Practice problems for Test 2

Solutions

1. (Note: feel free to show me your examples to make sure they are correct.)

group	order	abelian?	cyclic?
$\mathbb{Z}_5^*$	4	yes	yes
$\mathbb{Z}_6$	6	yes	yes
S_3	6	no	no
$\mathbb{Z}_4 \oplus \mathbb{Z}_2$	8	yes	no
$\mathbb{Z}$	∞	yes	yes
$GL_2(\mathbb{R})$	∞	no	no
$\{e\}$=trivial	1	yes	yes
D_5	10	no	no
$Mat_{2\times 3}(\mathbb{Z}_2)$	64	yes	no
$\mathbb{R}$	∞	yes	no

$\mathbb{Z}_5^*$ consists of all units in $\mathbb{Z}_5$, and it is a group under multiplication. $\mathbb{Z}_5^* = \{1,2,3,4\}$, so $|\mathbb{Z}_5^*| = 4$. It is abelian since multiplication of numbers is commutative. It is cyclic because it is generated by 2: $<2>= \{1,2,4,3\} = \mathbb{Z}_5^*$.

$\mathbb{Z}_6 = \{0,1,2,3,4,5\} =<1>$ is an abelian cyclic group (under addition) of order 6. (In general, $\mathbb{Z}_n = \{0,1,2,\ldots,n-1\} =<1>$ is an abelian cyclic group of order n.)

$S_3 = \{(1),(12),(13),(23),(123),(132)\}$ has order 6. It is not abelian because e.g. $(12)(13) = (132)$ and $(13)(12) = (123)$. (In general, S_n, the permutation group on a set of n elements, it has order $n!$ and is non-abelian.) It is not cyclic because every cyclic group is abelian and this one is not.

$\mathbb{Z}_4 \oplus \mathbb{Z}_2 = \{(x,y) \mid x \in \mathbb{Z}_4,\ y \in \mathbb{Z}_2\}$ is the set of all pairs, and it has order $4 \cdot 2 = 8$. It is abelian since both $\mathbb{Z}_4$ and $\mathbb{Z}_2$ are. It is not cyclic because it has no element of order 8: it is easy to check that the order of each element is ≤ 4.

$\mathbb{Z}$ is an infinite cyclic group consisting of all integer numbers (with addition). It is abelian since addition of numbers is abelian. It is cyclic because it is generated by 1.

$GL_2(\mathbb{R})$ is the group of 2×2 invertible matrices with real entries under multiplication. There are infinitely many such matrices, so its order is infinity. It is not abelian because e.g. $\begin{bmatrix} 0 & 2 \\ 3 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$ and $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 2 \\ 3 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$. It is not cyclic because every cyclic group is abelian.

$\{e\}$=trivial group has only one element. It is abelian (all elements commute), and cyclic (generated by e). It is a very uninteresting group, but I just wanted to give an example of a group of order 1.

D_5, dihedral group of order $2 \cdot 5 = 10$, is the group of rigid motions of a regular pentagon. Its elements are $e, a, a^2, a^3, a^4, b, ab, a^2b, a^3b, a^4b$. It is not abelian (e.g. $ba = a^4b \neq ab$), and hence not cyclic.

$Mat_{2\times 3}(\mathbb{Z}_2)$ is the group of all 2×3 matrices with entries in $\mathbb{Z}_2$ under addition. Its order is 64: each entry can be either 0 or 1, and there are 6 entries, so there are $2^6 = 64$ such matrices. It is abelian since addition in $\mathbb{Z}_2$ is commutative. But it is not cyclic: each non-zero element has order 2 because if you add an entry of a matrix to itself you'll get 0, thus any matrix added to itself gives the zero matrix. Therefore there is no element (matrix) of order 64.

$\mathbb{R}$ is an infinite group of real numbers under addition. It is abelian (addition of numbers is commutative) but not cyclic: every nonzero element generates a cyclic subgroup consisting of its own multiples, thus every cyclic subgroup has a smallest positive element. But $\mathbb{R}$ does not have any.

2. First notice that $\mathbb{R}$, $\mathbb{R}^*$, and $\mathbb{R}^+$ have infinite order, while $\mathbb{Z}_4 \oplus \mathbb{Z}_4$, $\mathbb{Z}_2 \oplus \mathbb{Z}_8$, $\mathbb{Z}_8 \oplus \mathbb{Z}_2$, and $\mathbb{Z}_{16}$ have order 16. So we only have to check the first 3 groups, and the last 4 groups, separately.

   Among the first 3 groups, $\mathbb{R}$ and $\mathbb{R}^+$ are isomorphic: let $f : \mathbb{R} \to \mathbb{R}^+$ be defined by $f(x) = e^x$. Then $f(x+y) = e^{x+y} = e^x e^y = f(x)f(y)$. f is one-to-one because if $f(x) = f(y)$ then $e^x = e^y$, then $\ln e^x = \ln e^y$ which implies $x = y$. Finally, f is onto because for any positive real z, let $x = \ln z$, then $f(x) = f(\ln z) = e^{\ln z} = z$. (See example 3.4.2 on p.115.)

