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Problems and Solutions

PROBLEMS

Proposals

To be considered for publication, solutions should be received by November 1, 2022.

2146. *Proposed by Kenneth Fogarty, Bronx Community College (emeritus), Bronx, NY.*

Let a and d be integers with $d > 0$. We say that (a, d) is *good* if there is an arithmetic sequence with initial term a and difference d that can be split into two sequences of consecutive terms with the same sum. In other words, there exist integers k and n with $0 < k < n$ such that

$$\sum_{i=0}^{k-1} (a + di) = \sum_{i=k}^{n-1} (a + di) .$$

If there is no such arithmetic sequence, we say that (a, d) is *bad*.

(a) Show that if $2a > d$, then (a, d) is good.
(b) Show that if $2a = d$, then (a, d) is bad.
(c) Show that if $a = 0$ (and hence $2a < d$), then (a, d) is good.
(d) Show that if $2a < d$ and $a \neq 0$, then there is a d such that (a, d) is good and a d such that (a, d) is bad.

2147. *Proposed by Lokman Gökçe, Istanbul, Turkey.*

Evaluate

$$\prod_{n=2}^{\infty} \frac{n^4 + 4}{n^4 - 1}.$$

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We invite readers to submit original problems appealing to students and teachers of advanced undergraduate mathematics. Proposals must always be accompanied by a solution and any relevant bibliographical information that will assist the editors and referees. A problem submitted as a Quickie should have an unexpected, succinct solution. Submitted problems should not be under consideration for publication elsewhere.

Proposals and solutions should be written in a style appropriate for this Magazine.

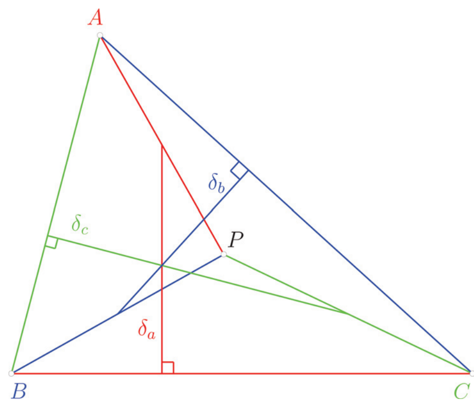
Authors of proposals and solutions should send their contributions using the Magazine's submissions system hosted at http://mathematicsmagazine.submittable.com. *More detailed instructions are available there. We encourage submissions in PDF format, ideally accompanied by LaTeX source. General inquiries to the editors should be sent to* mathmagproblems@maa.org.

2148. *Proposed by Tran Quang Hung, Hanoi, Vietnam.*

Let P be an interior point of triangle ABC. Denote by δ_a, δ_b, and δ_c the distances from midpoints of segments PA, PB, and PC to the lines BC, CA, and AB. Prove that

$$PA + PB + PC \geq \delta_a + \delta_b + \delta_c.$$

Show that equality holds if and only if triangle ABC is equilateral and P is its center.



2149. *Proposed by Ioan Băetu, Botoşani, Romania.*

Let $a_1, a_2, \ldots$ be a sequence of integers greater than 1. The series

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{\prod_{i=1}^{k} a_i} = 1 - \frac{1}{a_1} + \frac{1}{a_1 a_2} - \frac{1}{a_1 a_2 a_3} + \cdots$$

converges by the alternating series test.

(a) If the sequence $a_1, a_2, \ldots$ is unbounded, show that the sum of the series is irrational.

(b) Give an example of a bounded sequence of a_i's such that the sum of the series is irrational.

2150. *Proposed by Matthew McMullen, Otterbein University, Westerville, OH.*

Find the maximum area of a triangle whose vertices lie on the cardioid $r = 1 + \cos\theta$.

Quickies

1121. *Proposed by Salem Malikic, Bethesda, MD.*

For integers $n \geq 0$, let a_n and b_n be the unique real numbers such that

$$a_n + b_n i = (2 + i)^n.$$

Evaluate

$$\sum_{n=0}^{\infty} \frac{a_n b_n}{2^n \left(a_n^2 + b_n^2\right)}.$$

1122. *Proposed by the Columbus State University Problem Solving Group, Columbus State University, Columbus, GA.*

Show that there are infinitely many nonsimilar triangles having integer side lengths such that the angle measures are in arithmetic progression.

