and the same number of ways to distribute the shoes. Thus there are $(8!)^2$ distributions total.

    (b) There are 16! ways to order all socks and all shoes. However, this includes the orders in which some socks come after their corresponding shoes. In exactly half of the orders the sock for leg 1 comes before the shoe for leg 1, and in the other half it comes after. So we divide the total number of orders by 2 to obtain the number of orders in which the order of sock 1 and shoe 1 is correct. Then, using the same reasoning, we divide by 2 to obtain the number of orders in which the order of sock 2 and shoe 2 is correct. Continuing in this manner for all eight legs, we obtain $\dfrac{16!}{2^8}$ possible orders in which all socks come before their corresponding shoes.

49. Place all 20 balls in a row. Distributing them into six boxes is equivalent to placing five sticks somewhere among the balls so that the balls before the first stick would be placed into the first box, the balls between the first and second sticks would be placed into the second box, and so on. The balls after the fifth stick would be placed into the sixth box. This can be depicted by a diagram using circles for balls and vertical bars for sticks. For example, if three balls are put in box 1, no balls in box 2, seven balls in box 3, eight balls in box 4, two balls in box 5, and no balls in box 6, then we get the following diagram.

Now we have to count the number of such diagrams with 20 circles and 5 vertical bars. Since there are 25 objects total and we have to choose 5 positions for the bars, the number of such diagrams is $\dbinom{25}{5}$. This technique is often referred to as "sticks and stones" or "circles and bars" or "stars and bars" (some sources use stars instead of circles).

51. Place 10 counters in a row (these will represent the total sum, 10). Writing 10 as a sum $a+b+c$ is equivalent to placing two sticks between the counters so that there a counters before the first stick, b counters between the first and the second, and c counters after the second. For example, $10=2+1+7$ can be represented as follows.



Thus, the number of ways to write 10 as a sum of three natural numbers is the same as the number of ways to choose locations for the two sticks. There are nine possible locations for these, and there are $\dbinom{9}{2}=36$ ways to choose two of them.

53. Hint: use the same idea as in problem 51. There are twelve counters and two sticks, so the total number of ways is $\binom{11}{2} = 55$.

55. Let's first count the total number of ways to write 13 as a sum of three positive integers. Using the same idea as in problem 51, we have 13 counters and need two sticks which can be placed anywhere in the 12 possible locations. Thus, the number of ways is $\binom{12}{2} = 66$. However, this counts all possible ways, including the ones containing the number 11 (there are three of them, namely, $1+1+11$, $1+11+1$, and $11+1+1$) which we want to exclude. Thus the number of ways to write 13 as a sum of three positive integers not exceeding 10 is $66 - 3 = 63$.

57. Let's first put $2n$ ones. Since no two zeros may be consecutive, at most one zero can be placed in the very beginning, between any two consecutive ones, or at the very end. The number of ways to choose n positions for the zeros out of $2n+1$ possible positions is $\binom{2n+1}{n}$.

59. *Solution 1.* For the first group, there are $\binom{30}{10}$ ways to select students, and for the second group, there are $\binom{20}{10}$ ways. The remaining 10 students will form the third group. So, there are $\binom{30}{10}\binom{20}{10} = \frac{30!}{10!^3}$ ways to select students for the three groups (group 1, group 2, group 3). However, since the order in which we formed the groups does not matter, we divide by 3! to get $\frac{30!}{(10!)^3 3!} = \frac{30!}{6(10!)^3}$.

    Solution 2. Choose any student. There are $\binom{29}{9}$ ways to choose the students who will be in the same group. Then choose any of the remaining students. There are $\binom{19}{9}$ ways to choose the students who will be in the same group as this student. The remaining ten students will form the third group. Thus the total number of ways to divide the students into three groups of ten is

$$\binom{29}{9}\binom{19}{9} = \frac{29!}{9!20!} \cdot \frac{19!}{9!10!} = \frac{29!}{20 \cdot 9!9!10!} = \frac{30!}{6(10!)^3}.$$

61. *Solution 1.* There are $\binom{20}{5}\binom{15}{5}\binom{10}{5} = \frac{20!}{5!5!5!5!}$ ways total to visit four cities, each of them five times. Let's determine the number of ways to do this if we both start and end with a specific city, let's call it city A. In this case, we are free to choose the order in which we make the remaining 18 visits. Because three of those visits will be to city A (and five to each other city, say, B, C, and D), this can be done in $\binom{18}{3}\binom{15}{5}\binom{10}{5} = \frac{18!}{3!5!5!5!}$ ways. We can count the number of arrangements that start and end with B, start and end with C, or start and end with D the same way. So, the total number of ways to visit the four cities while starting and ending with different

