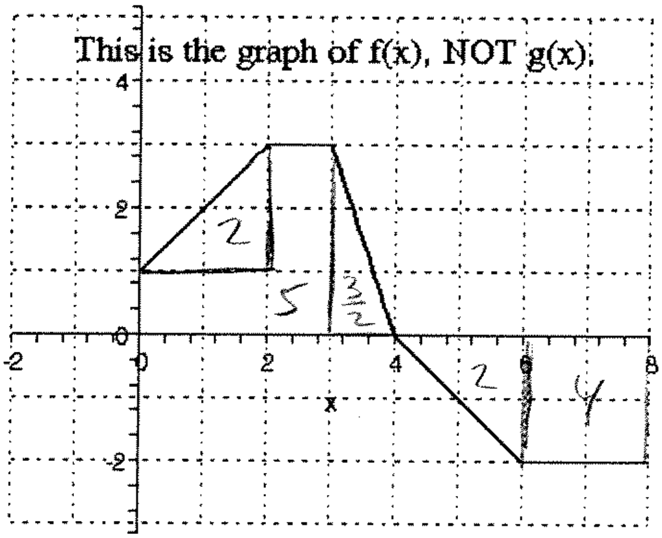
Calculus I, Drs. Nogin and Wyels, F'06 Name Answer Key Score ____ / 150

Sample Final Exam
(Use your study guide for a complete reference!)

Directions: Show, in an organized fashion, any work you want considered. Erase or cross out scratch work that you do not want considered. (Partial credit will be awarded for partially correct work or reasoning; correct answers with no work shown will receive no credit.)

10 1. Define $g(x) = \int_0^x f(t)\,dt$, and use the graph of $f(x)$ (shown) to determine $g(0)$, $g(4)$, and $g(8)$.



Using areas:

$g(0) = \int_0^0 f(t)\,dt = 0$

$g(4) = \int_0^4 f(t)\,dt = 2+5+\frac{3}{2} = 8.5$

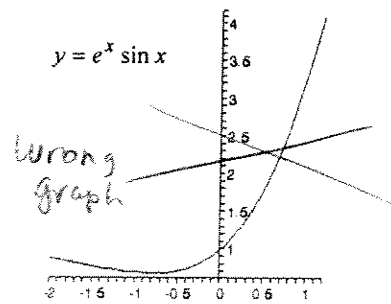
$g(8) = \int_0^8 f(t)\,dt = 8.5-2-4 = 2.5$

10 2. Find the equation of the tangent line to the curve $y = e^x \sin x$ at the point (0,1).

$y' = e^x \sin x + e^x \cos x$

$y'(0) = e^0 \cdot \sin 0 + e^0 \cdot \cos 0 = 1$

so slope = 1



$y - 1 = 1(x-0)$

$\boxed{y = x+1}$

(**Directions**: Show any work you want considered. Correct answers with no work shown will receive no credit.)

20 3. Evaluate the limits.

a) $\lim_{x\to 2} x^3 - x + 3$ $= 2^3 - 2 + 3 = 8 - 2 + 3 = 9$

b) $\lim_{x\to\infty} \frac{3x^3 - x^2 - 5}{x^3 + 2}$ $= \lim_{x\to\infty} \frac{3 - \frac{1}{x} - \frac{5}{x^3}}{1 + \frac{2}{x^3}} = \frac{3}{1} = 3$

c) $\lim_{x\to\infty} \frac{x^2}{e^x}$ $= \lim_{x\to\infty} \frac{2x}{e^x} = \lim_{x\to\infty} \frac{2}{e^x} = 0$

(using L'Hospital's Rule)

d) $\lim_{x\to\infty} \cos x$ DNE (because $\cos x$ does not approach any number as x becomes larger)

e) $\lim_{x\to 0} \frac{e^{-x}}{x}$ DNE because

$$\left.\begin{aligned} \lim_{x\to 0^+} \frac{e^{-x}}{x} &= +\infty \\ \lim_{x\to 0^-} \frac{e^{-x}}{x} &= -\infty \end{aligned}\right\} \text{different}$$

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20 4. Differentiation

a) Find the derivative of $f(x) = x^4 \sin x$.

$$f'(x) = 4x^3 \sin x + x^4 \cos x$$

b) Use the Quotient Rule to show that $\frac{d}{dx}\cot x = -\csc^2 x$.