Solutions

Evaluate the definite integral **June 2021**

2121. *Proposed by Seán M. Stewart, Bomaderry, Australia.*

Evaluate

$$\int_0^{\frac{1}{2}} \frac{\arctan x}{x^2 - x - 1}\, dx.$$

Solution by Lixing Han, University of Michigan-Flint, Flint, MI and Xinjia Tang, Changzhou University, Changzhou, China.
Using the substitution

$$x = \frac{\frac{1}{2} - t}{1 + \frac{1}{2}t} = \frac{1 - 2t}{2 + t},$$

we obtain

$$\begin{aligned}
\int_0^{\frac{1}{2}} \frac{\arctan x}{x^2 - x - 1} dx &= \int_{\frac{1}{2}}^0 \frac{\arctan\left(\frac{\frac{1}{2} - t}{1 + \frac{1}{2}t}\right)}{\left(\frac{1-2t}{2+t}\right)^2 - \frac{1-2t}{2+t} - 1} \cdot \frac{-5}{(2+t)^2} dt \\
&= \int_0^{\frac{1}{2}} \frac{\arctan\left(\frac{1}{2}\right) - \arctan t}{t^2 - t - 1} dt \\
&= \int_0^{\frac{1}{2}} \frac{\arctan\left(\frac{1}{2}\right)}{t^2 - t - 1} dt - \int_0^{\frac{1}{2}} \frac{\arctan t}{t^2 - t - 1} dt.
\end{aligned}$$

Thus, we have

$$\begin{aligned}
\int_0^{\frac{1}{2}} \frac{\arctan x}{x^2 - x - 1} dx &= \frac{1}{2} \arctan\left(\frac{1}{2}\right) \int_0^{\frac{1}{2}} \frac{dt}{t^2 - t - 1} \\
&= \frac{1}{2} \arctan\left(\frac{1}{2}\right) \frac{1}{\sqrt{5}} \ln\left(\left|\frac{2t - \sqrt{5} - 1}{2t + \sqrt{5} - 1}\right|\right)\Bigg|_0^{1/2} \\
&= -\frac{1}{2\sqrt{5}} \arctan\left(\frac{1}{2}\right) \ln\left(\frac{\sqrt{5} + 1}{\sqrt{5} - 1}\right) \\
&= -\frac{1}{\sqrt{5}} \arctan\left(\frac{1}{2}\right) \ln\left(\frac{\sqrt{5} + 1}{2}\right).
\end{aligned}$$

Also solved by Brian Bradie, Hongwei Chen, Herevé Grandmontagne (France), Eugene A. Herman, Omran Kouba (Syria), Kee-Wai Lau (China), Albert Natian, Moobinool Omarjee (France), Didier Pichon (France), Albert Stadler (Switzerland), Fejéntaláltuka Szöged (Hungary), and the proposer. There were four incomplete or incorrect solutions.

Find the maximum gcd **June 2021**

2122. *Proposed by Ahmad Sabihi, Isfahan, Iran.*

Let

$$G(m,k) = \max\{\gcd((n+1)^m + k, n^m + k) | n \in \mathbb{N}\}.$$

Compute $G(2, k)$ and $G(3, k)$.

Solution by Michael Reid, University of Central Florida, Orlando, FL.
We show that for $k \in \mathbb{Z}$, $G(2, k) = |4k + 1|$, and

$$G(3,k) = \begin{cases} 27k^2 + 1 & \text{if } k \text{ is even,} \\ \left(27k^2 + 1\right)/4 & \text{if } k \text{ is odd.} \end{cases}$$

The polynomial identity

$$(2n+3)(n^2+k) - (2n-1)((n+1)^2+k) = 4k+1$$

shows that

$$\gcd((n+1)^2 + k, n^2 + k) \text{ divides } 4k + 1,$$

and thus is at most $|4k + 1|$. Hence, $G(2, k) \le |4k + 1|$.

