Math 171 Solutions to homework problems Spring 2005

Section 2.1

1(b) Using the method of Example 2.2, prove that $2\left(1-\frac{1}{n}\right) -> 2$ as $n \to \infty$.

Given $\epsilon > 0$, choose $N > \frac{2}{\epsilon}$. Then for any $n \geq N$,

$$\left|2\left(1-\frac{1}{n}\right)-2\right| = \left|-\frac{2}{n}\right| = \frac{2}{n} \leq \frac{2}{N} < \epsilon.$$

6(a) Suppose that $\{x_n\}$ and $\{y_n\}$ converge to the same point. Prove that $x_n - y_n \to 0$ as $n \to \infty$.

Suppose that $\{x_n\}$ and $\{y_n\}$ converge to a. Then for any $\epsilon > 0$, choose N_1 such that $\forall n \geq N_1$, $|x_n - a| < \frac{\epsilon}{2}$, and choose N_2 such that $\forall n \geq N_2$, $|y_n - a| < \frac{\epsilon}{2}$. Let $N = max(N_1, N_2)$. Then $\forall n \geq N$ we have

$$|x_n - y_n| = |x_n - a + a - y_n| \leq |x_n - a| + |a - y_n| = |x_n - a| + |y_n - a| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

6(b) Prove that the sequence $\{n\}$ does not converge.

Suppose that the sequence $\{n\}$ converges. Every convergent sequence is bounded. Then $\{n\}$ is bounded, i.e. there exists a real number M such that for any $n \in \mathbb{N}$, $-M < n < M$. However, by Archimedean principle, for any real number M there exists a natural number n such that $n > M$. Contradiction. Therefore our assumption that $\{n\}$ converges was false.

6(c) Show that the converse of part (a) is false.

The converse of (a) is: if $\{x_n\}$ and $\{y_n\}$ are sequences such that $x_n - y_n \to 0$ as $n \to \infty$ then $\{x_n\}$ and $\{y_n\}$ converge to the same point.

Counterexample: Let $x_n = y_n = n$, then $x_n - y_n = 0 \to 0$, but the sequences $\{x_n\}$ and $\{y_n\}$ do not converge.

7(a) Let a be a fixed real number and define $\{x_n = a\}$ for $n \in \mathbb{N}$. Prove that the "constant" sequence x_n converges. (b) What does $\{x_n\}$ converge to?

The constant sequence is $a, a, a, \ldots$. We guess that this sequence converges to a. Proof:

given $\epsilon > 0$, let $N = 1$. Then $\forall n \geq N$ we have $|x_n - a| = |a - a| = 0 < \epsilon$. So $\lim_{n\to\infty} x_n = a$.

Section 2.2

5 Prove that given $x \in \mathbb{R}$ there is a sequence $r_n \in \mathbb{Q}$ such that $r_n \to x$ as $n \to \infty$.

By the density of rationals theorem, for any $n \in \mathbb{N}$ there exists a rational number r_n such that $x - \frac{1}{n} < r_n < x + \frac{1}{n}$. (That is, for each $n \in \mathbb{N}$ we choose such a rational number r_n, and so we get a sequence $\{r_n\}$.) Since $\lim_{n\to\infty}\left(x - \frac{1}{n}\right) = \lim_{n\to\infty}\left(x + \frac{1}{n}\right) = x$, by the Squeeze theorem $\lim_{n\to\infty} r_n = x$.

3 Prove that if $\{x_n\}$ and $\{y_n\}$ are convergent real sequences such that $y_n \neq 0$ and $\lim_{n\to\infty} y_n \neq 0$, then $\lim_{n\to\infty} \frac{x_n}{y_n} = \frac{\lim_{n\to\infty} x_n}{\lim_{n\to\infty} y_n}$.

Let $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y$. First we will prove that $\lim_{n\to\infty} \frac{1}{y_n} = \frac{1}{y}$. Let $\epsilon > 0$.

Since $y_n \to y$, $\exists N_1 \in \mathbb{N}$ such that $\forall n \geq N_1$ we have $|y_n - y| < \frac{|y|^2\epsilon}{2}$.

