Math 171 Solutions to homework problems Spring 2005

Section 4.1

1(b) For $f(x) = \frac{1}{x}$, $a \neq 0$, use Definition 4.1 directly to prove that $f'(a)$ exists.

$$f'(a) = \lim_{h\to 0} \frac{\frac{1}{a+h} - \frac{1}{a}}{h} = \lim_{h\to 0} \frac{\left(\frac{a-(a+h)}{a(a+h)}\right)}{h} = \lim_{h\to 0} \frac{-h}{ha(a+h)} = \lim_{h\to 0} \frac{-1}{a(a+h)} = -\frac{1}{a^2}$$

3 Let I be an open interval, $f : I \to \mathbb{R}$, and $c \in I$. The function f is said to have a local maximum at c if and only if there is a $\delta > 0$ such that $f(c) \geq f(x)$ holds for all $|x - c| < \delta$. (a) If f has a local maximum at c, prove that $\frac{f(c+h) - f(c)}{h} \leq 0$ and $\frac{f(c+H) - f(c)}{H} \geq 0$ for $h > 0$ and $H < 0$ sufficiently small.

Since f has a local maximum at c, $f(c) \geq f(c+h)$ for h sufficiently small. Therefore $f(c+h) - f(c) \leq 0$. For $h > 0$, we have $\frac{f(c+h) - f(c)}{h} \leq 0$.

Also, we can write $f(c) \geq f(c+H)$ for H sufficiently small. Therefore $f(c+H) - f(c) \leq 0$. For $H < 0$, we have $\frac{f(c+H) - f(c)}{H} \geq 0$.

3(b) If f is differentiable and has a local maximum at c, prove that $f'(c) = 0$.

Since f is differentiable, $f'(c) = \lim_{h\to 0} \frac{f(c+h) - f(c)}{h}$ exists. Therefore $\lim_{h\to 0^+} \frac{f(c+h) - f(c)}{h}$ and $\lim_{H\to 0^-} \frac{f(c+H) - f(c)}{H}$ exist and are equal. By the comparison theorem, part (a) implies that $\lim_{h\to 0^+} \frac{f(c+h) - f(c)}{h} \leq 0$ and $\lim_{H\to 0^-} \frac{f(c+H) - f(c)}{H} \geq 0$. Since these two limits must be equal, they are both equal to 0. Thus $f'(c) = 0$.

3(c) Make and prove analogous statements for local minima.

Definition. Let I be an open interval, $f : I \to \mathbb{R}$, and $c \in I$. The function f is said to have a local minimum at c if and only if there is a $\delta > 0$ such that $f(c) \leq f(x)$ holds for all $|x - c| < \delta$.
Statement a. If f has a local minimum at c, then $\frac{f(c+h) - f(c)}{h} \geq 0$ and $\frac{f(c+H) - f(c)}{H} \leq 0$ for $h > 0$ and $H < 0$ sufficiently small.
Statement b. If f is differentiable and has a local minimum at c, then $f'(c) = 0$.

Proof of a. Since f has a local minimum at c, $f(c) \leq f(c+h)$ for h sufficiently small. Therefore $f(c+h) - f(c) \geq 0$. For $h > 0$, we have $\frac{f(c+h) - f(c)}{h} \geq 0$.

Also, we can write $f(c) \leq f(c+H)$ for H sufficiently small. Therefore $f(c+H) - f(c) \geq 0$. For $H < 0$, we have $\frac{f(c+H) - f(c)}{H} \leq 0$.

Proof of b. Since f is differentiable, $\lim_{h\to 0} \frac{f(c+h) - f(c)}{h}$ exists. Therefore $\lim_{h\to 0^+} \frac{f(c+h) - f(c)}{h}$ and $\lim_{H\to 0^-} \frac{f(c+H) - f(c)}{H}$ exist and are equal. By the comparison theorem, part (a) implies that $\lim_{h\to 0^+} \frac{f(c+h) - f(c)}{h} \geq 0$ and $\lim_{H\to 0^-} \frac{f(c+H) - f(c)}{H} \leq 0$. Since these two limits must be equal, they are both equal to 0. Thus $f'(c) = 0$.