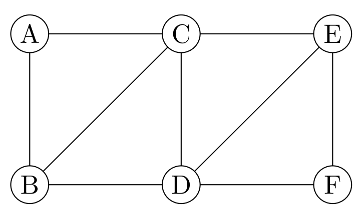
cities is

$$\begin{aligned}\frac{20!}{5!5!5!5!} - 4\frac{18!}{3!5!5!5!} &= \frac{18!\cdot 19\cdot 20}{5!5!5!5!} - 4\frac{18!}{3!5!5!5!} \\ &= \frac{18!\cdot 19}{3!5!5!5!} - \frac{18!\cdot 4}{3!5!5!5!} \\ &= \frac{18!\cdot 15}{3!5!5!5!} \\ &= \frac{18!}{2\cdot 4!5!5!}.\end{aligned}$$

Solution 2. There are 4 ways to choose the city that will be visited first, and once it has been chosen, there are 3 ways to choose the city that will be visited last. Then out of the remaining 18 visits, the first city must be visited four more times, the last city must also be visited four more times, and the other cities must be visited 5 times each, so there are $\binom{18}{4}\binom{14}{4}\binom{10}{5} = \frac{18!}{4!4!5!5!}$ ways to arrange these 18 visits. Thus the total number of ways to visit the cities is

$$\frac{4\cdot 3\cdot 18!}{4!4!5!5!} = \frac{18!}{2\cdot 4!5!5!}.$$

63. Label the vertices as follows.



(a) There are three ways to choose a color for A, then two ways to choose a color for B. After that, all colors are determined uniquely as C has to be the remaining color, then D has to be different from B and C, then E different from C and D, and, finally, F different from D and E. Thus there are only six ways to color the vertices.

(b) There are four ways to choose a color for A, then three ways to choose a color for B. After that, if we color the vertices in alphabetical order, each vertex has two choices as it only has to be different from the previous two. Thus there are $4\cdot 3\cdot 2^4 = 192$ ways to color the vertices.

65. Hint: use the Inclusion-Exclusion Principle to count the number of integers that are divisible by 5 or 7. (This is similar to Example 11.8.)
Answer: $100 - (20 + 14 - 2) = 68$.

67. Hint: use the Inclusion-Exclusion Principle to count the number of strings that do not contain 4 or do not contain 7.
Answer: $10^5 - (9^5 + 9^5 - 8^5) = 10^5 - 2\cdot 9^5 + 8^5 = 14670$.

69. Hint: use the Inclusion-Exclusion Principle to count the number of strings that miss at least one letter.
Answer: $3^7 - (2^7 + 2^7 + 2^7 - 1 - 1 - 1) = 3(3^6 - 2^7 + 1) = 3\cdot 602 = 1806$.

sphere and above the cube is the volume of one "cap" minus the volume of four thin pieces, i.e. $\left(\frac{2}{3}-\frac{8}{9\sqrt{3}}\right)\pi-\frac{4}{12}\left(\frac{8}{3\sqrt{3}}+\frac{8}{3}\pi-\frac{16}{3\sqrt{3}}\pi\right)=\frac{8}{9\sqrt{3}}\pi-\frac{8}{9\sqrt{3}}-\frac{2}{9}\pi.$

13 Calculus

1. *Solution 1.* The derivative of a cubic polynomial is a quadratic polynomial which has at most two real roots. However, the given curve has four maximum/minimum points at which the derivative is equal to zero.

   Solution 2. By the Fundamental Theorem of Algebra, a polynomial of degree n has exactly n zeros (not all of which have to be real), counting with multiplicity. The function represented by the given curve has five zeros, so cannot be a cubic polynomial.

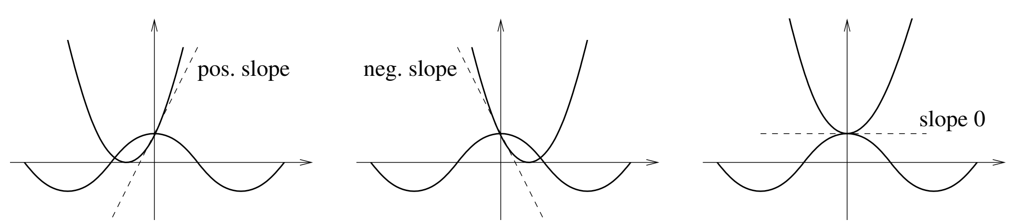
3. *Hint.* Any cross-section of the ellipsoid that passes through the origin is an ellipse.

   Proof. Without loss of generality, we can assume that a, b, and c are all positive.

   Case 1. At least two of a, b, and c are equal. Without loss of generality we can assume that $a=b$. Then the intersection of the xy-plane and the ellipsoid is a circle whose equation is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, $z=0$.