$$\frac{d}{dx}\cot x = \frac{d}{dx}\left(\frac{\cos x}{\sin x}\right) = \frac{-\sin x \cdot \sin x - \cos x \cdot \cos x}{\sin^2 x} =$$

$$\frac{-\sin^2 x - \cos^2 x}{\sin^2 x} = \frac{-1}{\sin^2 x} = -\csc^2 x$$

c) Find $\frac{d}{dx}\cos(x^3)$. $= -\sin(x^3)\cdot 3x^2 = -3x^2 \sin(x^3)$

d) Find $\frac{d}{dx}\left(6x - \frac{1}{\sqrt{x}}\right)$. $= \frac{d}{dx}\left(6x - x^{-\frac{1}{2}}\right) = 6 - \left(-\frac{1}{2}\right)x^{-\frac{3}{2}}$

$$= 6 + \frac{1}{2}x^{-\frac{3}{2}}$$

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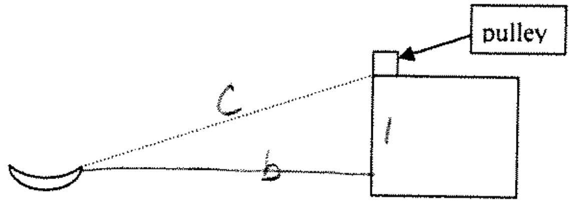
10 5. A boat is pulled into a dock by a rope attached to the bow of the boat and passing through a pulley on the dock that is 1 m higher than the bow of the boat. If the rope is pulled in at a rate of 1 m/s, how fast is the boat approaching the dock when it is 8 m from the dock?



$b^2 + 1 = c^2$

$2bb' = 2cc'$

$bb' = cc'$

given: $b = 8, \ c' = -1$

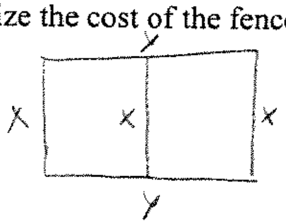
$c = \sqrt{65}$

so $8b' = -\sqrt{65}$

$b' = -\frac{\sqrt{65}}{8}$

Ans: approaching at $\frac{\sqrt{65}}{8}$ m/s.

10 6. A farmer wants to fence an area of 80 square feet in a rectangular garden and then divide it in half with a fence parallel to one of the sides of the rectangle. How can she do this so as to minimize the cost of the fence?



$xy = 80$

$L = 3x + 2y \leftarrow$ minimize

$y = \frac{80}{x}$

$L = 3x + 2 \cdot \frac{80}{x} = 3x + \frac{160}{x}$

$L' = 3 - \frac{160}{x^2} = 0$

$3 = \frac{160}{x^2}$

$3x^2 = 160$

$x^2 = \frac{160}{3}$

$x = \sqrt{\frac{160}{3}} \left(= \frac{4\sqrt{10}}{\sqrt{3}} \right)$

$y = \frac{80}{\sqrt{160/3}} = \frac{80\sqrt{3}}{\sqrt{160}} = \frac{80\sqrt{3}}{4\sqrt{10}}$

$y = \frac{20\sqrt{3}}{\sqrt{10}} = 2\sqrt{10}\sqrt{3} = 2\sqrt{30}$

(Directions: Show any work you want considered. Correct answers with no work shown will receive no credit.)

10 7. Water flows into a storage tank at a rate of $r(t) = 200 - 4t$ liters per minute, where $0 \le t \le 50$. Find the amount of water that flows from the tank during the first 10 minutes.

$0 \le t \le 10$

$$\text{Amount} = \int_0^{10} (200 - 4t)\, dt = \left(200t - 2t^2\right)\Big|_0^{10}$$

$$= (2000 - 200) - (0 - 0)$$

$$= 1800$$

10 8. Consider the integral $\int_2^7 \frac{1}{\sqrt{x}}\, dx$.

a) Set up an expression that would allow you to estimate the value of this integral by using a right-hand Riemann sum with 10 rectangles. (Do not evaluate your expression.)

b) Write an expression that would allow you to find the value of this integral by using the definition of a definite integral. (Do not evaluate your expression.)