Suppose $k > 0$, and let $n = 2k \in \mathbb{N}$. We have

$$n^2 + k = k(4k+1) \text{ and } (n+1)^2 + k = (k+1)(4k+1),$$

both of which are divisible by $4k + 1$. Thus

$$\gcd((n+1)^2 + k, n^2 + k) = 4k + 1 = |4k + 1|,$$

so $G(2, k) = |4k + 1|$ in this case.

For $k = 0$, we have $\gcd((n + 1)^2, n^2) = 1$ for all $n \in \mathbb{N}$, so $G(2, 0) = 1 = |4k + 1|$ in this case.

Suppose $k < 0$, and consider $n = -(2k + 1) \in \mathbb{N}$. Then

$$n^2 + k = (k+1)(4k+1) \text{ and } (n+1)^2 + k = k(4k+1)$$

are each divisible by $4k + 1$. Thus

$$\gcd((n+1)^2 + k, n^2 + k) = |4k + 1|,$$

so $G(2, k) = |4k + 1|$ in this case as well.

Now we consider $G(3, k)$. The polynomial identity

$$(6n^2 - 9nk - 3n + 9k + 1)((n+1)^3 + k)$$
$$- (6n^2 - 9nk + 15n - 18k + 10)(n^3 + k) = 27k^2 + 1$$

shows that

$$\gcd((n+1)^3 + k, n^3 + k) \text{ divides } 27k^2 + 1. \tag{1}$$

For all n, $(n+1)^3+k$ and n^3+k have opposite parity, so their greatest common divisor is odd. If k is odd, then $27k^2+1 = 4\big((27k^2+1)/4\big)$ is a product of two integers. Since the greatest common divisor is odd, and divides this product,

$$\gcd((n+1)^3+k, n^3+k) \text{ divides } \frac{27k^2+1}{4}. \tag{2}$$

For $k=0$, we have $\gcd((n+1)^3, n^3) = 1$ for all n, so $G(3,0) = 27k^2+1 = 1$.

For nonzero k, take $n = 3k(9k-1)/2$, which is a positive integer. We calculate

$$n^3+k = (27k^2+1)\left(\frac{(729k^3-243k^2+8)k}{8}\right)$$

and

$$(n+1)^3+k = (27k^2+1)\left(\frac{729k^4-243k^3+162k^2-28k+8}{8}\right).$$

If k is even, each factor above is an integer, which shows that

$$27k^2+1 \text{ divides } \gcd((n+1)^3+k, n^3+k).$$

With (1), we have

$$\gcd((n+1)^3+k, n^3+k) = 27k^2+1,$$

so $G(3,k) = 27k^2+1$ when k is even.

If k is odd, rewrite the above factorizations as

$$n^3+k = \left(\frac{27k^2+1}{4}\right)\left(\frac{(729k^3-243k^2+8)k}{2}\right)$$

and

$$(n+1)^3+k = \left(\frac{27k^2+1}{4}\right)\left(\frac{729k^4-243k^3+162k^2-28k+8}{2}\right),$$

again, all factors being integers. Therefore

$$\frac{27k^2+1}{4} \text{ divides } \gcd((n+1)^3+k, n^3+k).$$

With (2), we conclude that

$$\gcd((n+1)^3+k, n^3+k) = \frac{27k^2+1}{4},$$

so $G(3,k) = (27k^2+1)/4$ when k is odd.

Also solved by Hongwei Chen, Eagle Problem Solvers (Georgia Southern University), Dmitry Fleischman, George Washington University Math Problem Solving Group, Eugene A. Herman, Walther Janous (Austria), Didier Pinchon (France), Albert Stadler (Switzerland), Enrique Treviño, and the proposer. There were two incomplete or incorrect solutions.

Find the expected winnings **June 2021**

2123. *Proposed by Albert Natian, Los Angeles Valley College, Valley Glen, CA.*

An urn contains n balls. Each ball is labeled with exactly one number from the set

$$\{a_1, a_2, \ldots, a_n\}, a_1 > a_2 > \cdots > a_n$$

(so no two balls have the same number). Balls are randomly selected from the urn and discarded. At each turn, if the number on the ball drawn was the largest number remaining in the urn, you win the dollar amount of that ball. Otherwise, you win nothing. Find the expected value of your total winnings after n draws.