   Case 2. The numbers a, b, and c are all distinct. Without loss of generality we can assume that $a>b>c$. The intersection of the xy-plane and the ellipsoid is an ellipse whose equation is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, $z=0$. The intersection of the zy-plane and the ellipsoid is an ellipse whose equation is $\frac{z^2}{c^2}+\frac{y^2}{b^2}=1$, $x=0$. Now rotate the xy-plane until it coincides with the zy-plane so that the y-axis is always in it. The intersection of this rotating plane with the ellipsoid is always an ellipse. Moreover, one of its axes is always b, and the other one is changing continuously from $a>b$ to $c<b$. Because of continuity, at some point the changing axis is equal to b.

5. Consider the graphs of $f(x)=x^2+ax+1$ and $g(x)=\cos x$. Both graphs pass through the point $(0,1)$. The graph of $g(x)=\cos x$ has slope 0 at that point. If the slope of $f(x)=x^2+ax+1$ at $(0,1)$ is positive, then for some small negative values of x we will have $x^2+ax+1<\cos x$. If the slope of $f(x)=x^2+ax+1$ at $(0,1)$ is negative, then for some small positive values of x we will have $x^2+ax+1<\cos x$. The only case in which $x^2+ax+1\geq\cos x$ for all real x is when the slope of $f(x)=x^2+ax+1$ at $(0,1)$ is 0. The derivative of $f(x)$ is $f'(x)=2x+a$, thus the slope at $(0,1)$ is $f'(0)=a$. Therefore $a=0$ is the only such value of a.



7. At $x=0$ we have $\frac{x}{x^2+1}=\arctan x=x$.

Let's compare the derivatives of these functions for $x > 0$. The derivatives are:

$$\left(\frac{x}{x^2+1}\right)' = \frac{1-x^2}{(x^2+1)^2},$$
$$(\arctan x)' = \frac{1}{x^2+1} = \frac{1+x^2}{(x^2+1)^2},$$
$$(x)' = 1 = \frac{(1+x^2)^2}{(x^2+1)^2}.$$

We see that for all $x > 0$, $\left(\frac{x}{x^2+1}\right)' < (\arctan x)' < (x)'$, therefore the curve $y = \frac{x}{x^2+1}$ lies below the curve $y = \arctan x$ which lies below the line $y = x$.

9. We need the polynomial to pass through the given points, and have slope (which is $p'(x) = 3ax^2+2bx+c$) equal to 0 at both points. Thus we have the following equations.
The value at 0: $d = 1$.
The value at 1: $a + b + c + d = 0$.
The slope at 0: $c = 0$.
The slope at 1: $3a + 2b + c = 0$.
Since $d = 1$ and $c = 0$, the second and fourth equations become $a + b = -1$ and $3a + 2b = 0$. Then $b = -\frac{3}{2}a$, and $a - \frac{3}{2}a = -1$. This gives $a = 2$. Then $b = -3$.
So $p(x) = 2x^3 - 3x^2 + 1$.

11. Let the given line be tangent to the parabola at the point $(a, a-1)$. Then first, the parabola passes through $(a, a-1)$, thus

$$a - 1 = ca^2.$$

Second, the line and the parabola have the same slope at this point:

$$1 = 2ca.$$

From the second equation we have $c = \frac{1}{2a}$. Substituting this for c in the first equation, we have

$$a - 1 = \frac{a^2}{2a}.$$

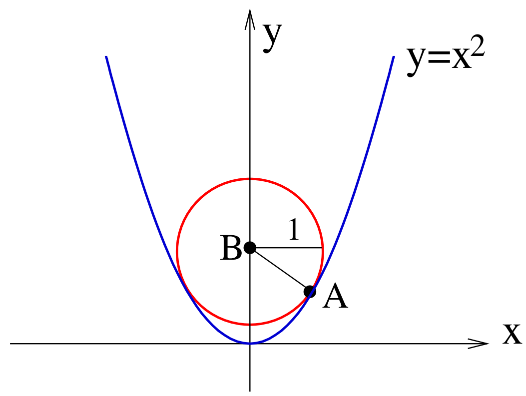
Solving for a, we have $a = 2$. Then $c = \frac{1}{4}$.

13. Draw a picture so that you see what's going on.

Let the slope of such a tangent line be m, then its equation is $y = mx$. Let (a, ma) be the touching point. Since this point lies on the parabola, $ma = a^2 + 2$. The slope of the parabola at the touching point must be m, therefore $2a = m$. Substituting this into the above equation gives $2a^2 = a^2 + 2$. Then $a = \pm\sqrt{2}$, and $m = \pm 2\sqrt{2}$. Thus the equations of the tangent lines are $y = 2\sqrt{2}x$ and $y = -2\sqrt{2}x$.

