Math 151 Spring 2004

Practice problems for Test 3

The actual test will consist of 6 problems. You will have 1 hour to complete the test.

1. Let $\mathbb{Z}_2(i) = \{a + bi \mid a, b \in \mathbb{Z}_2\}$ with addition and multiplication given by

$$(a + bi) + (c + di) = (a + c) + (b + d)i$$

$$(a + bi)(c + di) = (ab - bd) + (ad + bc)i$$

   where $a, b, c, d \in \mathbb{Z}_2$. Is $\mathbb{Z}_2(i)$ a ring? If so, is it a commutative ring? Does it have a multiplicative inverse? Is it an integral domain? Is it a field? (Justify your answers: say which axioms are or are not satisfied, and why.)

2. Find the quotient and remainder when $x^5 + 3x + 1$ is divided by $x^2 + 2$ over $\mathbb{Q}$.
3. Let $f(x) = x^5 + 4x^4 + 6x^3 + 6x^2 + 5x + 2$ and $g(x) = x^4 + 3x^2 + 3x + 6$.
   (a) Find the greatest common divisor $d(x)$ of $f(x)$ and $g(x)$ over $\mathbb{Z}_7$.
   (b) Find polynomials $a(x)$ and $b(x)$ in $\mathbb{Z}_7[x]$ such that $d(x) = a(x)f(x)+b(x)g(x)$.
4. Find the multiplicative inverse of $[x + 4]$ in $\mathbb{Z}_5/< x^3 + x + 1 >$.
5. Find all integer roots of $x^4 + 4x^3 + 8x + 32 = 0$.
6. Find the irreducible factors of $x^3 - 2$ over $\mathbb{Z}$; $\mathbb{Q}$; $\mathbb{R}$; $\mathbb{C}$; $\mathbb{Z}_3$.
7. Find all irreducible polynomials of degree 3 over $\mathbb{Z}_2$.
8. Use Eisenstein's criterion to show that the polynomial $3x^4 + 30x - 60$ is irreducible over $\mathbb{Q}$.
9. (a) Give an example of a commutative ring R with a multiplicative identity e, and a subring S of R with a multiplicative identity $e' \neq e$.
   (b) Prove that if R is an integral domain with a multiplicative identity e, and S is a subring of R with a multiplicative identity e' then $e = e'$.
10. Find all units in $\mathbb{Z}_6 \oplus \mathbb{Z}_8$.
11. Let R be a commutative ring with ideals I and J. Let

$$I + J = \{x \in R \mid x = a + b \text{ for some } a \in I, b \in J\}.$$

   Show that $I + J$ is an ideal of R.