Section 3.3

1(b) Prove that there is at least one $x \in \mathbb{R}$ that satisfies $e^x = \cos x + 1$.

Rewrite the given equation as $e^x - \cos x - 1 = 0$. Let $f(x) = e^x - \cos x - 1$. This function is continuous on $\mathbb{R}$. $f(0) = e^0 - \cos(0) - 1 = -1$ and $f(\pi) = e^\pi - \cos(\pi) - 1 = e^\pi > 0$. By the intermediate value theorem there exists a number $c \in (0, \pi)$ such that $f(c) = 0$. Thus the equation has a root.

4 Suppose that f is a real-valued function of a real variable. If f is continuous at a with $f(a) < M$ for some $M \in \mathbb{R}$ prove that there is an open interval I containing a such that $f(x) < M$ for all $x \in I$.

Consider the function $g(x) = M - f(x)$. Then $g(x)$ is continuous and $g(a) = M - f(a) > 0$. By the sign preserving property $g(x) > \varepsilon$ for some $\varepsilon > 0$ on some open interval containing a. So $M - f(x) = g(x) > 0$, and therefore $f(x) < M$ on that interval.

10 If $f : \mathbb{R} \to \mathbb{R}$ is continuous and $\lim\limits_{x\to\infty} f(x) = \lim\limits_{x\to-\infty} f(x) = \infty$, prove that f has a minimum on $\mathbb{R}$, i.e. there is an $x_m \in \mathbb{R}$ such that $f(x_m) = \inf_{x\in\mathbb{R}} f(x)$.

Let a be any real number, and let $M = f(a)$.

Since $\lim\limits_{x\to\infty} f(x) = \infty$, there exists M_1 such that $x > M_1$ implies $f(x) > M$. Since $f(a) = M$, $a \leq M_1$.

Since $\lim\limits_{x\to-\infty} f(x) = \infty$, there exists M_2 such that $x < M_2$ implies $f(x) > M$. Since $f(a) = M$, $a \geq M_2$. So $a \in [M_2, M_1]$.

By the extreme value theorem f attains its minimum m on the interval $[M_2, M_1]$, i.e. there exists $x_m \in [M_2, M_1]$ such that $f(x) \geq f(x_m)$ for all $x \in [M_2, M_1]$.

For $x \notin [M_2, M_1]$ we have $f(x) > M = f(a) \geq f(x_m)$. Thus we have $f(x) \geq f(x_m)$ for all $x \in \mathbb{R}$, so x_m is an absolute minimum of f.

Section 3.4

2(c) Prove that any polinomial $f(x)$ is uniformly continuous on $(0, 1)$.

Last week (see problem 4(b) in 3.2) we proved that for any polynomial function $f(x)$ and any real number a, $\lim\limits_{x\to a} f(x) = f(a)$. By remark 3.20, $f(x)$ is continuous at every point a. In particular, it is continuous on $[0, 1]$. By theorem 3.39, it is uniformly continuous on $[0, 1]$. Then it is uniformly continuous on $(0, 1)$ (since it is a subset of $[0, 1]$).

5(a) Let I be a boundend interval. Prove that if $f : I \to \mathbb{R}$ is uniformly continuous on I, then f is bounded on I.

If I is closed, then f is bounded by theorem 3.26. If I is (a, b), $(a, b]$, or $[a, b)$, then since f is uniformly continuous on I, it is uniformly continuous on (a, b). Then by theorem 3.40 there exists a continuous function g on $[a, b]$ such that $g(x) = f(x)$ on (a, b). Moreover, if I includes a or b then $f = g$ at that point, so $f = g$ on I. By theorem 3.26 g is bounded on $[a, b]$. Therefore g is bounded on I, and so f is bounded on I.

3 5(b) Prove that (a) may be false if I is unbounded or if f is merely continuous.

Counterexample 1 (if I is unbounded). Let $f(x) = x$ on R. It is uniformly continuous, but not bounded.

Counterexample 2 (if f is merely continuous). Let $f(x) = \dfrac{1}{x}$ on $(0, 1)$. The interval is bounded, but f is not bounded.