Influence of Triangulation and Refinement on the Chromatic Number of Archimedean Solid Graphs

Introduction

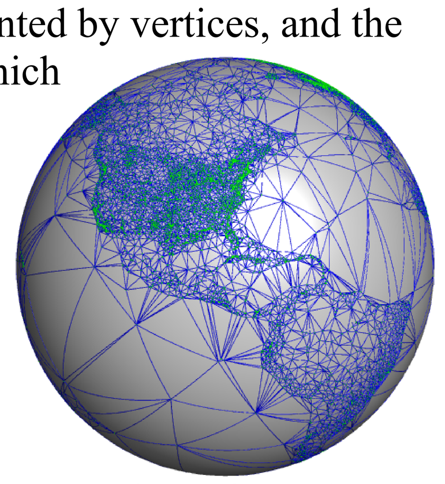
Finding the chromatic number of a large graph is a long and strenuous process, especially when it has hundreds or even thousands of vertices, however, it is often useful. Because this takes a long time, finding a shortcut for this process would help. In this project, I am trying to find a pattern in the chromatic numbers of graphs of polyhedra as they are triangulated and refined.

Definitions

A graph is an object that consists of vertices (nodes) and edges (connections between vertices). Each graph has a chromatic number. It is the smallest number of colors needed to color all of the vertices so that no two vertices connected by an edge are the same color.

Application

Graphs are often used to model various communication networks. A communication network is a series of terminals and links, to enable communication between terminals. These networks can be represented using graphs, to see them from a mathematical point of view. Usually, the terminals are represented by vertices, and the links are represented by edges, through which the data flows. The chromatic number represents the smallest number of frequencies that need to be used because if two terminals that are close to each other use the same frequency, they will interfere with each other. Triangulated graphs of Archimedean solids provide a good starting point for triangulating the spherical planet. Since they do not yet have enough vertices, they are often refined to obtain more terminals, so finding a pattern in chromatic numbers throughout triangulation and refinements would be useful.



Hypothesis

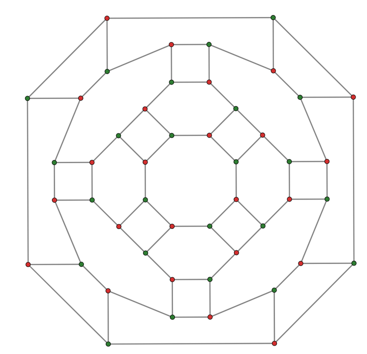
My hypothesis is that the chromatic number will never decrease during triangulation or refinement, because during both procedures vertices are only being added. More precisely, I think that from the original graph to the graph after the triangulation, the chromatic number will in most cases increase by one, but from triangulation to refinement, it will almost always stay the same. I also predict that if the original graph has a chromatic number of 3, then it will sometimes stay as 3 throughout both triangulation and refinement.

Some Known Facts

The chromatic number of any planar graph (a graph that can be drawn in a plane without any edge intersections) is at most four. Since the graph of any polyhedron is planar, none of the chromatic numbers that I find will exceed four.