Solution by Enrique Treviño, Lake Forest College, Lake Forest, IL.
Let X be the random variable described. Then $X = a_{i_1} + a_{i_2} + \cdots + a_{i_j}$ with $1 = i_1 < i_2 < \cdots < i_j \leq n$. Therefore, the expected value will be

$$\mathbb{E}[X] = \sum_{k=1}^{n} c_k a_k,$$

where c_k is the probability that the summand a_k appears in X. For a_k to appear, the ball labeled a_k must be drawn after those labeled $a_1, a_2, \ldots, a_{k-1}$, but this only happens if the permutation of $\{a_1, \ldots, a_k\}$ ends in a_k. This occurs with probability $1/k$. Therefore

$$\mathbb{E}[X] = a_1 + \frac{1}{2}a_2 + \frac{1}{3}a_3 + \cdots + \frac{1}{n}a_n.$$

Also solved by Robert A. Agnew, Alan E. Berger, Brian Bradie, Elton Bojaxhiu (Germany) & Enkel Hysnelaj (Australia), Paul Budney, Michael P. Cohen, Eagle Problem Solvers (Georgia Southern University), John Fitch, Dmitry Fleischman, Fresno State Journal Problem Solving Group, GWstat Problem Solving Group, George Washington University Problems Group, Victoria Gudkova (student) (Russia), Stephen Herschkorn, Shing Hin Jimmy Pa (Canada), David Huckaby, Walther Janous (Austria), Omran Kouba (Syria), Ken Levasseur, Reiner Martin (Germany), Kelly D. McLenithan, José Nieto (Venezuela), Didier Pinchon (France), Michael Reid, Edward Schmeichel, Albert Stadler (Switzerland), Fejéntaláltuka Szöged, and the proposer. There were two incomplete or incorrect solutions.

A sum over the partitions of *n* **June 2021**

2124. *Proposed by Mircea Merca, University of Craiova, Craiova, Romania.*

For a positive integer n, prove that

$$\sum_{\substack{\lambda_1+\lambda_2+\cdots+\lambda_k=n \\ \lambda_1\geqslant\lambda_2\geqslant\cdots\geqslant\lambda_k>0}} (-1)^{n-\lambda_1} \frac{\binom{\lambda_1}{\lambda_2}\binom{\lambda_2}{\lambda_3}\cdots\binom{\lambda_k}{0}}{1^{\lambda_1}2^{\lambda_2}\cdots k^{\lambda_k}} = \frac{1}{n!},$$

where the sum runs over all the partitions of n.

Solution by José Heber Nieto, Universidad del Zulia, Maracaibo, Venezuela.
Put $s_1 = \lambda_1 - \lambda_2$, $s_2 = \lambda_2 - \lambda_3, \ldots, s_{k-1} = \lambda_{k-1} - \lambda_k$, $s_k = \lambda_k$. Clearly, we have $s_i \geq 0$, $s_1 + s_2 + \cdots + s_k = \lambda_1$, and $s_1 + 2s_2 + 3s_3 + \cdots + ks_k = n$. Moreover, for fixed λ_1, if we vary k and $\lambda_2, \lambda_3, \ldots, \lambda_k$ satisfying the conditions $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k > 0$

and $\lambda_1 + \lambda_2 + \cdots + \lambda_k = n$, we obtain all the sequences of s_i's satisfying $s_i \geq 0$, $s_1 + s_2 + \cdots + s_k = \lambda_1$ and $s_1 + 2s_2 + 3s_3 + \cdots + ks_k = n$.

