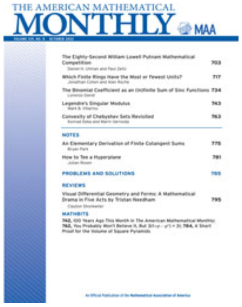

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Problems and Solutions

Daniel H. Ullman, Daniel J. Velleman, Stan Wagon, Douglas B. West & with the collaboration of Paul Bracken, Ezra A. Brown, Hongwei Chen, Zachary Franco, George Gilbert, László Lipták, Rick Luttmann, Hosam Mahmoud, Frank B. Miles, Lenhard Ng, Rajesh Pereira, Kenneth Stolarsky, Richard Stong, Lawrence Washington, and Li Zhou.

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PROBLEMS AND SOLUTIONS

Edited by **Daniel H. Ullman, Daniel J. Velleman, Stan Wagon, and Douglas B. West**

with the collaboration of Paul Bracken, Ezra A. Brown, Hongwei Chen, Zachary Franco, George Gilbert, László Lipták, Rick Luttmann, Hosam Mahmoud, Frank B. Miles, Lenhard Ng, Rajesh Pereira, Kenneth Stolarsky, Richard Stong, Lawrence Washington, and Li Zhou.

Proposed problems, solutions, and classics should be submitted online at americanmathematicalmonthly.submittable.com/submit.
Proposed problems must not be under consideration concurrently at any other journal, nor should they be posted to the internet before the deadline date for solutions. Proposed solutions to the problems below must be submitted by February 28, 2023. Proposed classics should include the problem statement, solution, and references. More detailed instructions are available online. An asterisk (*) after the number of a problem or a part of a problem indicates that no solution is currently available.

PROBLEMS

12342. *Proposed by George Stoica, Saint John, NB, Canada.* Let $v_1, \ldots, v_n$ be unit vectors in $\mathbb{R}^d$. Prove that if u maximizes $\prod_{i=1}^n |v_i \cdot u|$ over all unit vectors $u \in \mathbb{R}^d$, then for all i, $|v_i \cdot u| \geq \sin(\pi/(2n))$.

12343. *Proposed by Tran Quang Hung, Hanoi, Vietnam.* Let $ABCD$ be a convex quadrilateral with $AB = a$, $BC = b$, $CD = c$, $DA = d$, $AC = e$, and $BD = f$. Prove that $ABCD$ is a cyclic quadrilateral (i.e., the four vertices lie on a circle) if and only if

$$\frac{f^2 - e^2}{ac + bd} = \frac{\left(a^2 - c^2\right)\left(b^2 - d^2\right)}{(ab + cd)(ad + bc)}.$$

12344. *Proposed by Brian Bradie, Christopher Newport University, Newport News, VA.* Evaluate

$$\int_{-1}^{1} \frac{\arccos x}{x^2 + x + 1}\, dx.$$

12345. *Proposed by Donald E. Knuth, Stanford University, Stanford, CA.* Fix a probability p with $0 < p < 1$, and consider a random directed graph on vertices $\{1, \ldots, n\}$ where each arc ij for $1 \leq i < j \leq n$ is independently present with probability p. (For $j \leq i$ there is no arc ij.) A *source* vertex is one with no predecessors; a *sink* vertex is one with no successors. Let S_n be the number of sources and let T_n be the number of sinks. Find an exact formula for the correlation coefficient between S_n and T_n, and determine its asymptotic behavior as n approaches infinity.

12346. *Proposed by Nguyen Quang Minh, Hwa Chong Institution, Bukit Timah, Singapore.* Prove that there are infinitely many integers A such that, for every nonzero integer x and distinct positive odd integers m and n, the integer $x^m + Ax^n$ is not a perfect square.

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12347. *Proposed by Marian Tetiva, Gheorghe Roşca Codreanu National College, Bîrlad, Romania.* Let a and b be real numbers with $0 < a < 1 < b$. Find all continuous functions $f : \mathbb{R} \to \mathbb{R}$ such that $f(0) = 0$ and $f(f(x)) - (a+b)f(x) + abx = 0$ for all $x \in \mathbb{R}$.