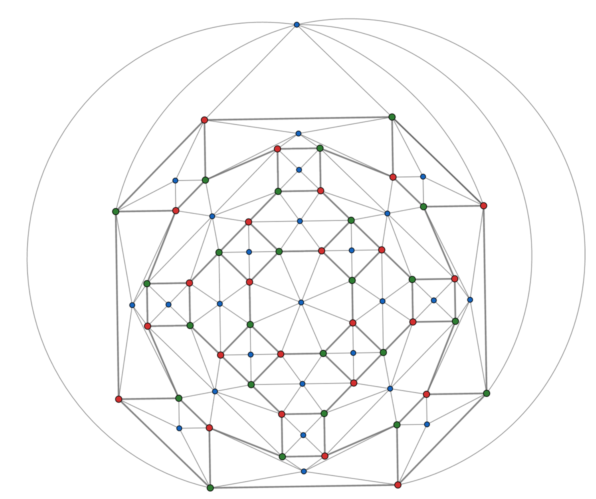
Procedure

1. I drew the graphs of all 13 Archimedean solids.
2. I found their chromatic numbers, and wrote short proofs.



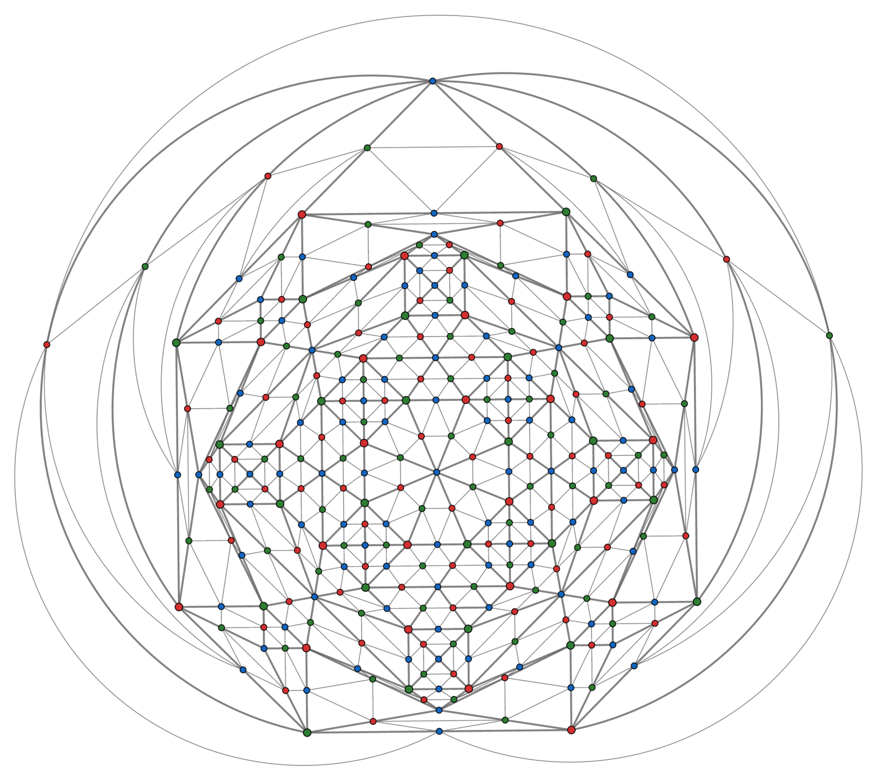
Original Graph of Truncated Cuboctahedron