Now

$$\frac{\binom{\lambda_1}{\lambda_2}\binom{\lambda_2}{\lambda_3}\cdots\binom{\lambda_k}{0}}{1^{\lambda_1}2^{\lambda_2}\cdots k^{\lambda_k}} = \frac{\lambda_1!}{s_1!s_2!\cdots s_k!(1!)^{s_1}(2!)^{s_2}\cdots(k!)^{s_k}}.$$

We note that

$$\frac{n!}{s_1!s_2!\cdots s_k!(1!)^{s_1}(2!)^{s_2}\cdots(k!)^{s_k}}$$

is the number of partitions of the set $\{1, 2, \ldots, n\}$ into s_i blocks of size i, for $i = 1, 2, \ldots, k$. For fixed λ_1, if we sum these expressions for all values of the s_i's and k such that $s_i \geq 0$, $s_1 + s_2 + \cdots + s_k = \lambda_1$ and $s_1 + 2s_2 + 3s_3 + \cdots + ks_k = n$, we obtain the number of partitions of the set $\{1, 2, \ldots, n\}$ into λ_1 blocks, that is the Stirling number of second kind $\left\{ {n \atop \lambda_1} \right\}$. Therefore

$$\sum_{\substack{\lambda_1+\lambda_2+\cdots+\lambda_k=n \\ \lambda_1\geq\lambda_2\geq\cdots\geq\lambda_k>0}} (-1)^{n-\lambda_1} \frac{\binom{\lambda_1}{\lambda_2}\binom{\lambda_2}{\lambda_3}\cdots\binom{\lambda_k}{0}}{1^{\lambda_1}2^{\lambda_2}\cdots k^{\lambda_k}} = \frac{1}{n!}\sum_{\lambda_1=1}^{n}(-1)^{n-\lambda_1}\lambda_1!\left\{ {n \atop \lambda_1} \right\}. \tag{1}$$

It is well known that

$$\sum_{\lambda_1=1}^{n}\left\{ {n \atop \lambda_1} \right\}x(x-1)(x-2)\cdots(x-\lambda_1+1) = x^n.$$

Substituting $-x$ for x we obtain

$$\sum_{\lambda_1=1}^{n}(-1)^{n-\lambda_1}\left\{ {n \atop \lambda_1} \right\}x(x+1)(x+2)\cdots(x+\lambda_1-1) = x^n.$$

For $x = 1$, we have

$$\sum_{\lambda_1=1}^{n}(-1)^{n-\lambda_1}\left\{ {n \atop \lambda_1} \right\}\lambda_1! = 1,$$

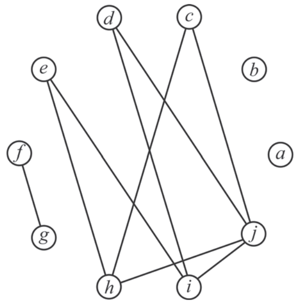
hence the right-hand side of (1) is $1/n!$ and we are done.

Also solved by Albert Stadler (Switzerland) and the proposer.

A graph involving a partition of 100 into ten parts **June 2021**

2125. *Proposed by Freddy Barrera, Colombia Aprendiendo, and Bernardo Recamán, Universidad Sergio Arboleda, Bogotá, Colombia.*

Given a collection of positive integers, not necessarily distinct, a graph is formed as follows. The vertices are these integers and two vertices are connected if and only if they have a common divisor greater than 1. Find an assignment of ten positive integers totaling 100 that results in the graph shown below.



Solution by Eagle Problem Solvers, Georgia Southern University, Statesboro, GA and Savannah, GA.

With the labeling above,

$$(a, b, c, d, e, f, g, h, i, j) = (1, 1, 7, 9, 10, 11, 11, 14, 15, 21)$$

is a solution. Note that each of e, h, i, and j must have at least two prime divisors, since each is adjacent to two vertices that are not adjacent to each other. The simplest option is $e = pq, h = qr, i = rs$, and $j = ps$ with p, q, r, and s prime. Assuming $\{p, q, r, s\} = \{2, 3, 5, 7\}$, the vertices e, h, i, and j must consist of two of the three pairs (6, 35), (10, 21), and (14, 15). The possibility with the smallest sum is $\{e, h, i, j\} = \{10, 14, 15, 21\}$. If we take $a = b = 1$ and $f = g = 11$, this forces $c + d = 16$. Assuming that c and d are powers of distinct primes from $\{2, 3, 5, 7\}$, we must have $(c, d) = (7, 9)$ or $(c, d) = (9, 7)$. The former forces $(e, h, i, j) = (10, 14, 15, 21)$, which yields the solution above. The latter gives a solution with $(e, h, i, j) = (10, 15, 14, 21)$.