15. Let point A (with positive x-coordinate) where the circle touches the parabola be (a, a^2), and the center B of the circle be $(0, b)$. Then the distance between these points is 1, thus

$$a^2 + (b - a^2)^2 = 1.$$



The slope of the parabola at the point (a, a^2) is $2a$ (the derivative of x^2 at $x = a$), then the slope of AB is $-\dfrac{1}{2a}$ (since AB and the parabola are orthogonal at (a, a^2)). Thus we have

$$\frac{b - a^2}{0 - a} = -\frac{1}{2a}.$$

This equation gives $b - a^2 = \dfrac{1}{2}$, then from the first equation we have $a^2 + \dfrac{1}{4} = 1$, so $a^2 = \dfrac{3}{4}$. Then $b = a^2 + \dfrac{1}{2} = \dfrac{3}{4} + \dfrac{1}{2} = \dfrac{5}{4}$. So the center of the circle is at $\left(0, \dfrac{5}{4}\right)$.

17. *Hint.* Use partial fraction decomposition. Recall that since $x^2 + x = x(x+1)$, the partial fraction decomposition has the form $\dfrac{A}{x} + \dfrac{B}{x+1}$.

Solution. Let's find the partial fraction decomposition, i.e. A and B such that

$$\frac{1}{x^2 + x} = \frac{A}{x} + \frac{B}{x+1}.$$

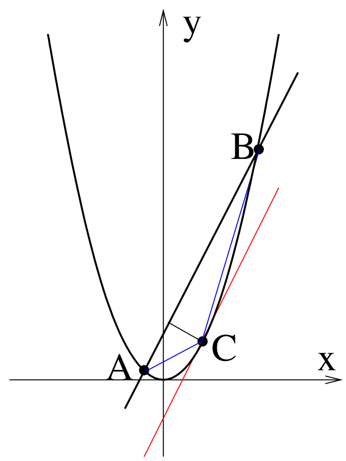
Multiplying both sides of this equation by $x^2 + x$, we get $1 = A(x+1) + Bx$. Then $1 = (A+B)x + A$. It follows that $A + B = 0$ and $A = 1$, then $B = -1$. Thus

$$\begin{aligned}
f(x) &= \frac{1}{x^2+x} = \frac{1}{x} - \frac{1}{x+1} = x^{-1} - (x+1)^{-1}\\
f'(x) &= -x^{-2} + (x+1)^{-2}\\
f''(x) &= 2x^{-3} - 2(x+1)^{-3}\\
f'''(x) &= -2 \cdot 3x^{-4} + 2 \cdot 3(x+1)^{-4}\\
&\vdots\\
f^{(n)}(x) &= (-1)^n n! x^{-n-1} - (-1)^n n! (x+1)^{-n-1}
\end{aligned}$$

Note: the above formula can be proved by Mathematical Induction.

19. Since A and B are given, the length of AB is fixed. To maximize the area of $\triangle ABC$, we have to maximize the height h_c. To do this, the point C must lie on the tangent line

parallel to the line AB. Thus the slope of the parabola at C must be equal to m. Then the x-coordinate of C is $\frac{m}{2}$ (since the slope at point (x, x^2) is $2x$). The y-coordinate of C is then $\frac{m^2}{4}$. It can be verified that this point is always between the intersection points A and B.



21. Draw a picture to see the triangle and the regions.

Let m be the slope of the line with the required property. Since it passes through the point $(1, 1)$, its equation is $y - 1 = m(x - 1)$. To find its x-intercept, we let $y = 0$ and solve for x:

$$\begin{aligned} -1 &= m(x-1), \\ x &= -\frac{1}{m} + 1 = \frac{m-1}{m}. \end{aligned}$$

To find the y-intercept, we let $x = 0$ and solve for y:

$$\begin{aligned} y - 1 &= m(-1), \\ y &= -m + 1. \end{aligned}$$

The area of the triangle bounded by this line and the axes is

$$\frac{1}{2} \cdot \frac{m-1}{m}(-m+1) = -\frac{(m-1)^2}{2m}.$$

The area of the region bounded by this line, the parabola $y = x^2$, and the y-axis is

$$\begin{aligned} \int_0^1 (m(x-1) + 1 - x^2)dx &= \int_0^1 (-x^2 + mx - m + 1)dx \\ &= \left(-\frac{x^3}{3} + \frac{mx^2}{2} + (-m+1)x \right) \Big|_0^1 \\ &= -\frac{1}{3} + \frac{m}{2} + (-m+1) \\ &= -\frac{m}{2} + \frac{2}{3}. \end{aligned}$$