12348. *Proposed by Erik Vigren, Uppsala, Sweden, and Hans Rullgård, Kungälv, Sweden.* We have n people in a circle, numbered from 1 to n clockwise. They are removed one at a time as follows, until just one remains. At each step, remove the nth person among those remaining, where the count starts at the lowest-numbered person remaining and procedes clockwise. Let $W(n)$ be the number of the last person remaining. For example, with $n = 5$, we remove in order the people numbered 5, 1, 3, and 2, and so $W(5) = 4$. (This is a variation of the classic Josephus problem.)
(a) What is $W(10^{12})$?
(b) For $n \geq 5$, show that $W(n) = n - 4$ if and only if $n/2$ is a Sophie Germain prime (i.e., $n/2$ and $n + 1$ are prime).
(c) Find the smallest even number that does not equal $W(n)$ for any n.

SOLUTIONS

A Double Sum for Apéry's Constant

12222 [2020, 945]. *Proposed by Roberto Tauraso, Università di Roma "Tor Vergata," Rome, Italy.* Prove

$$\sum_{k=1}^{\infty} \frac{(-1)^k}{k^2} \sum_{n=k}^{\infty} \frac{1}{n2^n} = -\frac{13\,\zeta(3)}{24},$$

where $\zeta(3)$ is Apéry's constant $\sum_{k=1}^{\infty} 1/k^3$.

Composite solution by Brian Bradie and Hongwei Chen, Christopher Newport University, Newport News, VA. In general, $\zeta(m) = \sum_{k=1}^{\infty} 1/k^m$. In working with expressions involving reciprocal powers, it is useful to have the gamma function integral and its logarithmic version

$$\frac{n!}{k^{n+1}} = \int_0^{\infty} \mathrm{e}^{-kt} t^n \, dt = (-1)^n \int_0^1 x^{k-1} (\ln x)^n \, dx, \tag{1}$$

where the latter integral is obtained from the former by setting $t = -\ln x$.

Let S be the desired double sum. After interchanging the order of summation, we invoke (1) with $n = 1$ to obtain

$$\begin{aligned} S &= \sum_{n=1}^{\infty} \frac{1}{n2^n} \sum_{k=1}^{n} \frac{(-1)^k}{k^2} = \sum_{n=1}^{\infty} \frac{1}{n2^n} \sum_{k=1}^{n} (-1)^{k+1} \int_0^1 x^{k-1} \ln x \, dx \\ &= \sum_{n=1}^{\infty} \frac{1}{n2^n} \int_0^1 \left(\sum_{k=1}^{n} (-1)^{k+1} x^{k-1} \right) \ln x \, dx = \sum_{n=1}^{\infty} \frac{1}{n2^n} \int_0^1 \frac{1 - (-x)^n}{1 + x} \ln x \, dx. \end{aligned}$$

Because the integrand in this last expression is nonpositive for every x in [0, 1] and every n, one can interchange the summation and integration to obtain

$$S = \int_0^1 \frac{-\ln(1 - 1/2) + \ln(1 + x/2)}{1 + x} \ln x \, dx = \int_0^1 \frac{\ln(2 + x) \ln x}{x + 1} \, dx.$$

We break the integral for S into three integrals by applying the polarization identity $ab = \frac{1}{2}(a^2 + b^2 - (a - b)^2)$ to the numerator of the integrand, using $a = \ln x$ and $b = \ln(2 + x)$. Letting

$$J(f(x)) = \int_0^1 \frac{(\ln f(x))^2}{1+x} dx,$$

we obtain

$$2S = J(x) + J(x+2) - J(x/(2+x)). \tag{2}$$

Expanding $1/(1+x)$ into a geometric series and applying (1) with $n = 2$ yields

$$J(x) = \sum_{k=0}^{\infty} (-1)^k \int_0^1 x^k (\ln x)^2 dx = 2 \sum_{k=0}^{\infty} \frac{(-1)^k}{(k+1)^3}.$$

