Math 171 Solutions to homework problems Spring 2005

Section 1.3

1(f) Find the infimum and supremum of each of $E = \{x \in \mathbb{R} \mid x = \frac{1}{n} - (-1)^n$ for $n \in \mathbb{N}\}$.

First find the first few elements of E: $2, \frac{1}{2} - 1, \frac{1}{3} + 1, \frac{1}{4} - 1, \frac{1}{5} + 1, \frac{1}{6} - 1, \ldots$

It appears that sup $E = 2$ and inf $E = -1$. We use the definitions of sup E and inf E to prove these results.

For any $x \in E$ we have $x = \frac{1}{n} - (-1)^n = \frac{1}{n} + (-1)^{n+1} \leq 1 + 1 = 2$, so 2 is an upper bound of E. Since $2 \in E$, for any upper bound M of E we have $M \geq 2$. This means that 2 is the supremum of E.

For any $x \in E$ we have $x = \frac{1}{n} - (-1)^n = \frac{1}{n} + (-1)^{n+1} > 0 + (-1) = -1$, so -1 is a lower bound of E. Now we have to show that for any lower bound m of E, $m \leq -1$. Let $m > -1$, then $m + 1 > 0$, and $\frac{1}{m+1} > 0$. By Archimedean principle there exists $k \in \mathbb{N}$ such that $k > \frac{1}{m+1}$. If k is even let $n = k$. If k is odd let $n = k + 1$. So we have $n \in \mathbb{N}$ such that n is even and $n > \frac{1}{m+1}$. Then $\frac{1}{n} < m + 1$, therefore $\frac{1}{n} - (-1)^n = \frac{1}{n} - 1 < m$. Thus any number $m > -1$ is not a lower bound, so any lower bound is less that or equal to -1, which means that -1 is the infimum of E.

2 Show that if E is a nonempty bounded subset of $\mathbb{Z}$, then both sup E and inf E exist and belong to E.

For sup E, the proof of Theorem 1.21 can be applied here because nowhere in that proof we use that the numbers a, b, s, etc. are positive. For inf E, we have to either modify the proof for sup E or find another proof.

Suppose that $s = \sup E$ and apply the Approximation Property (Theorem 1.20) with $\epsilon = 1$ to choose an $a \in E$ such that $s - 1 < a \leq s$.
Case I. $s = a$. Then $s \in E$, and we are done.
Case II. $a < s$. Apply the Approximation Property again, this time with $\epsilon = s - a$, to choose a b such that $a < b \leq s$. Since both a and b are integers and $b > a$, $b \geq a + 1$ (the next integer after a). Adding 1 to the above inequality $s - 1 < a$ we have $s < a + 1$. So $s < a + 1 \leq b \leq s$, therefore $s < s$. Contradiction. Therefore this case is not possible.

Since E is bounded, there exist $m, M \in \mathbb{R}$ such that for any $x \in E$, $m \leq x \leq M$. Then $-M \leq -x \leq -m$, therefore $-E$ is bounded. By the completeness axiom, $-E$ has a supremum, and then by Theorem 1.28 E has an infimum. Moreover, by Theorem 1.28 $\sup(-E) = -\inf E$. Since $-E \subset \mathbb{Z}$, $\sup(-E) \in (-E)$. Then $\inf E = -\sup(-E) \in E$.

3 (Density of Irrationals) Prove that if $a < b$ are real numbers, then there is an irrational $\xi \in \mathbb{R}$ such that $a < \xi < b$.

Since $a < b$, $a - \sqrt{2} < b - \sqrt{2}$. By the density of rationals theorem there exists a rational number q such that $a - \sqrt{2} < q < b - \sqrt{2}$. Then $a < q + \sqrt{2} < b$.
The number $\xi = q + \sqrt{2}$ is irrational because if it were rational then $\sqrt{2} = \xi - q$ would be rational, but it is a very well known fact that $\sqrt{2}$ is irrational (ask me for a proof if you don't know it!).

5(a) (Approximation property for infima) By modifying the proof of Theorem 1.20, prove that if a set $E \subset \mathbb{R}$ has a finite infimum and $\epsilon > 0$ is any positive number, then there is a point $a \in E$ such that $\inf E + \epsilon > a \geq \inf E$.