Also, $\exists N_2 \in \mathbb{N}$ such that $\forall n \geq N_2$ we have $|y_n - y| < \frac{|y|}{2}$. If y is positive this implies that $\frac{y}{2} < y_n < \frac{3y}{2}$. If y is negative then $\frac{3y}{2} < y_n < \frac{y}{2}$. In either case, $|y_n| > \frac{|y|}{2}$.

Let $N = max(N_1, N_2)$. Then $\forall n \geq N$ we have

$$\left|\frac{1}{y_n} - \frac{1}{y}\right| = \left|\frac{y - y_n}{y_n y}\right| = \frac{|y - y_n|}{|y_n||y|} = \frac{|y_n - y|}{|y_n||y|} < \frac{\frac{|y|^2\epsilon}{2}}{\frac{|y|}{2}|y|} = \epsilon.$$

Finally, $\lim_{n\to\infty} \frac{x_n}{y_n} = \lim_{n\to\infty} \left(x_n \cdot \frac{1}{y_n}\right) = \lim_{n\to\infty} x_n \cdot \lim_{n\to\infty} \frac{1}{y_n} = x \cdot \frac{1}{y} = \frac{x}{y}$.

Another approach is to show that $\lim_{n\to\infty} (x_n y - x y_n) = 0$ and $\left\{\frac{1}{y_n}\right\}$ is bounded, then
$\lim_{n\to\infty} \left(\frac{x_n}{y_n} - \frac{x}{y}\right) = \lim_{n\to\infty} \frac{x_n y - x y_n}{y_n y} = \lim_{n\to\infty} (x_n y - x y_n)\frac{1}{y}\frac{1}{y_n} = 0.$

9 Interpret a decimal expansion $0.a_1a_2\ldots$ as $0.a_1a_2\ldots = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{a_k}{10^k}$.

Prove that (a) $0.5 = 0.4999\ldots$ and (b) $1 = 0.999\ldots$.

The limit on the right is the limit of the following sequence:

$x_1 = \sum_{k=1}^{1} \frac{a_k}{10^k} = \frac{a_1}{10^1} = 0.a_1$

$x_2 = \sum_{k=1}^{2} \frac{a_k}{10^k} = \frac{a_1}{10^1} + \frac{a_2}{10^2} = 0.a_1a_2$

$x_3 = \sum_{k=1}^{3} \frac{a_k}{10^k} = \frac{a_1}{10^1} + \frac{a_2}{10^2} + \frac{a_3}{10^3} = 0.a_1a_2a_3$

$x_4 = \sum_{k=1}^{4} \frac{a_k}{10^k} = \frac{a_1}{10^1} + \frac{a_2}{10^2} + \frac{a_3}{10^3} + \frac{a_4}{10^4} = 0.a_1a_2a_3a_4$

which is the infinite decimal $0.a_1a_2a_3a_4\ldots$.

(a) If $a_1 = 4$ and $a_k = 9$ for $k \geq 2$, we have

$0.4999\ldots = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{a_k}{10^k} = \frac{4}{10} + \lim_{n\to\infty} \sum_{k=2}^{n} \frac{9}{10^k} = 0.4 + \frac{1}{10} \lim_{n\to\infty} \sum_{k=1}^{n-1} \frac{9}{10^k}$

$= 0.4 + \frac{1}{10} \lim_{n\to\infty} \sum_{k=1}^{n-1} \frac{10-1}{10^k}$ (use exercise 1(c) on p. 17) $= 0.4 + \frac{1}{10} \lim_{n\to\infty} \left(1 - \frac{1}{10^{n-1}}\right)$

(a proof of $\lim_{n\to\infty} \frac{1}{10^{n-1}} = 0$ is similar to ex. 2 on p. 36) $= 0.4 + \frac{1}{10} \cdot 1 = 0.5.$

(b) $0.9999\ldots = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{9}{10^k} = \lim_{n\to\infty} \frac{10-1}{10^k}$ (use exercise 1(c) on p. 17)

$= \lim_{n\to\infty} \left(1 - \frac{1}{10^n}\right) = 1.$