To evaluate $J(x+2)$, we substitute $t = 1/(x+2)$. Since $1/(x+1) = t/(1-t)$, we obtain $dx/(1+x) = dt/(t(t-1))$. Using partial fraction expansion and then another geometric series,

$$J(x+2) = \int_{1/3}^{1/2} \left(\frac{1}{t} + \frac{1}{1-t} \right) (\ln t)^2 dt = \frac{(\ln 3)^3 - (\ln 2)^3}{3} + \sum_{k=0}^{\infty} \int_{1/3}^{1/2} t^k (\ln t)^2 dt.$$

Integrating by parts twice yields

$$\int_{1/3}^{1/2} t^k (\ln t)^2 dt = t^{k+1} \left(\frac{(\ln t)^2}{(k+1)} - \frac{2 \ln t}{(k+1)^2} + \frac{2}{(k+1)^3} \right) \Bigg|_{1/3}^{1/2}. \tag{3}$$

Summing over k, we now have $J(x+2)$ expressed in terms of polylogarithms, where the *polylogarithm* $\mathrm{Li}_s(z)$ is defined by $\mathrm{Li}_s(z) = \sum_{k=1}^{\infty} z^k/k^s$. Note that $\mathrm{Li}_1(z) = -\ln(1-z)$. The function Li_2 is called the *dilogarithm*, and Li_3 is called the *trilogarithm*. In particular, $J(x) = -2\,\mathrm{Li}_3(-1)$ and

$$\begin{aligned}
J(x+2) &= \frac{(\ln 3)^3 - (\ln 2)^3}{3} + \sum_{k=1}^{\infty} (1/2)^k \left(\frac{(\ln(1/2))^2}{k} - \frac{2\ln(1/2)}{k^2} + \frac{2}{k^3} \right) \\
&\quad - \sum_{k=1}^{\infty} (1/3)^k \left(\frac{(\ln(1/3))^2}{k} - \frac{2\ln(1/3)}{k^2} + \frac{2}{k^3} \right) \\
&= \frac{(\ln 3)^3 - (\ln 2)^3}{3} + (\ln 2)^2 \,\mathrm{Li}_1(1/2) + 2 \ln 2 \,\mathrm{Li}_2(1/2) + 2\,\mathrm{Li}_3(1/2) \\
&\quad - (\ln 3)^2 \,\mathrm{Li}_1(1/3) - 2 \ln 3 \,\mathrm{Li}_2(1/3) - 2\,\mathrm{Li}_3(1/3) \\
&= \frac{(\ln 2)^3 - (\ln 3)^3}{3/2} + 2 \ln 2 \,\mathrm{Li}_2(1/2) + 2\,\mathrm{Li}_3(1/2) \\
&\quad - 2 \ln 3 \,\mathrm{Li}_2(1/3) - 2\,\mathrm{Li}_3(1/3) + (\ln 3)^2 \ln 2,
\end{aligned}$$

where the last step uses $\mathrm{Li}_1(z) = -\ln(1-z)$.

To evaluate $J(x/(2+x))$, we substitute $t = x/(2+x)$, which yields $x = 2t/(1-t)$, $1 + x = (1+t)/(1-t)$, $dx = 2\,dt/(1-t)^2$, and $dx/(1+x) = 2\,dt/(1-t^2)$. Integrating as we did in (3) after expanding a geometric sum yields

$$\begin{aligned}
J(x/(2+x)) &= 2 \int_0^{1/3} \frac{1}{1-t^2} (\ln t)^2 dt \\
&= 2 \sum_{k=0}^{\infty} \left(\frac{1}{3} \right)^{2k+1} \left(\frac{(\ln 3)^2}{2k+1} + \frac{2 \ln 3}{(2k+1)^2} + \frac{2}{(2k+1)^3} \right).
\end{aligned}$$

The odd terms in a Taylor series $T(x)$ at 0 sum to $\big(T(x) - T(-x)\big)/2$, so

$$J(x/(2+x)) = (\ln 3)^2 \ln 2 + 2\ln 3\big(\mathrm{Li}_2(1/3) - \mathrm{Li}_2(-1/3)\big) + 2\big(\mathrm{Li}_3(1/3) - \mathrm{Li}_3(-1/3)\big).$$