3. I triangulated the graphs.
4. I found their chromatic numbers (with proofs).



Graph of Triangulated Truncated Cuboctahedron

5. I refined the triangulated graphs.
6. I found their chromatic numbers (with proofs).



Refined Graph of Triangulated Truncated Cuboctahedron

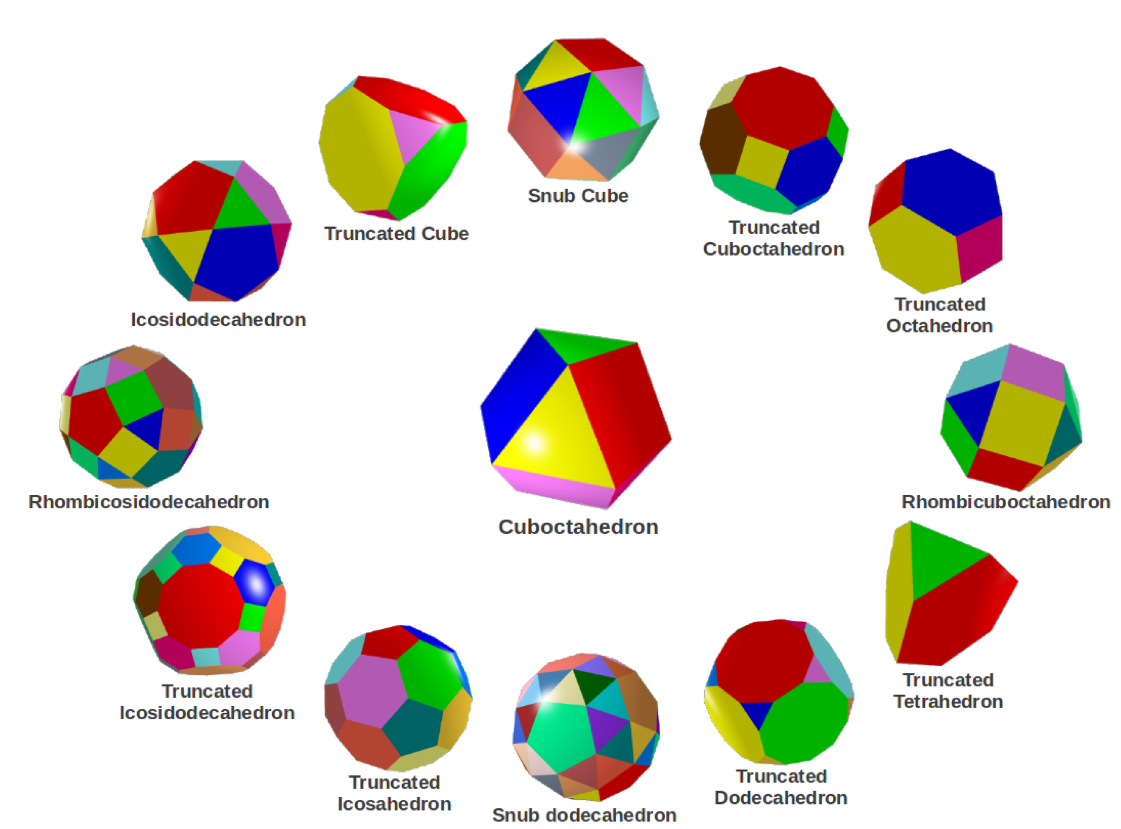
7. I put all of my data into a table and I looked at how the chromatic numbers changed as the graphs were triangulated and refined.

Variables

Independent: Process of triangulation, process of refinement
Dependent: Chromatic number
Controls: Type of triangulation, type of refinement

Data

Archimedean Solid	Chromatic Number of Original Graph	Chromatic Number of Triangulation	Chromatic Number of Refinement
Truncated Tetrahedron	3	4	4
Cuboctahedron	3	3	3
Truncated Cube	3	4	4
Truncated Octahedron	2	3	3
Rhombicub-octahedron	3	4	4
Truncated Cuboctahedron	2	3	3
Snub Cube	3	3	3
Icosidodecahedron	3	4	4
Truncated Dodecahedron	3	4	4
Truncated Icosahedron	3	4	4
Rhombicosi-dodecahedron	3	4	4
Truncated Icosidodecahdron	2	3	3
Snub Dodecahdron	4	4	4



Findings

After analyzing my data, I have found that the ones that started with a chromatic number of 2 had chromatic numbers of 3 after triangulation and refinement. The ones that started with a chromatic number of 3 most of the time had chromatic numbers of 4 after triangulation and refinement, however, some stayed at 3. None of them went from 3 to 3 to 4, none went from 2 to 3, and only one started with a chromatic number of 4, which also stayed at 4.

Conclusion

My hypothesis was correct that none of the chromatic numbers decreased during triangulation and refinement. Doing this experiment helped me figure out that during triangulation and refinement by degree 2 the chromatic number can never decrease, and I now also have a proof. My results also show that in most cases triangulating graphs increases the chromatic number by one while refining them doesn't affect their chromatic number. This is very close to my hypothesis, although I wasn't sure earlier on if the chromatic number would change after refinement. I also didn't know whether or not the chromatic number would ever start as 4, and I found that rarely, it does.

Recommendations

To anyone doing this project, I would recommend looking for shapes that have an odd number of vertices, excluding triangles, as they can help find the chromatic number more efficiently. The main shapes for this would be pentagons and heptagons. You could also use other solids as well, such as the five Platonic solids in addition to the thirteen Archimedean solids, to get a larger data set.

Ideas for Future Research

For a continuation of this project, I could explore different types of triangulation and refinement. In this project, I used the most common form of triangulation, and I also only refined my graphs by a degree of two. I would like to know how other triangulations and refinements of degree three, four, or even any natural number, would affect the chromatic number.

Works Cited

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