Practice Test 1 Solutions

1. Prove that for any natural n,

$$1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \cdots + n(n+1) = \frac{n(n+1)(n+2)}{3}.$$

Proof by Mathematical Induction.
Basis step: if $n = 1$, then $1 \cdot 2 = \frac{1 \cdot 2 \cdot 3}{3}$ is true.
Inductive step: assume the statement holds for $n = k$ for some natural number k. We will show that it holds for $n = k + 1$. In other words, we assume that

$$1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \cdots + k(k+1) = \frac{k(k+1)(k+2)}{3}$$

holds, and we will prove that

$$1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \cdots + (k+1)(k+2) = \frac{(k+1)(k+2)(k+3)}{3}$$

holds.
We have: $1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \cdots + (k+1)(k+2) =$

$(1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \cdots + k(k+1)) + (k+1)(k+2) =$

$\dfrac{k(k+1)(k+2)}{3} + (k+1)(k+2) = \dfrac{k(k+1)(k+2) + 3(k+1)(k+2)}{3} =$
$\dfrac{(k+1)(k+2)(k+3)}{3}.$

2. Let $\{F_0,\ F_1,\ F_2,\ \ldots\}$ be the Fibonacci sequence defined by $F_0 = 0$, $F_1 = 1$, and $F_{n+1} = F_n + F_{n-1}$, $n \geq 1$. Prove that $F_{n-1}^2 + F_n^2 = F_{2n-1}$
Proof by Strong Mathematical Induction.
Basis step. If $n = 1$, then the identity says that $F_0^2 + F_1^2 = F_1^2$, or $0^2 + 1^2 = 1^2$ which is true.
Inductive step. Assume that it holds for all $1 \leq n \leq k$. Namely, we will use that it holds for $n = k$ and $n = k - 1$, i.e.

$$F_{k-1}^2 + F_k^2 = F_{2k-1}$$

and $F^2_{(k-1)-1} + F^2_{(k-1)} = F_{2(k-1)-1}$, or equivalently,

$$F^2_{k-2} + F^2_{k-1} = F_{2k-3}.$$

We want to prove that it holds for $n = k+1$, i.e. $F^2_{(k+1)-1} + F^2_{k+1} = F_{2(k+1)-1}$, or, equivalently,

$$F^2_k + F^2_{k+1} = F_{2k+1}.$$

It may be easier here to work from the right hand side: $F_{2k+1} = F_{2k} + F_{2k-1} = F_{2k-1} + F_{2k-2} + F_{2k-1} = 2F_{2k-1} + F_{2k-2} = 2F_{2k-1} + F_{2k-1} - F_{2k-3} = 3F_{2k-1} - F_{2k-3} = 3(F^2_{k-1} + F^2_k) - (F^2_{k-2} + F^2_{k-1}) = 3F^2_{k-1} + 3F^2_k - F^2_{k-2} - F^2_{k-1} = 2F^2_{k-1} + 3F^2_k - F^2_{k-2} = 2F^2_{k-1} + 3F^2_k - (F_k - F_{k-1})^2 = 2F^2_{k-1} + 3F^2_k - F^2_k + 2F_kF_{k-1} - F^2_{k-1} = F^2_{k-1} + 2F^2_k + 2F_kF_{k-1} = F_{k-1}(F_{k-1} + F_k) + F_k(F_k + F_{k-1}) + F^2_k = F_{k-1}F_{k+1} + F_kF_{k+1} + F^2_k = (F_{k-1} + F_k)F_{k+1} + F^2_k = F^2_k + F^2_{k+1}$.

3. Kevin is paid every other week on Friday. Show that every year, in some month he is paid three times.
   Solution. Since there are 52 whole weeks in a year, Kevin is paid at least 26 times a year. Since there are 12 months, by generalized Dirichlet's box principle, at least one month will contain 3 pay days.

4. Let f be a one-to-one function from $X = \{1, 2, 3, 4, 5\}$ onto X. Let $f^k = \underbrace{f \circ f \circ \cdots \circ f}_{k\times}$ denote the k-fold composition of f with itself. Show that for some positive integer m, $f^m(x) = x$ for all $x \in X$.
   Proof. Note that for each k, the function f^k is a permutation of the set X and there are $5! = 120$ different permutations of the set X. Consider f, f^2, ..., f^{121}. By Dirichlet's box principle, at least two of these are equal, i.e. $f^a = f^b$ for some $a < b$, $a, b \in$. Then f^{b-a} is the identity function, i.e. $f^{b-a}(x) = x$ for all $x \in X$.

5. Six integer numbers, a_1, a_2, a_3, a_4, a_5, and a_6 are chosen randomly. Prove that $\prod_{1\leq i<j\leq 6} (a_i - a_j)$ is divisible by 10.
   Proof. There are two possible remainders (0 and 1) upon division by 2. Since there are more numbers than possible remainders, by Dirichlet's box principle at least two numbers have the same remainder. Then their difference is divisible by 2. Therefore the product of all differences is divisible by 2.
   There are five possible remainders (0, 1, 2, 3, 4) upon division by 5. Since there are more numbers than possible remainders, by Dirichlet's box principle at least two numbers have the same remainder. Then their difference is divisible by 5. Therefore the product of all differences is divisible by 5.
   Since the product is divisible by both 2 and 5 and these are distinct primes, the product is divisible by 10.

6. Show that $2^{457} + 3^{457}$ is divisible by 5.
Poof. By Theorem 4.22, since the exponent is odd, the given expression factors as $(2 + 3)$ times an integer. Therefore, it is divisible by $2 + 3$, i.e. divsible by 5.

7. Solve for x: $|x + 1| + 5 - x^2 \geq 0$
Case I. $x + 1 \geq 0$. Then $|x + 1| = x + 1$, so the inequality becomes

$$\begin{aligned} x + 1 + 5 - x^2 &\geq 0 \\ x + 6 - x^2 &\geq 0 \\ x^2 - x - 6 &\leq 0 \\ (x - 3)(x + 2) &\leq 0 \\ -2 \leq x &\leq 3. \end{aligned}$$

The condition $x + 1 \geq 0$ implies $x \geq -1$, so the solution set in this case is $[-1, 3]$.
Case II. $x + 1 < 0$. Then $|x + 1| = -(x + 1)$, so the inequality becomes

$$\begin{aligned} -(x + 1) + 5 - x^2 &\geq 0 \\ -x + 4 - x^2 &\geq 0 \\ x^2 + x - 4 &\leq 0 \\ \left(x - \frac{-1 + \sqrt{17}}{2}\right)\left(x - \frac{-1 - \sqrt{17}}{2}\right) &\leq 0 \\ \frac{-1 - \sqrt{17}}{2} \leq x &\leq \frac{-1 + \sqrt{17}}{2}. \end{aligned}$$

The condition $x + 1 < 0$ implies $x < -1$, so the solution set in this case is $\left[\frac{-1 - \sqrt{17}}{2}, -1\right)$.

Answer: $\left[\frac{-1 - \sqrt{17}}{2}, 3\right]$.