Substituting these expressions for $J(x)$, $J(x+2)$, and $J(x/(2+x))$ into (2) and combining like terms yields

$$S = \frac{(\ln 2)^3 - (\ln 3)^3}{3} - \ln 3\big(2\,\mathrm{Li}_2(1/3) - \mathrm{Li}_2(-1/3)\big) - \big(2\,\mathrm{Li}_3(1/3) - \mathrm{Li}_3(-1/3)\big) + \ln 2\,\mathrm{Li}_2(1/2) - \mathrm{Li}_3(-1) + \mathrm{Li}_3(1/2).$$

The following are known evaluations of dilogarithms and trilogarithms at -1, $1/2$, and $\pm 1/3$:

$$\mathrm{Li}_3(-1) = -\frac{3}{4}\zeta(3)$$

$$\mathrm{Li}_2(1/2) = \frac{\pi^2}{12} - \frac{(\ln 2)^2}{2}$$

$$\mathrm{Li}_3(1/2) = \frac{-\pi^2 \ln 2}{12} + \frac{(\ln 2)^3}{6} + \frac{7}{8}\zeta(3)$$

$$2\,\mathrm{Li}_2(1/3) - \mathrm{Li}_2(-1/3) = \frac{\pi^2}{6} - \frac{(\ln 3)^2}{2}$$

$$2\,\mathrm{Li}_3(1/3) - \mathrm{Li}_3(-1/3) = -\frac{\pi^2 \ln 3}{6} + \frac{(\ln 3)^3}{6} + \frac{13}{6}\zeta(3).$$

After substituting these evaluations into the last expression for S, remarkably all terms not involving $\zeta(3)$ cancel, leaving

$$S = \frac{3}{4}\zeta(3) + \frac{7}{8}\zeta(3) - \frac{13}{6}\zeta(3) = -\frac{13}{24}\zeta(3).$$

Editorial comment. The generation of many terms not involving $\zeta(3)$, which then cancel, suggests that there should be a shorter solution not involving polylogarithms, but no solver was able to contribute such a solution. Some solvers replaced the original 2 by $1/x$, differentiated, summed, integrated, and thereby reduced the desired sum to

$$\int_0^{1/2} \frac{\mathrm{Li}_2(-x)}{x(1-x)}\,dx.$$

However, this also does not seem to lead to a shorter solution.

A standard reference for polylogarithms and their evaluations is L. Lewin (1981), *Polylogarithms and Associated Functions*, Amsterdam: North-Holland. For further examples of series summing to $\zeta(3)$ and historical background, see A. van der Poorten (1979), A proof that Euler missed, *Math. Intelligencer* 1: 195–203, and W. Dunham (2021), Euler and the cubic Basel problem, this MONTHLY 128: 291–301.

Also solved by N. Bhandari (Nepal), R. Boukharfane (Morocco), G. Fera (Italy), M. L. Glasser, P. W. Lindstrom, M. Omarjee (France), A. Stadler (Switzerland), S. M. Stewart (Australia), R. Stong, and the proposer.

Collinear Intersection Points

12224 [2021, 88]. *Proposed by Cherng-tiao Perng, Norfolk State University, Norfolk, VA.* Let ABC be a triangle, with D and E on AB and AC, respectively. For a point F in the plane, let DF intersect BC at G and let EF intersect BC at H. Furthermore, let AF

intersect BC at I, let DH intersect EG at J, and let BE intersect CD at K. Prove that I, J, and K are collinear.

