Math 151

Spring 2012

Final Exam

- Show all your work. Prove all your claims. Provide as many details as possible. Give specific examples and counterexamples.
- You may consult your notes and any printed and on-line materials. However, no communication on any topic related to Abstract Algebra with anybody but your instructor is allowed. (In case you need an exception to this rule due to other classes, research projects, etc., please let your instructor know in advance.) You are welcome to address any questions to your instructor (please be prepared that some questions may not be answered until the final is due, but it never hurts to ask!)
- Extra credit (up to 10 points) will be given if your solutions are typed. Ten points will be awarded if all solutions including all math formulas, diagrams, etc. are typed, and all pictures (if necessary) are computer-generated. Partial credit will be given if the exam is partially typed. You are especially encouraged to use LaTeX, but you may use any software you like and/or are comfortable with. You may consult any materials and communicate with other people regarding how to type in LaTeX or use other computer software of your choice. However, you must then be careful not to show your solutions to your classmates.

1. (6 pts) Let a, b be nonzero integers. Prove that $(a, b) = 1$ if and only if $(a + b, ab) = 1$.
2. (8 pts) Let $a = 517$ and $b = 2012$.
   (a) Find the greatest common divisor d of a and b.
   (b) Express d as a linear combination of a and b.
   (c) Find the multiplicative inverse of $[a]$ in $\mathbb{Z}_b$ if it exists. If it does not exist, explain why not.

3. (6 pts) Give an example of a nonabelian group G of order 12 with a subgroup H of order 3. Is your group G cyclic? Is H cyclic? (Explain!)

4. (6 pts) Prove that if G is a finite abelian group and a prime p divides $|G|$, then G has an element of order p.

5. (6 pts) Is $H = \{A \in \text{Mat}_{n\times n}(\mathbb{R}) \mid \det(A) \geq 1\}$ is a subgroup of $GL_n(\mathbb{R})$? (Prove your claim!)

6. (8 pts)

   (a) How many functions are there from $\mathbb{Z}_6$ to $\mathbb{Z}_4$? (Explain!)

   (b) How many of the functions in part (a) are bijections? (Explain!)

   (c) Find all group homomorphisms from $\mathbb{Z}_6$ to $\mathbb{Z}_4$. For each of them, find
      - its kernel,
      - its image,
      - whether or not it is a ring homomorphism. (Explain!)

7. (6 pts) Given a group G and its subgroup H, determine if H is normal in G. If so, compute the factor group G/H (i.e. determine what familiar group it is isomorphic to). If not, show that it is not normal.

   (a) $G = S_5$, $H = S_3$

   (b) $G = \mathbb{Z}$, $H =< 4 >$

8. (8 pts) Let $x, y, z \in S$ where S is

   (a) a multiplicative group,

   (b) a field,

   (c) a ring and $x \neq 0$,

   (d) an integral domain and $x \neq 0$.

   Is it true that $xy = xz$ implies $y = z$?
   (In each case, prove or give a counterexample.)

9. (6 pts) Factor (into a product of irreducible polynomials) $f(x) = x^4 + 4x^3 + 3x + 1$ over $\mathbb{Z}_5$.