MATH 149 **Spring 2015**

Study Guide and Sample Problems for Test 2

Note: the actual test will consist of five problems, some of which will be computational, some will ask for a brief explanation, and some may require a rigorous detailed proof. Some of the problems will be very similar to homework problems and/or those discussed in class, but some will be different. So make sure that you understand well all the concepts discussed, know precise definitions and basic properties, rather than memorize how to solve specific problems.

1. Groups, abelian groups, rings, commutative rings, fields.
   - Give the definition of a group; abelian group. Give an example of a non-ableian group.
   - Give an example of a ring that is not a field.
   - Which of the following sets with the usual addition and multiplication are fields: $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}^+$ (the set of positive real numbers), $\mathbb{R}$? For those the are not fields, say which axioms do not hold.
   - Give an example of a set of familiar objects where the following property does not hold: if $ab = 0$, then $a = 0$ or $b = 0$.
2. Rational and irrational numbers.
   - Prove that the quotient of two nonzero rational numbers is rational.
   - Prove that the following statement is false: "the quotient of two nonzero irrational numbers is irrational".
   - Prove that $\sqrt{5}$ is irrational.
   - Convert the following decimals into fractions: 12.345, 12.3(45).
   - What is the 149th digit after the decimal point in the decimal representation of $\frac{1}{7}$?
   - Explain why a decimal represents a rational number if and only if it is either terminating or periodic.
3. Real numbers.
   - Give the definition of a convergent sequence.
   - Give the definition of a Cauchy sequence. What important property do all Cauchy sequences have?
   - Explain why the infinite decimal 0.(9) represents the same number as 1.
   - Show that $\sqrt{2}$ cannot be written in the form $a + b\sqrt{3}$ where $a, b \in \mathbb{Q}$ (i.e. $\sqrt{2} \notin \mathbb{Q}(\sqrt{3})$).

4. Complex numbers.
    - Find the additive and multiplicative inverses of the element $4+3i$ in $\mathbb{C}$.
    - Compute: i^{149}, $(\sqrt{3}+i)^9$.
    - Find at least one complex solution to $x^4=-1$.
    - Give the definition of a convergent sequence.
    - Find the sum of the infinite series $1+\frac{1}{2}i+\frac{1}{4}+\frac{1}{8}i+\frac{1}{16}+\frac{1}{32}i+\ldots$.
5. Functions
    - What is a function?
    - List at least 4 ways to define/represent a function.
    - Give the definitions of: domain, range, intercepts, asymptotes, roots, graph.
    - Is the set of all non-zero functions from $\mathbb{R}$ to $\mathbb{R}$ with the usual function addition and multiplication a field? If not, which axioms do not hold?
    - Determine which of the following statements are true, which are false, and which do not make sense or are incorrectly worded. For the last group, explain what exactly is incorrect, and, if possible, propose a correction.
        - The point $(2,3)$ satisfies the function $f(x)=x^2-1$.
        - If the function $f:\mathbb{R}\to\mathbb{R}$ is defined by $f(x)=\frac{1}{x}$, then the value $f(0)$ does not exist.
        - The point $(3,4)$ lies on the graph of the function $x^2+y^2=25$.
        - The graph of the equation $y=x+5$ has two intercepts.
        - One way to find all horizontal intercepts of the function $f(x)$ is to solve the equation $f(x)=0$.
        - The graph of a function $f:\mathbb{R}\to\mathbb{R}$ can have any number of vertical asymptotes, from 0 to infinitely many.
        - The graph of a function $f:\mathbb{R}\to\mathbb{R}$ can have any number of horizontal asymptotes, from 0 to infinitely many.
        - A function $f:\mathbb{R}\to\mathbb{R}$ can have any number of roots, from 0 to infinitely many.
        - The graph of a function $f:\mathbb{R}\to\mathbb{R}$ can have any number of x-intercepts, from 0 to infinitely many.
        - The graph of a function $f:\mathbb{R}\to\mathbb{R}$ can have any number of y-intercepts, from 0 to infinitely many.