Math 151

Solutions to selected homework problems

Section 4.3, Problem 5:

Let $\phi : F_1 \to F_2$ be an isomorphism of fields. Prove that $\phi(1) = 1$ (that is, prove that ϕ must map the multiplicative identity of F_1 to the multiplicative identity of F_2).

Solution:

Since $\phi(0) = 0$, $1 \neq 0$, and ϕ is a bijection, $\phi(1) \neq 0$. Therefore $\phi(1)$ has a multiplicative inverse.

Then $\phi(1) = \phi(1) \cdot 1 = \phi(1)\phi(1)(\phi(1))^{-1} = phi(1 \cdot 1)(\phi(1))^{-1} = phi(1)(\phi(1))^{-1} = 1$.

Section 4.3, Problem 9:

Prove that $\mathbb{R}[x]/ < x^2 + x + 1 >$ is isomorphic to $\mathbb{C}$.

Hint:

We need to construct an isomorphism between these to fields, say, $\phi : \mathbb{C} \to \mathbb{R}[x]/ < x^2 + x + 1 >$.

Since the multiplicative identity must be mapped to the multiplicative identity, we must have $\phi(1) = [1]$. We need to determine $\phi(i)$. Since $i^2 = -1$, we must send i to a class whose square is equal to $[-1]$. So let $\phi(i) = ax + b$, and we need $[ax + b]^2 = [-1]$:
$[a^2x^2 + 2abx + b^2] = [-1]$
$[a^2(-x - 1) + 2abx + b^2] = [-1]$
$[(2ab - a^2)x + (b^2 - a^2)] = [-1]$
$2ab - a^2 = 0$, $b^2 - a^2 = -1$. Solve this system, and then define $\phi(c + di) = [c + d(ax + b)]$.
Show that this function is a bijection, preserves addition, and preserves multiplication.

Section 4.3, Problem 21(b):

Find the multiplicative inverse of $[a + bx]$ in $\mathbb{Q}[x]/ < x^2 - 2 >$.

Solution:

Case I: $b = 0$, $a = 0$. Then $[a + bx] = [0]$ does not have a multiplicative inverse.

Case II: $b = 0$, $a \neq 0$. Then $[a]^{-1} = [a^{-1}]$.

Case III: $b \neq 0$.

Dividing $x^2 - 2$ by $a + bx$ gives:

$$x^2 - 2 = (\frac{1}{b}x - \frac{a}{b^2})(bx + a) + \frac{a^2 - 2b^2}{b^2}$$

Case IIIA: $a^2 - 2b^2 \neq 0$, The remainder is a nonzero constant, therefore the gcd of $x^2 - 2$

and $a+bx$ is 1. Then

$$\frac{b^2}{a^2-2b^2}(x^2-2) = \frac{b^2}{a^2-2b^2}\left(\frac{1}{b}x - \frac{a}{b^2}\right)(bx+a)+1$$

$$\frac{b^2}{2b^2-a^2}\left(\frac{1}{b}x - \frac{a}{b^2}\right)(bx+a) = 1 - \frac{b^2}{a^2-2b^2}(x^2-2)$$

So $[bx+a]^{-1} = \left[\frac{b^2}{2b^2-a^2}\left(\frac{1}{b}x - \frac{a}{b^2}\right)\right] = \left[\frac{bx-a}{2b^2-a^2}\right]$.

Case IIIB: $a^2-2b^2=0$. Then $a^2=2b^2$, and $[a+bx][a-bx][a^2-b^2x^2] = [2b^2-b^2x^2] = [b^2(2-x^2)] = [0]$, and $[a+bx]$ does not have a multiplicative inverse.