The triangle is divided by the parabola into two regions of equal area if and only if the area of the above region is half of the area of the whole triangle, that is,

$$-\frac{m}{2}+\frac{2}{3}=-\frac{(m-1)^2}{4m}.$$

Solving this equation for m and disregarding the positive root gives $m=\frac{1-\sqrt{10}}{3}$. Thus an equation of the line is

$$y-1=\frac{1-\sqrt{10}}{3}(x-1),$$

or, equivalently,

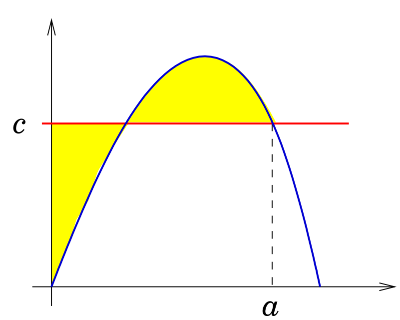
$$y=\frac{1-\sqrt{10}}{3}x+\frac{\sqrt{10}+2}{3}.$$

23. To find the intersection points of the line $y=ax$ and the parabola $y=x^2$, we solve $ax=x^2$. The roots are $x=0$ and $x=a$, thus the intersection points are $(0,0)$ and (a,a^2).

If $a>0$, the area is $\int_0^a (ax-x^2)dx = \left(a\frac{x^2}{2}-\frac{x^3}{3}\right)\Big|_0^a = \frac{a^3}{2}-\frac{a^3}{3}=\frac{a^3}{6}$. The area is equal to 1 when $\frac{a^3}{6}=1$. Then $a^3=6$, and $a=\sqrt[3]{6}$.

If $a<0$, then the area is $\int_a^0 (ax-x^2)dx=-\frac{a^3}{6}$, so $a=-\sqrt[3]{6}$.

25. Let the right intersection point have coordinates (a,c). Then $c=8a-27a^3$.



If the areas of the shaded regions are equal then the area of the region under the given cubic curve from $x=0$ to $x=a$ is equal to the area of the rectangle with width a and height $c = 8a - 27a^3$. Thus we have

$$\begin{aligned}
\int_0^a 8x - 27x^3 &= a(8a - 27a^3) \\
\left(4x^2 - \frac{27}{4}x^4\right)\Big|_0^a &= 8a^2 - 27a^4 \\
4a^2 - \frac{27}{4}a^4 &= 8a^2 - 27a^4 \\
\frac{81}{4}a^4 &= 4a^2 \\
81a^4 &= 16a^2 \\
81a^2 &= 16 \text{ (since } a \neq 0) \\
a &= \sqrt{\frac{16}{81}} = \frac{4}{9}.
\end{aligned}$$

Then $c = 8a - 27a^3 = \frac{32}{9} - \frac{27 \cdot 4^3}{9^3} = \frac{32}{9} - \frac{64}{27} = \frac{96 - 64}{27} = \frac{32}{27}$.

27. Let the curve be given by $y = f(x)$. Since it passes through $(3, 2)$, $f(3) = 2$.

At a point $P(a, f(a))$, the tangent line has slope $f'(a)$ and equation $y - f(a) = f'(a)(x - a)$. Its x-intercept is $\left(-\frac{f(a)}{f'(a)} + a, 0\right)$. The part of the tangent line that lies in the first quadrant is bisected by P if and only if $2a = -\frac{f(a)}{f'(a)} + a$. Thus $af'(a) = -f(a)$. Since this must be true for every point on the curve in the first quadrant, we have the differential equation $xf'(x) = -f(x)$. Any function of the form $f(x) = \frac{c}{x}$ is a solution of this equation. Using the condition $f(3) = 2$, we find $c = 6$. So $f(x) = \frac{6}{x}$ satisfies the required condition.

29. Observe that $\int_0^1 (\sqrt[3]{1 - x^7} - \sqrt[7]{1 - x^3})dx = \int_0^1 \sqrt[3]{1 - x^7}dx - \int_0^1 \sqrt[7]{1 - x^3}dx$.

The integral $\int_0^1 \sqrt[3]{1 - x^7}dx$ is equal to the area of the region bounded by $y = \sqrt[3]{1 - x^7}$, the x-axis, and the y-axis. The integral $\int_0^1 \sqrt[7]{1 - x^3}dx$ is equal to the area of the region bounded by $y = \sqrt[7]{1 - x^3}$, the x-axis, and the y-axis.