Suppose that there exists $\epsilon > 0$ such that there is no point $a \in E$ satisfying
$\inf E \le a \le \inf E + \epsilon$. Then $\inf E + \epsilon$ is a lower bound of E. By definition of $\inf E$, any lower bound is less than or equal to $\inf E$, so we have $\inf E + \epsilon \le \inf E$. This implies that $\epsilon \le 0$. Contradiction.

5(b) Give a second proof of the Approximation Property for Infima by using Theorem 1.28.

By Theorem 1.28, $-\inf E = \sup (-E)$. By the Approximation Property for Suprema, for any $\epsilon > 0$ there exists a point $b \in (-E)$ such that $\sup (-E) - \epsilon < b \le \sup (-E)$. Let $a = -b$, then $b = -a$, $a \in E$ and $-\inf E - \epsilon < -a \le -\inf E$. Multiplying this inequality by -1 gives $\inf E + \epsilon > a \ge \inf E$.

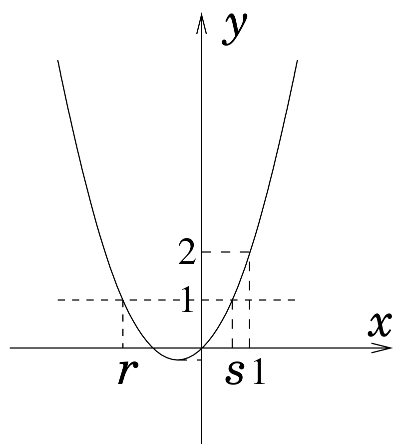
Section 1.4

3 Prove that the set $\{1, 3, \ldots\}$ is countable.

To show that the given set (let's call it E) is countable we need to construct a bijection between $\mathbb{N}$ and E. Let $f : \mathbb{N} \to E$ be given by $f(n) = 2n - 1$. This function is 1-1 because if $f(n_1) = f(n_2)$ then $2n_1 - 1 = 2n_2 - 1$, which implies $2n_1 = 2n_2$, and then $n_1 = n_2$. This function is onto because any odd integer is of the form $2k - 1$, and it is positive if and only if $k \ge 1$.

4(c) Find $f(E)$ and $f^{-1}(E)$ for $f(x) = x^2 + x$, $E = [-2, 1)$.

Sketch the graph of $f(x)$:



By definition, $f(E)$ is the set of all values of $f(x)$ for $x \in E$. We see from the graph that $f([-2, 1)) = \left[-\frac{1}{4}, 2\right]$ (the vertex of the parabola has x-coordinate $-\frac{1}{2}$, and it is easy to compute the y-coordinate).

The inverse image $f^{-1}(E)$ of E under f is the set of all $x \in E$ such that $f(x) \in E$ (see Definition 1.42). Again, from the graph we see that $f^{-1}([-2, 1))$ is the interval (r, s) where r and s are the roots of $f(x) = 1$. So we solve $x^2 + x = 1$ to find these roots.
$x^2 + x - 1 = 0$
$x = \dfrac{-1 \pm \sqrt{1+4}}{2}$, so $r = \dfrac{-1 - \sqrt{5}}{2}$ and $s = \dfrac{-1 + \sqrt{5}}{2}$. Answer: $\left(\dfrac{-1 - \sqrt{5}}{2}, \dfrac{-1 + \sqrt{5}}{2} \right)$.

Note: Another (purely algebraic) way to find the inverse image of $[-2, 1)$ under f is to solve $-2 \le x^2 + x < 1$.

7 Prove that $\left(\bigcap_{\alpha \in A} E_\alpha \right)^c = \bigcup_{\alpha \in A} E_\alpha^c$.

$x \in \left(\bigcap_{\alpha \in A} E_\alpha \right)^c$ iff $x \notin \bigcap_{\alpha \in A} E_\alpha$ iff $x \notin E_\alpha$ for some $\alpha \in A$ iff $x \in E_\alpha^c$ for some $\alpha \in A$ iff $x \in \bigcup_{\alpha \in A} E_\alpha^c$