Solution I by Nigel Hodges, Cheltenham, UK. We use $XY.ZW$ to denote the intersection of lines XY and ZW. Let $L = AG.DI$, $M = AH.EI$, and $N = BC.DE$. Lines EH, AI, and GD concur at F. Therefore, by the theorem of Desargues, the points $EA.HI$, $EG.HD$, and $AG.ID$ are collinear. Since E lies on AC, and since H and I lie on BC, we have $EA.HI = C$, and by definition, $EG.HD = J$ and $AG.ID = L$. Thus, we have

$$C, J, \text{ and } L \text{ are collinear.} \tag{1}$$

Similarly, applying the theorem of Desargues to EH, IA, and GD we conclude that

$$M, J, \text{ and } B \text{ are collinear,} \tag{2}$$

and using EH, IA, and DG we get

$$M, N, \text{ and } L \text{ are collinear.} \tag{3}$$

Statement (3) implies that lines LM, DE, and CB concur at N, so by one more application of the theorem of Desargues we conclude that $LD.ME$, $LC.MB$, and $DC.EB$ are collinear. But L lies on DI and M lies on EI, so $LD.ME = I$, (1) and (2) imply that $LC.MB = J$, and $DC.EB = K$ by definition. Thus I, J, and K are collinear.

Solution II by O. P. Lossers, Eindhoven University of Technology, Eindhoven, Netherlands. We use homogeneous coordinates with $A = (1 : 0 : 0)$, $B = (0 : 1 : 0)$, $C = (0 : 0 : 1)$, and $K = (1 : 1 : 1)$. This gives $D = (1 : 1 : 0)$ and $E = (1 : 0 : 1)$. Let $F = (a : b : c)$. Since G lies on BC and DF, we have $G = (0 : b - a : c)$. Similarly,

$$H = (0 : b : c - a),\ I = (0 : b : c), \quad \text{and} \quad J = (a : a - b : a - c),$$

so it follows that I, J, and K are collinear.

Also solved by M. Bataille (France), J. Cade, C. Curtis, I. Dimitrić, G. Fera (Italy), R. Frank (Germany), O. Geupel (Germany), J.-P. Grivaux (France), E. A. Herman, W. Janous (Austria), J. H. Lindsey II, C. R. Pranesachar (India), C. Schacht, V. Schindler (Germany), A. Stadler (Switzerland), R. Stong, R. Tauraso (Italy), T. Wiandt, L. Zhou, Davis Problem Solving Group, The Zurich Logic-Coffee (Switzerland), and the proposer.

Gamma at Reciprocals of Positive Integers

12225 [2021, 88]. *Proposed by Pakawut Jiradilok, Massachusetts Institute of Technology, Cambridge, MA, and Wijit Yangjit, University of Michigan, Ann Arbor, MI.* Let Γ denote the gamma function, defined by $\Gamma(x) = \int_0^\infty e^{-t} t^{x-1}\, dt$ for $x > 0$.
(a) Prove that $\lceil \Gamma\,(1/n) \rceil = n$ for every positive integer n, where $\lceil y \rceil$ denotes the smallest integer greater than or equal to y.
(b) Find the smallest constant c such that $\Gamma\,(1/n) \geq n - c$ for every positive integer n.

Solution by Missouri State University Problem Solving Group, Springfield, MO. We use three facts about the gamma function: (i) $\Gamma(x + 1) = x\Gamma(x)$, (ii) $\Gamma'(1) = -\gamma$, where γ is the Euler–Mascheroni constant, and (iii) the gamma function is convex on $(0, \infty)$.
(a) The equation of the line tangent to $y = \Gamma(x + 1)$ at the point $(0, 1)$ is

$$y = 1 + \Gamma'(1)x = 1 - \gamma x.$$

Since the gamma function is convex, this implies that for $x > -1$,

$$\Gamma(x + 1) \geq 1 - \gamma x.$$

Applying this with $x = 1/n$ yields

$$\Gamma(1/n) = n\Gamma(1/n+1) \geq n(1-\gamma/n) = n - \gamma.$$

Also, since $\Gamma(1) = \Gamma(2) = 1$, by convexity $\Gamma(x+1) \leq 1$ for $0 \leq x \leq 1$. Hence

$$\Gamma(1/n) = n\Gamma(1/n+1) \leq n.$$

Since $n - \gamma \leq \Gamma(1/n) \leq n$ and $\gamma < 1$, we conclude that $\lceil \Gamma(1/n) \rceil = n$.
(b) The solution to part **(a)** shows that γ satisfies the required condition. Now let c be any constant such that $\Gamma(1/n) \geq n - c$ for all n. We have

$$c \geq n - \Gamma(1/n) = n - n\Gamma(1/n+1) = -\frac{\Gamma(1+1/n)-1}{1/n}.$$

Letting n approach ∞ yields

$$c \geq \lim_{n\to\infty} -\frac{\Gamma(1+1/n)-1}{1/n} = -\Gamma'(1) = \gamma.$$

Thus, γ is the smallest such c.