Equations $y = \sqrt[3]{1 - x^7}$ and $y = \sqrt[7]{1 - x^3}$ can be rewritten as $x^7 + y^3 = 1$ and $x^3 + y^7 = 1$, respectively. It is easy to see that both curves pass through $(1, 0)$ and through $(0, 1)$, and these two curves are symmetric about the line $y = x$. Thus the areas of the two regions described above are equal, therefore the difference of the integrals $\int_0^1 \sqrt[3]{1 - x^7}dx$ and $\int_0^1 \sqrt[7]{1 - x^3}dx$ is 0. Thus $\int_0^1 (\sqrt[3]{1 - x^7} - \sqrt[7]{1 - x^3})dx = 0$.

31. *Hint.* Use areas.

Solution. Since $\arcsin(x)$ (defined on $[-1, 1]$) is the inverse function of $\sin(x)$ on $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, their graphs are symmetric about the line $y = x$, thus the value of

$\int_0^1 \arcsin(x)dx$, the area of the region under the graph of $\arcsin(x)$ on $[0,1]$, is equal to the area of the region between $y = \sin(x)$ and $y = 1$ from $x = 0$ to $x = \frac{\pi}{2}$. (Draw the graphs to see this!) The latter area can be calculated by $\int_0^{\frac{\pi}{2}} (1 - \sin(x))dx = (x + \cos(x)) \Big|_0^{\frac{\pi}{2}} = \frac{\pi}{2} - 1$.

33. If the function $a_1 \cos x + a_2 \cos(2x) + \cdots + a_{30} \cos(30x)$ takes on only positive values, then its integral over any interval must be positive (because it is the area of the region under the graph of the function). However,

$$\begin{aligned}
&\int_0^{2\pi} (a_1 \cos x + a_2 \cos(2x) + \cdots + a_{30} \cos(30x))dx \\
&= \left(a_1 \sin x + \frac{a_2}{2} \sin(2x) + \cdots + \frac{a_{30}}{30} \sin(30x)\right)\Big|_0^{2\pi} = 0.
\end{aligned}$$

34. *Hint.* Try the special cases $k = 1$ and $k = 2$ first, and then generalize.

35. Since $f(0) = 0$ and $\sin(0) = 0$, $|f(x)| < |\sin(x)|$ for all x, and the slope of $y = \sin(x)$ at $(0,0)$ is 1, we have $|f'(0)| < 1$.

    Since $|f'(0)| = |a_1 + 2a_2 + \cdots + na_n|$, the required inequality follows.

36. *Hint.* Interpret the sum as a Riemann sum of a function. Then the limit as n approaches infinity is the value of an integral.

37. *Hint.* Factor out $\frac{1}{2}$ and notice that $2^{2n} = 4^n$.

    Solution. Observe that

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{1}{2^{2n+1}} &= \frac{1}{2} + \frac{1}{2^3} + \frac{1}{2^5} + \frac{1}{2^7} + \ldots \\
&= \frac{1}{2}\left(1 + \frac{1}{2^2} + \frac{1}{2^4} + \frac{1}{2^6} + \ldots\right) \\
&= \frac{1}{2}\left(1 + \frac{1}{4} + \frac{1}{4^2} + \frac{1}{4^3} + \ldots\right) \\
&= \frac{1}{2} \cdot \frac{1}{1 - \frac{1}{4}} \\
&= \frac{2}{3}.
\end{aligned}$$

39. This is a well-known theorem. It is easy to find proofs online, e.g. see wikipedia.

41. The area of the region under the graph of f is $\int_0^t f(x)dx$, so we have $\int_0^t f(x)dx = t^3$ for all $t > 0$. Differentiating both sides, we have $f(t) = 3t^2$. Thus $f(x) = 3x^2$.

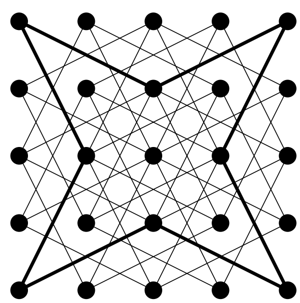
43. The 5-dimensional volume of a 5-dimensional ball is proportional to the fourth power of its radius. Suppose $V = cr^5$ where c is a constant. Then the 3-dimensional volume of the boundary of this ball is $S = V' = 5cr^4$. Therefore $\frac{V}{S} = \frac{cr^5}{5cr^4} = \frac{r}{5}$.

14 Various problems

1. First notice that years 2008, 2012, 2016, ..., 2048 are leap years, so there are 11 leap years between 2005 and 2050. The other 34 years from 2006 to 2050 (including these) are non-leap years. Since non-leap years contain 365 days and leap years contain 366 days, the number of days that pass between December 25, 2005 and December 25, 2050 is $34 \cdot 365 + 11 \cdot 366 \equiv 6 \cdot 1 + 4 \cdot 2 \equiv 14 \equiv 0 \pmod 7$, therefore December 25, 2025 is a Sunday again.