   $\mathbb{R}$ and $\mathbb{R}^*$ are not isomorphic because the first group has no element of order 2, and the second group has an element of order 2, namely $-1 : (-1)^2 = 1$. $\mathbb{R}^+$ and $\mathbb{R}^*$ are not isomorphic for the same reason. (See example 3.4.3 on p.116.)

   Among the last 4 groups, $\mathbb{Z}_2 \oplus \mathbb{Z}_8$ and $\mathbb{Z}_8 \oplus \mathbb{Z}_2$ are isomorphic: define $f : \mathbb{Z}_2 \oplus \mathbb{Z}_8 \to \mathbb{Z}_8 \oplus \mathbb{Z}_2$ by $f((x,y)) = (y,x)$. Obviously this is a 1-1 correspondence, and it is a homomorphism because $f((x,y)+(z,w)) = f((x+z,y+w)) = (y+w,x+z) = (y,x)+(w,z) = f((x,y)) + f((z,w))$.

   All other pairs are not isomorphic: $\mathbb{Z}_4 \oplus \mathbb{Z}_4$ only has elements of order ≤ 4; $\mathbb{Z}_2 \oplus \mathbb{Z}_8$ and $\mathbb{Z}_8 \oplus \mathbb{Z}_2$ have elements of order 8 but no elements of order 16; $\mathbb{Z}_{16}$ has elements of order 16.

3. Let's denote this subset of G by H. We want to show that H is a subgroup.

   Closed under multiplication: if $a, b \in H$, then $a^2 = b^2 = e$. Then $(ab)^2 = a^2b^2 = e \cdot e = e$, so $ab \in H$.
   Identity: $\text{ord}(e) = 1 \leq 2$, so $e \in H$.
   Closed under inverses: if $a \in H$, then $a^2 = e$. Then $(a^{-1})^2 = (a^2)^{-1} = e^{-1} = e$, so $a^{-1} \in H$.

4. Let's denote the given matrix by A. We have to compute powers of A until we get the identity matrix. The smallest positive k such that $A^k = I$ is then the order of A, and the cyclic subgroup generated by A is $\{I, A, A^2, \ldots, A^{k-1}\}$. Notice that entries of our matrices are elements of $\mathbb{Z}_3$, so each time we multiply matrices, we have to reduce each entry of the product modulo 3. Then
   $< \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix} >= \left\{ I_2, \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 2 & 2 \\ 0 & 2 \end{bmatrix} \right\}.$
   Therefore the order of A is 6.

5. (a) Generators of $\mathbb{Z}_{24}$ are numbers (more precisely, classes of numbers) between 0 and 24 that are relatively prime to 24. There are 8 of them: 1, 5, 7, 11, 13, 17, 19, 23.
   (b) $H = \{0, 6, 12, 18\}$ is a cyclic subgroup. Generators: 6 and 18. 0 and 12 are not generators because the order of 0 is 1, and the order of 12 is 2.
   $K = \{0, 4, 8, 12, 16, 20\}$ is a cyclic subgroup. Generators: 4 and 20.
   $H \cap K = \{0, 12\}$ is a cyclic subgroup. Generator: 12.
   $H \cup K = \{0, 4, 6, 8, 12, 16, 18, 20\}$ is not a subgroup: it is not closed under addition, e.g., $4 + 6 = 10 \notin H \cup K$.
   $H + K = \{0, 2, 4, 6, 8, 10, 12, 14, 16, 18, 20, 22\}$ is a cyclic subgroup. Generators: 2, 10, 14, 22.

6. (a) $f : \mathbb{Z} \to \mathbb{Z}$, $f(x) = 3x$ is a homomorphism: $f(x+y) = 3(x+y) = 3x + 3y = f(x) + f(y)$. $\text{Ker}(f) = \{0\}$. Image $= 3\mathbb{Z}$, the set of all multiples of 3. It is one-to-one because $3x = 3y$ implies $x = y$. It is not onto because e.g. 1 is not in the image.

(b) $f : \mathbb{Z} \to \mathbb{Z}_4$, $f(x) = [x]_4$ is a homomorphism: $f(x+y) = [x+y]_4 = [x]_4 + [y]_4 = f(x) + f(y)$. $\text{Ker}(f) = 4\mathbb{Z}$, the set of all multiples of 4. Image $= \mathbb{Z}_4$. It is not one-to-one because e.g. $f(0) = [0]_4$ and $f(4) = [4]_4 = [0]_4$. It is onto: every element of $\mathbb{Z}_4$ is in the image since $[x]_4 = f(x)$.