Also solved by R. A. Agnew, K. F. Andersen (Canada), P. Bracken, H. Chen, G. Fera (Italy), D. Fleischman, J.-P. Grivaux (France), J. A. Grzesik (Canada), L. Han, N. Hodges (UK), O. Kouba (Syria), O. P. Lossers (Netherlands), I. Manzur (UK) & M. Graczyk (France), R. Molinari, M. Omarjee (France), A. Stadler (Switzerland), R. Stong, R. Tauraso (Italy), J. Vinuesa (Spain), M. Vowe (Switzerland), T. Wiandt, J. Yan (China), L. Zhou, and the proposer.

A Recursive Sequence That Is Convergent or Eventually Periodic

12226 [2021, 88]. *Proposed by Jovan Vukmirovic, Belgrade, Serbia.* Let x_1, x_2, and x_3 be real numbers, and define x_n for $n \geq 4$ recursively by $x_n = \max\{x_{n-3}, x_{n-1}\} - x_{n-2}$. Show that the sequence $x_1, x_2, \ldots$ is either convergent or eventually periodic, and find all triples (x_1, x_2, x_3) for which it is convergent.

Solution by O. P. Lossers, Eindhoven University of Technology, Eindhoven, Netherlands. Let λ_1 be the unique real root of $\lambda^3 + \lambda - 1$, so

$$\lambda_1 = \left(\frac{9+\sqrt{93}}{18}\right)^{1/3} + \left(\frac{9-\sqrt{93}}{18}\right)^{1/3} = 0.682327803828\ldots.$$

The sequence converges if and only if $(x_1, x_2, x_3) = (x_1, x_1\lambda_1, x_1\lambda_1^2)$ with $x_1 > 0$ or $(x_1, x_2, x_3) = (x_1, 0, 0)$ with $x_1 \leq 0$. Otherwise, it is eventually periodic with period 4.

Given such a sequence $x_1, x_2, \ldots$, let $i \in \mathbb{N}$ be of *type A* if $x_i \leq x_{i+2}$ and *type B* if $x_i > x_{i+2}$. We claim that if i is of type A and $i+1$ is of type B, then $x_j = x_{j+4}$ for $j \geq i+3$. To see this, let $(a, b, c) = (x_i, x_{i+1}, x_{i+2})$. We have $a \leq c$ and $x_{i+3} = c - b$, so $b > c - b$ and $x_{i+4} = b - c$.

If $c \leq b - c$, which with $b > c - b$ implies $b > c$, then the sequence continues

$$x_{i+5} = 2b - 2c, \quad x_{i+6} = b - c, \quad x_{i+7} = c - b, \quad x_{i+8} = b - c, \quad x_{i+9} = 2b - 2c.$$

With $(x_{i+7}, x_{i+8}, x_{i+9}) = (x_{i+3}, x_{i+4}, x_{i+5})$, the claim follows. If $c > b - c$, then the sequence continues

$$x_{i+5} = b, \quad x_{i+6} = c, \quad x_{i+7} = c - b,$$

yielding $(x_{i+5}, x_{i+6}, x_{i+7}) = (x_{i+1}, x_{i+2}, x_{i+3})$. In both cases, the sequence has period 4 beginning no later than x_{i+3} and hence does not converge.

If i of type A is never followed by $i+1$ of type B, then either all i are of type B or there exists some integer $k \geq 1$ such that i is of type A if and only if $i \geq k$. If all i are of type B, then $x_n = -x_{n-2} + x_{n-3}$ for $n \geq 4$. The characteristic polynomial $\lambda^3 + \lambda - 1$ is strictly increasing with unique real root λ_1 between 0 and 1. The complex conjugate roots λ_2 and λ_3 have magnitude greater than 1.