3. *Proof 1:* In a reentrant knight's tour black and white squares must alternate. However, a 5×5 chessboard has 13 squares of one color and 12 squares of the other color, so it is not possible to have a cycle in which the colors alternate.

   Proof 2: Draw a graph representing legal moves of a knight. Look at the corner vertices. They all have degree 2, thus, in order to visit the corner vertices, we must use both edges at each corner vertex. Those eight edges form a cycle. It is not possible to add more edges to this cycle, but the cycle misses many points. Therefore there is no Hamilton cycle, and thus there is no reentrant tour.



5. *Sketch.* Calculate first few terms and notice that $a_n = \dfrac{1}{n!}$. Prove this formula by Mathematical Induction. Then $\displaystyle\sum_{n=0}^{\infty} a_n = \sum_{n=0}^{\infty} \frac{1}{n!} = e$.

7. *Solution 1.* Consider non-negative and negative values of x separately.

   Case I. If $x \geq 0$, then $|x| = x$, and the inequality becomes $|6 - 2x| + x \leq 3$.

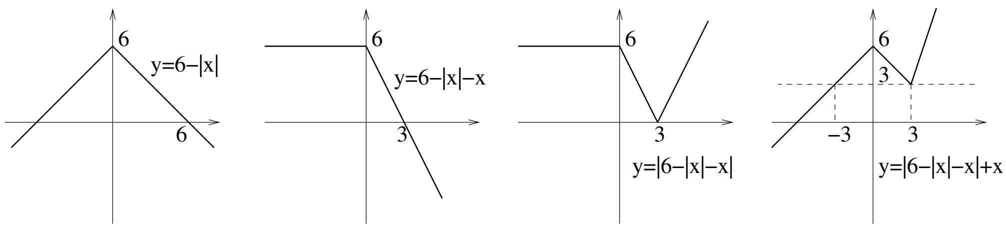
   Case Ia. If $6 - 2x \geq 0$, or $x \leq 3$, then $|6 - 2x| = 6 - 2x$, and we have $6 - 2x + x \leq 3$. This gives $x \geq 3$. Together with the condition $x \leq 3$, we get one root $x = 3$. This root satisfies the condition $x \geq 0$.

   Case Ib. If $6 - 2x < 0$, or $x > 3$, then $|6 - 2x| = -6 + 2x$, and we have $-6 + 2x + x \leq 3$. This gives $x \leq 3$ which contradicts the condition $x > 3$. Thus we have no roots in this case.

   Case II. If $x < 0$, then $|x| = -x$, and the inequality becomes $|6| + x \leq 3$, or $6 + x \leq 3$ since $|6| = 6$. Equivalently, $x \leq -3$. All the values of $x \leq -3$ satisfy the condition $x < 0$.

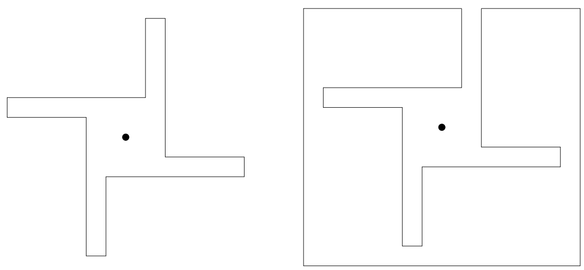
   Answer: $x = 3$ and $x \leq -3$. In the interval notation, $(-\infty, -3] \cup \{3\}$.

   Solution 2. Draw the graph of $f(x) = |6 - |x| - x| + x$.



We see that $f(x) \leq 3$ when $x = 3$ and when $x \leq 3$.

9. Below are some examples (but there are many others).



11. *Hint.* Recall problems 35 and 36 in Chapter 3. Generalize to a $(n+1)\times\left(\dfrac{(n+1)n^2}{2}+1\right)$ board colored with n colors. Then derive the desired result.

13. Suppose that the 9 bags contain different numbers of coins, and the total number of coins is 40. Let $a_1 < a_2 < \cdots < a_9$ be the numbers of coins in the 9 bags. Then $a_1 \geq 1$, $a_2 \geq 2$, $\ldots$, $a_9 \geq 9$, and $40 = a_1 + a_2 + \cdots + a_9 \geq 1 + 2 + \cdots + 9 = 45$. Since $40 < 45$, we get a contradiction, thus it is not possible for all bags to contain different numbers of coins.

15. *Solution 1.* Let's label the starting square (the lower-left corner) with S and both squares to which the camel can get to in one move with 1. Then we label all the squares to which the camel can get in two moves with 2.