(a) $\Delta x = \frac{7-2}{10} = \frac{5}{10} = \frac{1}{2}$

$x_i = 2 + \Delta x\, i = 2 + \frac{1}{2} i$

Riemann sum: $\sum_{i=1}^{10} f(x_i)\Delta x = \sum_{i=1}^{10} \frac{1}{\sqrt{2 + \frac{1}{2} i}} \cdot \frac{1}{2}$

(b) $\Delta x = \frac{7-2}{n} = \frac{5}{n}$

$x_i = 2 + \Delta x\, i = 2 + \frac{5}{n} i$

$$\int_2^7 \frac{1}{\sqrt{x}}\, dx = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i)\Delta x = \lim_{n \to \infty} \sum_{i=1}^{n} \frac{1}{\sqrt{2 + \frac{5}{n} i}} \cdot \frac{5}{n}$$

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25 9. Integration

a) Evaluate $\int \cos(3t-\pi)\,dt$. $= \frac{1}{3}\sin(3t-\pi)+C$

b) Evaluate $\int_0^1 2+x+x^2\,dx$. $= \left(2x+\frac{x^2}{2}+\frac{x^3}{3}\right)\Big|_0^1 = \left(2+\frac{1}{2}+\frac{1}{3}\right)-(0)$

$= \frac{12+3+2}{6} = \frac{17}{6}$

c) Find $\frac{d}{dx}\left(\int_0^x \tan(t^3-6t)\,dt\right)$. $= \tan(x^3-6x)$

(Fund Th of Calc Part I)

d) If $\int_0^3 f(t)\,dt = 10$ and $\int_0^5 f(t)\,dt = 7$, what is $\int_3^5 f(t)\,dt$?

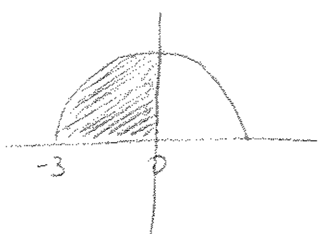
$\int_0^5 f(t)\,dt = \int_0^3 f(t)\,dt + \int_3^5 f(t)\,dt$

$7 = 10 + \int_3^5 f(t)\,dt$, so $\int_3^5 f(t)\,dt = -3$

e) Evaluate $\int_{-3}^0 \sqrt{x^2-9}\,dx$.

typo should be $9-x^2$ (because x^2-9 is negative on $(-3,0)$, so the function is undefined)

$\int_{-3}^0 \sqrt{9-x^2}\,dx = \text{Area} = \frac{1}{4}\pi\cdot 3^2 = \frac{9}{4}\pi$



$y=\sqrt{9-x^2}$
$y^2 = 9-x^2$
$x^2+y^2=9$
circle

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15 10. Let $f(x) = \frac{1}{4}x^4 - 6x^2 + 16x + c$. Note: c is an unknown constant.

(Hint: $x^3 - 12x + 16 = (x+4)(x-2)^2$.)

a) Specify the interval(s) on which $f(x)$ is increasing.

b) Specify the interval(s) on which $f(x)$ is concave up.

c) Identify the locations (x-values) of any local maxima and minima.

$f'(x) = x^3 - 12x + 16x = (x+4)(x-2)^2$

(a) $f(x)$ is increasing when $f'(x) > 0$, i.e. $x+4>0$

$\boxed{x > -4}$

(b) $f(x)$ is decreasing when $f'(x) < 0$, i.e. $x+4<0$

$\boxed{x < -4}$

(c) at $x=-4$ $f(x)$ changes from decreasing to increasing, therefore $x=-4$ is a loc. minimum.

no loc maxima

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10 11. Create a graph of the function described, including all the features specified.

$f(0) = 0$

$f(x) = 0$ for $x = 0, -4$

$\lim_{x\to\infty} f(x) = 0$

$\lim_{x\to-\infty} f(x) = 0$

y=0 hor. asympt.

$\lim_{x\to 2^-} f(x) = -\infty$

$\lim_{x\to 2^+} f(x) = +\infty$

x=2 vert. asympt.

$f'(x) = 0$ for $x = -5, -2$

$f''(x) = 0$ for $x = -6, -4, -1, 1$

$f'(x) > 0$ for x in $(-5,-2)$ increas.

$f'(x) < 0$ for x in $(-\infty,-5)$, $(-2,2)$ and $(2,\infty)$ decreas.

$f''(x) > 0$ for x in $(-6,-4)$, $(-1,1)$ and $(2,\infty)$ CU

$f''(x) < 0$ for x in $(-\infty,-6)$, $(-4,-1)$ and $(1,2)$ CD