It follows that $x_n = c_1\lambda_1^n + \Re(c_2\lambda_2^n)$ for some real c_1 and complex c_2, where $\Re(z)$ denotes the real part of z. Since $|\lambda_2| > 1$ and $x_{n-3} > x_{n-1}$ for $n \geq 4$, we conclude $c_2 = 0$ and therefore $x_n = c_1\lambda_1^n$, where $c_1 > 0$ to satisfy $x_n > x_{n+2}$. This is a strictly decreasing convergent solution, not eventually periodic.

Finally, if i is of type A if and only if $i \geq k$, then $x_{k+1}, x_{k+2}, \ldots$ satisfies $x_n = x_{n-1} - x_{n-2}$ for $n \geq k+3$. Therefore,

$$\begin{aligned}
x_{k+3} &= x_{k+2} - x_{k+1} \geq x_{k+1},\\
x_{k+4} &= -x_{k+1} \geq x_{k+2},\\
x_{k+5} &= -x_{k+2},\\
x_{k+6} &= x_{k+1} - x_{k+2} \geq -x_{k+1},\\
x_{k+7} &= x_{k+1} \geq -x_{k+2}.
\end{aligned}$$

From $-x_{k+1} \geq x_{k+2}$ and $x_{k+1} \geq -x_{k+2}$ we conclude $x_i = 0$ for $i \geq k+1$. Since k is of Type A, also $x_k \leq 0$. If $k > 1$, then $x_{k+2} = x_{k-1} - x_k > x_{k+1} - x_k = -x_k \geq 0$, which contradicts $x_{k+2} = 0$. Therefore, k must equal 1, and the convergent sequences that are also eventually periodic are given by $(x_1, x_2, x_3) = (x_1, 0, 0)$ with $x_1 \leq 0$.

Also solved by C. Curtis & J. Boswell, G. Fera (Italy), N. Hodges (UK), Y. J. Ionin, P. Lalonde (Canada), M. Reid, R. Stong, L. Zhou, and the proposer.

Sum of Reciprocals of Consecutive Integers

12227 [2021, 88]. *Proposed by Gregory Galperin, Eastern Illinois University, Charleston, IL, and Yury J. Ionin, Central Michigan University, Mount Pleasant, MI.* Prove that for any integer n with $n \geq 3$ there exist infinitely many pairs (A, B) such that A is a set of n consecutive positive integers, B is a set of fewer than n positive integers, A and B are disjoint, and $\sum_{k\in A} 1/k = \sum_{k \in B} 1/k$.

Solution by Rory Molinari, Beverly Hills, MI. For positive integers t and n, let

$$A_n(t) = \begin{cases} \{t-m, t-m+1, \ldots, t+m\} & \text{if } n = 2m+1,\\ \{t-m, t-m+1, \ldots, t+m-1\} & \text{if } n = 2m, \end{cases}$$

where m is an integer. For a set X of nonzero numbers, let $S(X) = \sum_{i\in X} 1/i$.

First consider the odd case: $n = 2m+1 \geq 3$. Fix a positive integer p. Using $1/(np) = 1/p - (n-1)/(np)$, we compute

$$\begin{aligned}
S(A_n(np)) &= \frac{1}{p} - \frac{n-1}{np} + \sum_{i=1}^{m}\left(\frac{1}{np-i} + \frac{1}{np+i}\right)\\
&= \frac{1}{p} + \sum_{i=1}^{m}\left(\frac{1}{np-i} + \frac{1}{np+i} - \frac{2}{np}\right)\\
&= \frac{1}{p} + \sum_{i=1}^{m}\frac{2i^2}{np(n^2p^2-i^2)} = \frac{1}{p} + \sum_{i=1}^{m}\frac{1}{b(np,i)},
\end{aligned}$$

where $b(x, y) = x(x^2 - y^2)/(2y^2)$. If we choose p to be a multiple of $2m!$, then $b(np, i)$ is an integer for $1 \leq i \leq m$. By taking $A = A_n(np)$ and $B = \{p, b(np, 1), \ldots, b(np, m)\}$,