Math 151

Solutions to selected homework problems

Section 3.2, Problem 8(ab):

Let $G = GL_2(\mathbb{R})$. For each of the following subsets of $M_2(\mathbb{R})$, determine whether or not the subset is a subgroup of G.

(a) $A = \left\{ \left[\begin{array}{cc} a & b \\ 0 & 0 \end{array} \right] \middle| ab \neq 0 \right\}$

(b) $B = \left\{ \left[\begin{array}{cc} 0 & b \\ c & 0 \end{array} \right] \middle| bc \neq 0 \right\}$

Solution:

(a) Explanation 1:

A matrix of the form $\left[\begin{array}{cc} a & b \\ 0 & 0 \end{array} \right]$ has determinant 0, therefore it is not invertible, and is not in the set $GL_2(\mathbb{R})$. So the set A is not a subset of $GL_2(\mathbb{R})$. Thus it is not a subgroup.

Explanation 2:

The set A does not contain the identity matrix $\left[\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right]$, therefore it is not a subgroup.

(b) Explanation 1:

Since $\left[\begin{array}{cc} 0 & b \\ c & 0 \end{array} \right] \left[\begin{array}{cc} 0 & d \\ e & 0 \end{array} \right] = \left[\begin{array}{cc} be & 0 \\ 0cd & \end{array} \right]$, the set B is not closed under multiplication. Therefore it is not a subgroup.

Explanation 2:

The set B does not contain the identity matrix $\left[\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right]$, therefore it is not a subgroup.

Section 3.3, Problem 5:

Prove that if G_1 and G_2 are abelian groups, then the direct product $G_1 \times G_2$ is abelian.

Solution:

Let $a, b \in G_1 \times G_2$, then $a = (a_1, a_2)$ and $b = (b_1, b_2)$ where $a_1, b_1 \in G_1$ and $a_2, b_2 \in G_2$. Since G_1 and G_2 are abelian, $a_1 b_1 = b_1 a_1$ and $a_2 b_2 = b_2 a_2$. Therefore $ab = (a_1, a_2)(b_1, b_2) = (a_1 b_1, a_2 b_2) = (b_1 a_1, b_2 a_2) = (b_1, b_2)(a_1, a_2) = ba$, thus $G_1 \times G_2$ is abelian.

Section 3.3, Problem 8:

Let G_1 and G_2 be groups, with subgroups H_1 and H_2, respectively. Show that $\{(x_1, x_2) \mid x_1 \in H_1,\ x_2 \in H_2\}$ is a subgroup of the direct product $G_1 \times G_2$.

Solution:

Let $H = \{(x_1, x_2) \mid x_1 \in H_1,\ x_2 \in H_2\}$. We will show that H is closed under multiplication, contains the identity element, and is closed under inverses.

If (x_1, x_2) and (y_1, y_2) are in H, i.e. $x_1, y_1 \in H_1$ and $x_2, y_2 \in H_2$, then $x_1y_1 \in H_1$ and $x_2y_2 \in H_2$ since H_1 and H_2 are subgroups (and therefore are closed under multiplication). Then $(x_1, x_2)(y_1, y_2) = (x_1y_1, x_2y_2) \in H$. So H is closed under multiplication.

Let e_1 and e_2 be the identity elements in G_1 and G_2, respectively. Then (e_1, e_2) is the identity element in $G_1 \times G_2$. Since H_1 and H_2 are subgroups, $e_1 \in H_1$ and $e_2 \in H_2$. Therefore $(e_1, e_2) \in H$, so H contains the identity element.

If $(x_1, x_2) \in H$, then $(x_1, x_2)^{-1} = (x_1^{-1}, x_2^{-1}) \in H$ since $x_1^{-1} \in H_1$ and $x_2^{-1} \in H_2$ (again, since H_1 and H_2 are subgroups).

Thus H is a subgroup of $G_1 \times G_2$.

Section 3.3, Problem 10:

Let $n > 2$ be an integer, and let $X \in S_n \times S_n$ be the set $X = \{(\sigma, \tau) \mid \sigma(1) = \tau(1)\}$. Show that X is not a subgroup of $S_n \times S_n$.

Solution:

Consider $\sigma = (123)$ and $\tau = (12)$. Since $\sigma(1) = 2 = \tau(1)$, $(\sigma, \tau) \in X$. However, we will show that $(\sigma, \tau)^{-1} = (\sigma^{-1}, \tau^{1}) \notin X$. Indeed, $\sigma^{-1} = (132)$ and $\tau^{-1} = (12)$, so $\sigma^{-1}(1) = 3$ and $\tau^{-1}(1) = 2$, so $\sigma^{-1}(1) \neq \tau^{-1}(1)$. Thus X is not closed under the inverses, and therefore is not a subgroup of $S_n \times S_n$.