Math 151

Solutions to selected homework problems

Section 3.4, Problem 2:

Show that the multiplicative group $\mathbb{Z}_7^\times$ is isormorphic to the additive group Z_6.

Solution:

Define $\phi : Z_6 \to \mathbb{Z}_7^\times$ by $\phi([x]_6) = [3]_7^x$.

First we will show that ϕ is well-defined. If $[x_1]_6 = [x_2]_6$, then $x_1 = x_2 + 6k$ for some $k \in \mathbb{Z}$. Then $\phi([x_1]_6) = [3]_7^{x_1} = [3]_7^{x_2+6k} = [3]_7^{x_2} \cdot [3]_7^{6k} = [3]_7^{x_2} \cdot ([3]_7^2)^{3k} = [3]_7^{x_2} \cdot ([9]_7)^{3k} = [3]_7^{x_2} \cdot ([2]_7)^{3k} = [3]_7^{x_2} \cdot ([2]_7^3)^k = [3]_7^{x_2} \cdot ([8]_7)^k = [3]_7^{x_2} \cdot ([1]_7)^k = [3]_7^{x_2} \cdot [1]_7 = [3]_7^{x_2} = \phi([x_2]_6)$.

To show that ϕ is a bijection, we compute the values of all elements in Z_6: $\phi([0]_6) = [3]_7^0 = [1]_7$; $\phi([1]_6) = [3]_7^1 = [3]_7$; $\phi([2]_6) = [3]_7^2 = [2]_7$; $\phi([3]_6) = [3]_7^3 = [6]_7$; $\phi([4]_6) = [3]_7^4 = [4]_7$; $\phi([5]_6) = [3]_7^5 = [5]_7$; Since all images are distinct and every element in $\mathbb{Z}_7^\times$ is the image of some element in Z_6, we have a bijection.

Finally, we will show that ϕ preserves the operation: $\phi([x_1]_6 + [x_2]_6) = \phi([x_1 + x_2]_6) = [3]_7^{x_1+x_2} = [3]_7^{x_1} \cdot [3]_7^{x_2} = \phi([x_1]_6) \cdot \phi([x_2]_6)$.

It follows from the above that ϕ is an isomorphism.

Section 3.4, Problem 6:

Let G_1 and G_2 be groups. Show that $G_2 \times G_1$ is isomorphic to $G_1 \times G_2$.

Solution:

Define $\phi : G_2 \times G_1 \to G_1 \times G_2$ by $\phi((y, x)) = (x, y)$ for all $(y, x) \in G_2 \times G_1$.

The function ϕ is one-to-one because if $\phi((y_1, x_1)) = \phi((y_2, x_2))$, then $(x_1, y_1) = (x_2, y_2)$, so $x_1 = x_2$ and $y_1 = y_2$. Thus $(y_1, x_1) = (y_2, x_2)$.

It is onto because for any $(x, y) \in G_1 \times G_2$, we have $\phi((y, x)) = (x, y)$.

Finally, it preserves the operation: $\phi((y_1, x_1)(y_2, x_2)) = \phi((y_1 y_2, x_1 x_2)) = (x_1 x_2, y_1 y_2) = (x_1, y_1)(x_2, y_2) = \phi((y_1, x_1))\phi((y_2, x_2))$.

Section 3.4, Problem 15:

Let G be any group, and let a be a fixed element of G. Define a function $\phi_a : G \to G$ by $\phi_a(x) = axa^{-1}$, for all $x \in G$. Show that ϕ_a is an isomorphism.

Solution:

First we will show that ϕ_a is one-to-one: if $\phi_a(x_1) = \phi_a(x_2)$, then $ax_1a^{-1} = ax_2a^{-1}$. Multiplying both sides of this equation by a on the right gives $ax_1a^{-1}a = ax_2a^{-1}a$, i.e. $ax_1 = ax_2$. Multiplying now by a^{-1} on the left gives $a^{-1}ax_1 = a^{-1}ax_2$, i.e. $x_1 = x_2$.

Next, ϕ is onto since for any $y \in G$, $\phi_a(a^{-1}ya) = aa^{-1}yaa^{-1} = y$.

Finally, ϕ preserves the operation: $\phi_a(x_1x_2) = ax_1x_2a^{-1} = ax_1a^{-1}ax_2a^{-1} = \phi_a(x_1)\phi_a(x_2)$.

Section 3.4, Problem 20:

Let G_1 and G_2 be groups. Show that G_1 is isomorphic to the subgroup of the direct product $G_1 \times G_2$ defined by $\{(x_1, x_2) \mid x_2 = e\}$.

Solution:

Let $H = \{(x_1, x_2) \mid x_2 = e\}$. Define $\phi : G_1 \to H$ by $\phi(x) = (x, e)$.

Then ϕ is one-to-one since if $\phi(x_1) = \phi(x_2)$, then $(x_1, e) = (x_2, e)$, so $x_1 = x_2$.

Also, ϕ is onto since for any element $(x_1, x_2) \in H$, $x_2 = e$, and thus $\phi(x_1) = (x_1, e) = (x_1, x_2)$.

Finally, ϕ preserves the operation since $\phi(x_1x_2) = (x_1x_2, e) = (x_1, e)(x_2, e) = \phi(x_1)\phi(x_2)$.

Thus ϕ is an isomorphism.

Section 3.5, Problem 8:

Find $< \pi >$ in $\mathbb{R}^\times$.

Solution:

Since $\pi > 1$, for any $k < n$ we have $\pi^k < \pi^n$. So all powers of π are distinct. Therefore $< \pi > = \{\dots, \pi^{-2}, \pi^{-1}, 1, \pi, \pi^2, \dots\}$.

Section 3.5, Problem 11:

Which of the groups $Z_7^\times$, $Z_{10}^\times$, $Z_{12}^\times$, $Z_{14}^\times$ are isomorphic?

Solution:

First we find the orders of the given groups: $|Z_7^\times| = |\{[1], [2], [3], [4], [5], [6]\}| = 6$,
$|Z_{10}^\times| = |\{[1], [3], [7], [9]\}| = 4$, $\quad |Z_{12}^\times| = |\{[1], [5], [7], [11]\}| = 4$,
$|Z_{14}^\times| = |\{[1], [3], [5], [9], [11], [13]\}| = 6$. Since isomorphic groups have the same order, we have to check two pairs: $Z_7^\times$ and $Z_{14}^\times$; $Z_{10}^\times$ and $Z_{12}^\times$.

Both $Z_7^\times$ and $Z_{14}^\times$ are cyclic of order 6 (we check below that both are generated by [3]), therefore they both are isomorphic to Z_6, and thus isomorphic to each other:

$< [3]_7 > = \{[1]_7, [3]_7, [2]_7, [6]_7, [4]_7, [5]_7\} = Z_7^\times$;

$< [3]_{14} > = \{[1]_{14}, [3]_{14}, [9]_{14}, [13]_{14}, [11]_{14}, [5]_{14}\} = Z_{14}^\times$.

However, the groups $Z_{10}^\times$ and $Z_{12}^\times$ are not isomophic because $Z_{10}^\times$ is cyclic (it is generated by [3]), but $Z_{12}^\times$ is not cyclic (the order of each element is either 1 or 2):

$< [3]_{10} > = \{[1]_{10}, [3]_{10}, [9]_{10}, [7]_{10}\} = Z_{10}^\times$;

$[5]_{12}^2 = [7]_{12}^2 = [11]_{12}^2 = [1]_{12}$.