**Math 151**

## Solutions to selected homework problems

**Section 1.2, Problem 8:**

Let $a$, $b$ be positive integers, and let $d = (a, b)$. Since $d|a$ and $d|b$, there exist integers $h$, $k$ such that $a = dh$ and $b = dk$. Show that $(h, k) = 1$.

**Solution:**

Since $d = (a, b)$, $d = ma + nb$ for some $m, n \in \mathbb{Z}$. Then $d = mdh + ndk$, so $1 = mh + nk$. Therefore (by Proposition 1.2.2) $(h, k) = 1$.

**Section 1.2, Problem 10:**

Show that $a\mathbb{Z} \cap b\mathbb{Z} = [a, b]\mathbb{Z}$.

**Solution:**

Let $x \in a\mathbb{Z} \cap b\mathbb{Z}$. Then $x \in a\mathbb{Z}$ and $x \in b\mathbb{Z}$, i.e. $x$ is a multiple of both $a$ and $b$. By definition of lcm, $x$ is a multiple of $[a, b]$. Therefore $x \in [a, b]\mathbb{Z}$. Thus we have $a\mathbb{Z} \cap b\mathbb{Z} \subseteq [a, b]\mathbb{Z}$.

Now let $x \in [a, b]\mathbb{Z}$. Then $x$ is a multiple of $[a, b]$. It follows that $x$ is a multiple of both $a$ and $b$, i.e. $x \in a\mathbb{Z}$ and $x \in b\mathbb{Z}$. Therefore $x \in a\mathbb{Z} \cap b\mathbb{Z}$. Thus we have $[a, b]\mathbb{Z} \subseteq a\mathbb{Z} \cap b\mathbb{Z}$.

Since $a\mathbb{Z} \cap b\mathbb{Z} \subseteq [a, b]\mathbb{Z}$ and $[a, b]\mathbb{Z} \subseteq a\mathbb{Z} \cap b\mathbb{Z}$, we have $a\mathbb{Z} \cap b\mathbb{Z} = [a, b]\mathbb{Z}$.

**Section 1.2, Problem 24:**

Show that $\log 2/\log 3$ is not a rational number.

**Solution:**

We will prove this statement by contradiction. Suppose $\log 2/\log 3$ is rational. Then $\log 2/\log 3 = m/n$ for some $m, n \in \mathbb{Z}$, $n > 0$. Since $\log 2/\log 3 > 0$, we also have $m > 0$. Using the formula $\log_a b/\log_a c = \log_c b$, we have $\log_3 2 = m/n$. It follows that $3^{m/n} = 2$, or $3^m = 2^n$. Since $m, n > 0$, $3^m = 2^n > 1$. Since both 2 and 3 are prime, this contradicts the unique prime factorization theorem.

Note: the idea of this proof is similar to that of the proof that $\sqrt{2}$ is irrational (on page xix in our book). Namely, we assume the number is rational, write it as a quotient of two positive integers, and then rewrite the equation so that only integer numbers are involved. Finally, we use some properties of integer numbers to get a contradiction (another property we could use is that $2^n$ is even while $3^m$ is odd).

**Section 1.3, Problem 14:**

Find the units digit of $3^{29} + 11^{12} + 15$.

**Solution:**

Note: There are many different ways to do this problem. Below is one.

$3^{29}+11^{12}+15 \equiv 3^{28}\cdot3+1^{12}+5 \equiv (3^4)^7\cdot3+1+5 \equiv 81^7\cdot3+6 \equiv 1^7\cdot3+6 \equiv 3+6 \equiv 9 \pmod{10}$, thus the units digit is 9.

**Section 1.3, Problem 28:**

Prove that there exist infinitely many prime numbers of the form $4m + 3$ (where $m$ is an integer).

**Solution:**

We will prove this statement by contradiction. Suppose there are finitely many prime numbers of the form $4m + 3$, say, $p_0 = 3$, $p_1$, $p_2$, $\ldots$, $p_n$. Consider the number $N = 4p_1p_2 \ldots p_n. + 3$. It is of the form $4m + 3$. If it is prime, we have another prime number and that contradicts to our assumption. It if is composite, it must have a prime factor. Since it is odd, 2 is not its factor. Further, it is not possible that all of its prime factors are of the form $4m + 1$ because a product of numbers of this form is also of this form $((4m_1 + 1)(4m_2 + 1) = 4(4m_1m_2 + m_1 + m_2) + 1)$. Thus it must have a prime factor of the form $4m + 3$, i.e. $p_i$ for some $1 \leq i \leq n$. Then $p_i|3$. Since $p_i > 3$, we again get a contradiction.