A more detailed analysis shows that, in fact, these are the only solutions.

Also solved by Brian D. Beasley, Elton Bojaxhiu (Germany) & Enkel Hysnelaj (Australia), Dmitry Fleischman, George Washington University Problems Group, Kelly D. McLenithan & Stephen C. Mortenson, Lane Nielsen, José Heber Nieto (Venezuela), Didier Pinchon (France), Randy K. Schwartz, Albert Stadler (Switzerland), and the proposers.

Answers

Solutions to the Quickies from page 243.

A1121. More generally, we will evaluate

$$\sum_{n=0}^{\infty} \frac{a_n b_n}{c^n \left(a_n^2 + b_n^2\right)},$$

where a_n, b_n, c, α, and β are real, $|c| > 1$, and

$$a_n + b_n i = (\alpha + \beta i)^n.$$

Note that

$$a_n^2 + b_n^2 = (a_n + b_n i)(a_n - b_n i) = (\alpha + \beta i)^n (\alpha - \beta i)^n = (\alpha^2 + \beta^2)^n,$$

and

$$a_n b_n = \frac{1}{2}\text{Im}((a_n + b_n i)^2) = \frac{1}{2}\text{Im}\left((\alpha + \beta i)^{2n}\right).$$

Since

$$\left|\frac{(\alpha+\beta i)^2}{c(\alpha^2+\beta^2)}\right| = \frac{1}{|c|} < 1,$$

we have

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{a_n b_n}{c^n\left(a_n^2+b_n^2\right)} &= \frac{1}{2}\mathrm{Im}\left(\sum_{n=0}^{\infty}\left(\frac{(\alpha+\beta i)^2}{c(\alpha^2+\beta^2)}\right)^n\right) \\
&= \frac{1}{2}\mathrm{Im}\left(\frac{1}{1-\frac{(\alpha+\beta i)^2}{c(\alpha^2+\beta^2)}}\right) \quad \text{(geometric series)} \\
&= \frac{1}{2}\mathrm{Im}\left(\frac{c(\alpha^2+\beta^2)}{c(\alpha^2+\beta^2)-(\alpha^2-\beta^2)-2\alpha\beta i}\right) \\
&= \frac{1}{2}\left(\frac{c(\alpha^2+\beta^2)2\alpha\beta}{\left(c(\alpha^2+\beta^2)-(\alpha^2-\beta^2)\right)^2+4\alpha^2\beta^2}\right) \\
&= \frac{c(\alpha^2+\beta^2)\alpha\beta}{c^2(\alpha^2+\beta^2)^2-2c(\alpha^2+\beta^2)(\alpha^2-\beta^2)+(\alpha^2+\beta^2)^2} \\
&= \frac{c\alpha\beta}{c^2(\alpha^2+\beta^2)-2c(\alpha^2-\beta^2)+(\alpha^2+\beta^2)} \\
&= \frac{c\alpha\beta}{(c-1)^2\alpha^2+(c+1)^2\beta^2}
\end{aligned}$$

For the original problem, $(\alpha, \beta, c) = (2, 1, 2)$ and the series sums to 4/13.

A1122. Since the angle sum of a triangle is 180°, the middle angle must have measure 60°. By the law of cosines, we have

$$a^2 + b^2 - ab = c^2.$$

Dividing by c^2, we have

$$x^2 - xy + y^2 = 1$$

with $x, y \in \mathbb{Q}$. The point $(1, 0)$ is clearly on this curve. The equation of a line with slope m passing through the point is $y = mx + 1$. We know that this line meets the conic section above in $(1, 0)$ and find that the other point of intersection is

$$(x, y) = \left(\frac{1-2m}{1-m+m^2}, \frac{1-m^2}{1-m+m^2}\right).$$

Taking relatively prime positive integers p and q with $2p < q$, letting $m = p/q$, and clearing denominators gives

$$a = q(q-2p), b = q^2 - p^2, c = p^2 - pq + q^2$$

as solutions.