	1						
			1				
S							

2		2					
		2		2			
	1						
2				2		2	
			1				
S		2				2	

Next, we label all the squares to which the camel can get in three moves with 3. Since it cannot get from any of these to the upper-right corner, it is not possible to get there in four or fewer moves. However, it is possible for the camel to get to that corner in five moves as is shown below.

	3		3		3		
2		2					
	3		3		3		3
		2		2			
	1		3		3		3
2				2		2	
	3		1		3		
S		2				2	

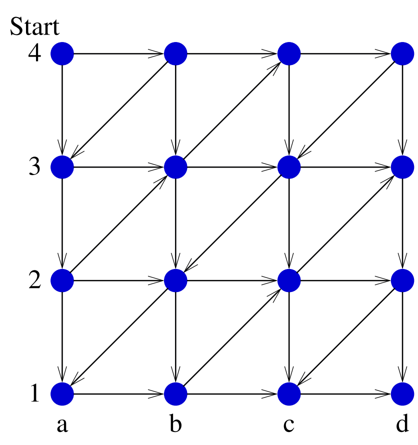
							5
				4			
							3
						2	
			1				
S							

Sketch of Solution 2. First prove that an odd number of moves would be necessary to get from the lower-left corner to the upper-right corner. Then only show the squares to which the camel can get in 1 or 2 moves. Since from none of these it can get to the upper-right corner, a total of three moves is not possible. Therefore, at least five moves are needed. The last picture above shows that five moves are sufficient.

17. *Hint.* Consider the remainders of Fibonacci numbers upon division by 8.

19. Let ABC be a three-digit number such that B is the average of A and C. Then $B = \dfrac{A+C}{2}$, or equivalently, $A+C = 2B$. Thus $A+C$ must be even, which means that A and C must have the same parity (both even or both odd). There are 9 choices for the digit A (since it can't be 0). After we choose A, there are 5 choices for C (because it must have the same parity as A, and there are 5 even digits and 5 odd digits, so whatever the parity of A is, C has 5 choices). Then, once we have chosen both A and C, the digit B is determined uniquely by $B = \dfrac{A+C}{2}$. Thus there are 45 such numbers total.

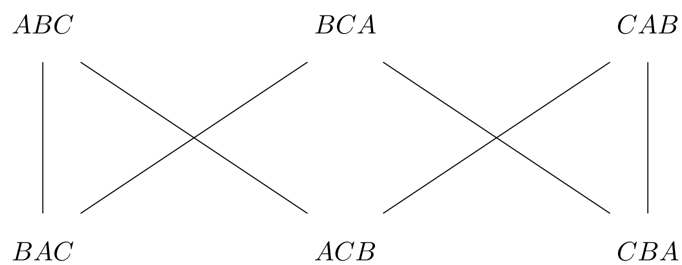
21. For convenience, let us label the columns a-d and the rows 1-4, similar to a chessboard.



We will determine all "good" and "bad" positions (a position is "good" if it is good for us to go there on our turn, and it is "bad" if we do not want to go there on our turn). We need to make sure that (1) from any bad position there is at least one move to a good one (that's where we will go on our turn), and (2) any move from a good position leads to a bad one (that's where our opponent will be forced to go on their turn). Working backwards, we have: d1 is good; c1 and d2 are bad; d3 is good; c2, c3, and d4 are bad; b1 is good; a1 and b2 are bad; c4 is good; b3 and b4 are bad; a2 is good; a3 is bad; a4 is good. Since the starting position is good, we should choose to go second. On each move, our opponent is forced to go to bad position, and then we will have a way to go to a good one.

23. *Answer:* 8.

25. No. Let's denote the three grasshoppers A, B, and C. There are six different orders (ABC, ACB, BAC, BCA, CAB, and CBA). The diagram below shows which position can change into which in one leap.



Notice that these six orders can be divided into two groups (the top row and the bottom row) such that the groups must alternate with every leap of a grasshopper. Since 11 is odd, their order after 11 leaps must be in a different group than the starting one. So they cannot return to their original order after 11 leaps.

27. *Sketch.* Calculate the number of pairs of numbers that can be formed form 21 numbers. Calculate the smallest and largest possible sum of two numbers from the given range and thus determine the number of possible values of the sum. Observe that there are more possible pairs than possible values of the sum and use Pigeonhole principle. Finally, show that if different two pairs have the same sum, then they are disjoint.

29. *Note.* There are many variations of this problem. In some versions cameras are restricted to the walls (edges of the polygon), or even to the corners (vertices of the polygon). Some versions require only the edges to be guarded. There is a whole book (Art Gallery Theorems and Algorithms, Joseph O'Rourke, Oxford University Press 1987) dedicated to this problem. There are also many references on the Internet.