3(d) Show by example that the converses of the statements in parts (b) and (c) are false. Namely, find an f such that $f'(0) = 0$ but f has neither a local maximum nor a local minimum at 0.

Let $f(x) = x^3$. Then $f'(x) = 3x^2$, so $f'(0) = 0$. However, $f(x)$ is positive for positive x, and $f(x)$ is negative for negative x, so $f(x)$ has neither a local maximum nor a local minimum at 0.

Section 4.2

1(d) Let $f(x) = |x^3 + 2x^2 - x - 2|$. Find all x for which $f'(x)$ exists and find a formula for f'.

Let $p(x) = x^3 + 2x^2 - x - 2$, then $f(x) = |p(x)| = \begin{cases} p(x) & if\ p(x) \geq 0 \\ -p(x) & if\ p(x) < 0 \end{cases}$.

So we need to find the intevals where $p(x)$ is positive and intervals where it is negative.

$p(x) = (x^3 + 2x^2) - (x + 2) = x^2(x + 2) - (x + 2) = (x^2 - 1)(x + 2) = (x - 1)(x + 1)(x + 2)$, so $p(x) = 0$ at -2, -1, and 1. Then it is easy to check that it is positive on $(-2, -1)$ and $(1, +\infty)$, and negative on $(-\infty, -2)$ and $(-1, 1)$.

Since $p'(x) = 3x^2 + 4x - 1$ and $-p'(x) = -3x^2 - 4x + 1$, we have

$$f'(x) = \begin{cases} p'(x) & if\ p(x) > 0 \\ -p'(x) & if\ p(x) < 0 \end{cases} = \begin{cases} 3x^2 + 4x - 1 & if\ x \in (-2, -1) \cup (1, +\infty) \\ -3x^2 - 4x + 1 & if\ x \in (-\infty, -2) \cup (-1, 1) \end{cases}$$

Note: since $p'(x) \neq -p'(x)$ at -2, -1, and 1, we conclude that $f'(x)$ does not exist at these points. (Sketch graphs of $p(x)$ and $f(x)$!)

5 Suppose that f is differentiable at a and $f(a) \neq 0$.
(a) Show that for h sufficiently small, $f(a + h) \neq 0$.

Since $f(x)$ is differentiable at a, it is continuous at a.

If $f(a) > 0$, by the sign-preserving property $f(x) > 0$ for x sufficiently close to a, i.e. $f(a+h) > 0$ for h sufficiently small.

If $f(a) < 0$, consider $g(x) = -f(x)$. Then $g(x)$ is continuous at a and $g(a) > 0$. So by the sign-preserving property, $g(a + h) > 0$ for h sufficiently small. Therefore $f(a + h) < 0$ for h sufficiently small.

5(b) Using Definition 4.1 directly, prove that $\dfrac{1}{f(x)}$ is differentiable at $x = a$ and $\left(\dfrac{1}{f}\right)'(a) = -\dfrac{f'(a)}{f^2(a)}$.

$$\left(\frac{1}{f}\right)'(a) = \lim_{h\to 0} \frac{\frac{1}{f(a+h)} - \frac{1}{f(a)}}{h} = \lim_{h\to 0} \frac{\left(\frac{f(a)-f(a+h)}{f(a)f(a+h)}\right)}{h} = \lim_{h\to 0} \frac{f(a) - f(a+h)}{hf(a)f(a+h)}$$

$$= -\lim_{h\to 0} \left(\frac{f(a+h) - f(a)}{h} \cdot \frac{1}{f(a)f(a+h)}\right) = -f'(a) \cdot \frac{1}{f^2(a)} = -\frac{f'(a)}{f^2(a)}$$

6 Use Exercise 5 and the Product Rule to prove the Quotient Rule.

$$\left(\frac{f}{g}\right)'(a) = \left(f \cdot \frac{1}{g}\right)'(a) = f'(a) \cdot \frac{1}{g(a)} + f(a)\left(\frac{1}{g}\right)'(a) = f'(a) \cdot \frac{1}{g(a)} - \frac{f(a)g'(a)}{g^2(a)}$$

$$= \frac{f'(a)g(a) - f(a)g'(a)}{g^2(a)}$$