(c) $f : \mathbb{Z} \to \mathbb{Z}_6$, $f(x) = [2x]_6$ is a homomorphism: $f(x+y) = [2(x+y)]_6 = [2x+2y]_6 = [2x]_6 + [2y]_6 = f(x) + f(y)$. $\text{Ker}(f) = 3\mathbb{Z}$, the set of all multiples of 3. Image $= 2\mathbb{Z}_6 = \{0, 2, 4\}$. It is not one-to-one because e.g. $f(0) = [0]_6$ and $f(3) = [6]_6 = [0]_6$. It is not onto because e.g. $[1]_6$ is not in the image.

(d) $f : \mathbb{Z}_2 \to \mathbb{Z}$, $f([x]_2) = x$ is not a homomorphism because it is not a well-defined function: $[0]_2 = [2]_2$, but $f([0]_2) = 0$, $f([2]_2) = 2$, and $0 \neq 2$.

(e) $f : \mathbb{R} \oplus \mathbb{R} \to \mathbb{R}$, $f((x,y)) = x + y$ is a homomorphism. The kernel consists of all pairs (x, y) for which $x + y = 0$, or $y = -x$. Therefore $\text{Ker}(f) = \{(x, -x)\}$. Image $= \mathbb{R}$: given $z \in \mathbb{R}$, $z = f((z, 0))$ is in the image. It is not one-to-one because e.g. $f((1,0)) = 1$ and $f((2,1)) = 1$. It is onto as shown above.

(f) $f : \mathbb{R} \oplus \mathbb{R} \to \mathbb{R}$, $f((x,y)) = xy$ is not a homomorphism. E.g., $f((1,1) + (1,0)) = f((2,1)) = 2$ but $f((1,1)) + f((1,0)) = 1 + 0 = 1$, so the function does not preserve addition.

(g) $f : \mathbb{R}^* \times \mathbb{R}^* \to GL_2(\mathbb{R})$, $f((x,y)) = \begin{bmatrix} 2x-y & y-x \\ 2x-2y & 2y-x \end{bmatrix}$ is a homomorphism:

$$f((x,y))f((z,w)) = \begin{bmatrix} 2x-y & y-x \\ 2x-2y & 2y-x \end{bmatrix} \begin{bmatrix} 2z-w & w-z \\ 2z-2w & 2w-z \end{bmatrix}$$
$$= \begin{bmatrix} (2x-y)(2z-w) + (y-x)(2z-2w) & (2x-y)(w-z) + (y-x)(2w-z) \\ (2x-2y)(2z-w) + (2y-x)(2z-2w) & (2x-2y)(w-z) + (2y-x)(2w-z) \end{bmatrix}$$
$$= \begin{bmatrix} 2xz-yw & yw-xz \\ 2xz-2yw & 2yw-xz \end{bmatrix} = f((xz, yw)) = f((x,y)(z,w)).$$

The kernel of f consists of all pairs for which $2x - y = 2y - x = 1$ and $2x - 2y = y - x = 0$. Solving this system gives $x = y = 1$.

The image of f consists of all matrices of the form $\begin{bmatrix} 2x-y & y-x \\ 2x-2y & 2y-x \end{bmatrix}$. It is OK to leave this matrix as is. However, I decided to give a slightly more explicit desctiption: let $a = 2x - y$ and $b = y - x$, then the other two entries can be expressed in terms of a and b, and Image $= \left\{ \begin{bmatrix} a & b \\ -2b & a+3b \end{bmatrix} \mid \text{this matrix must be invertible: } a^2 + 3ab + 4b \neq 0 \right\}$.

f is one-to-one since the kernel is trivial, and it is not onto because e.g. $\begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}$ is not in the image.

7. Let H and K be normal in G. We want to show that $H \cap K$ is normal (we already know that it is a subgroup: there was a homework problem in which we proved that the intersection of any collection of subgroups is a subgroup). Let $x \in H \cap K$, and let $g \in G$. Then $x \in H$ and $x \in K$. Since H is normal, $gxg^{-1} \in H$. Since K is normal, $gxg^{-1} \in K$. Then $gxg^{-1} \in H \cap K$, therefore $H \cap K$ is normal.

8. H is not normal in G because e.g. $\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} -1 & 4 \\ -1 & 3 \end{bmatrix} \notin H$.

K is normal in H because for any $\begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} \in K$ and $\begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \in H$,

$$\begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a & b \\ 0 & c \end{bmatrix}^{-1} = \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{a} & \frac{-b}{ac} \\ 0 & \frac{1}{c} \end{bmatrix} = \begin{bmatrix} a & ax+b \\ 0 & c \end{bmatrix} \begin{bmatrix} \frac{1}{a} & \frac{-b}{ac} \\ 0 & \frac{1}{c} \end{bmatrix}$$
$$= \begin{bmatrix} 1 & \frac{ax}{c} \\ 0 & 1 \end{bmatrix} \in K.$$

K is not normal in G because the counterexample above works in this case too.