

### Practice problems for Test 3

The actual test will consist of 6 problems. You will have 1 hour to complete the test.

- Find the quotient and remainder when  $x^5 + 3x + 1$  is divided by  $x^2 + 2$  over  $\mathbb{Q}$ .
- Let  $f(x) = x^5 + 4x^4 + 6x^3 + 6x^2 + 5x + 2$  and  $g(x) = x^4 + 3x^2 + 3x + 6$ .
  - Find the greatest common divisor  $d(x)$  of  $f(x)$  and  $g(x)$  over  $\mathbb{Z}_7$ .
  - Find polynomials  $a(x)$  and  $b(x)$  in  $\mathbb{Z}_7[x]$  such that  $d(x) = a(x)f(x) + b(x)g(x)$ .
- Find the multiplicative inverse of  $[x + 4]$  in  $\mathbb{Z}_5 / \langle x^3 + x + 1 \rangle$ .
- Find all integer roots of  $x^4 + 4x^3 + 8x + 32 = 0$ .
- Find the irreducible factors of  $x^3 - 2$  over  $\mathbb{Z}$ ;  $\mathbb{Q}$ ;  $\mathbb{R}$ ;  $\mathbb{C}$ ;  $\mathbb{Z}_3$ .
- Find all irreducible polynomials of degree 3 over  $\mathbb{Z}_2$ .
- Use Eisenstein's criterion to show that the polynomial  $3x^4 + 30x - 60$  is irreducible over  $\mathbb{Q}$ .
- Let  $\mathbb{Z}_2(i) = \{a + bi \mid a, b \in \mathbb{Z}_2\}$  with addition and multiplication given by

$$(a + bi) + (c + di) = (a + c) + (b + d)i$$

$$(a + bi)(c + di) = (ac - bd) + (ad + bc)i$$

where  $a, b, c, d \in \mathbb{Z}_2$ . Is  $\mathbb{Z}_2(i)$  a commutative ring? Is it an integral domain? Is it a field? (Justify your answers: say which axioms are or are not satisfied, and why.)

- Find all units in
  - $\mathbb{Z}_6 \oplus \mathbb{Z}_8$
  - $\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$
  - $\mathbb{R} \oplus \mathbb{R}$
- Find all ring homomorphisms from  $\mathbb{Z}$  to  $\mathbb{R}$ .
  - Find all ring isomorphisms from  $\mathbb{Z}$  to  $\mathbb{R}$ .
- Which of the following elements are algebraic over  $\mathbb{Q}$ ? (As always, remember to justify your answers.)
  - $\sqrt[4]{5}$
  - $\sqrt[4]{5} + 1$
  - $e$
  - $e + 1$