Math 171 Solutions to homework problems Spring 2005

Section 4.3

1(b) Evaluate the limit if it exists: $\lim_{x\to 0^+} \frac{\cos x - e^x}{\log(1+x^2)}$**.**

Since $\lim_{x\to 0^+} \cos x - e^x = \lim_{x\to 0^+} \log(1+x^2) = 0$*, we can use L'Hospital's Rule:*

$$\lim_{x\to 0^+} \frac{\cos x - e^x}{\log(1+x^2)} = \lim_{x\to 0^+} \frac{(\cos x - e^x)'}{(\log(1+x^2))'} = \lim_{x\to 0^+} \frac{-\sin x - e^x}{\left(\frac{2x}{1+x^2}\right)} = \lim_{x\to 0^+} \frac{-(\sin x + e^x)(1+x^2)}{2x} = -\infty$$

4(a) Using $(e^x)' = e^x$**,** $(\log x)' = \frac{1}{x}$**, and** $x^\alpha = e^{\alpha \log x}$**, show that** $(x^\alpha)' = \alpha x^{\alpha-1}$ **for all** $x > 0$**.**

$$(x^\alpha)' = \left(\left(e^{\log x}\right)^\alpha\right)' = \left(e^{\alpha \log x}\right)' = e^{\alpha \log x} \cdot \frac{\alpha}{x} = x^\alpha \cdot \frac{\alpha}{x} = \alpha x^{\alpha-1}$$

6 Let f **be differentiable on a nonempty, open interval** (a,b) **with** f' **bounded on** (a,b)**. Prove that** f **is uniformly continuous on** (a,b)**.**

Let $|f'| \le M$ *for some* $M > 0$*. Let* $\varepsilon > 0$ *be given. Set* $\delta = \frac{\varepsilon}{M}$*. Then by the Mean Value Theorem for any* $x_1 < x_2$ *in* (a,b) *such that* $|x_2 - x_1| < \delta$ *there exists* $c \in (x_1, x_2)$ *such that* $f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$*. Therefore* $|f(x_2) - f(x_1)| = |f'(c)| \cdot |x_2 - x_1| < M \cdot \delta = M \cdot \frac{\varepsilon}{M} = \varepsilon$*.*

8 Let f **be twice differentiable on** (a,b) **and let there be points** $x_1 < x_2 < x_3$ **in** (a,b) **such that** $f(x_1) > f(x_2)$ **and** $f(x_3) > f(x_2)$**. Prove that there is a point** $c \in (a,b)$ **such that** $f''(c) > 0$**.**

Since $x_1 < x_2$ *and* $f(x_1) > f(x_2)$*, by the Mean Value Theorem there exists a point* $a \in (x_1, x_2)$ *such that* $f'(a) = \frac{f(x_2) - f(x_1)}{x_2 - x_1} < 0$*.*

Since $x_2 < x_3$ *and* $f(x_3) > f(x_2)$*, by the Mean Value Theorem there exists a point* $b \in (x_2, x_3)$ *such that* $f'(b) = \frac{f(x_3) - f(x_2)}{x_3 - x_2} > 0$*.*

Then by the Mean Value Theorem (applied to $f'(x)$*) there exists a point* $c \in (a,b)$ *such that* $f''(c) = \frac{f'(b) - f'(a)}{b - a} > 0$*.*

Section 4.4

1(b) Find all $a \in \mathbb{R}$ **such that** $ax^2 + 3x + 5$ **is strictly increasing on the interval** $(1,2)$**.**

Let $f(x) = ax^2 + 3x + 5$*. Then* $f'(x) = 2ax + 3$ *is continuous. If* $f'(x)$ *is negative at some number, it is negative on some open interval, and then by the increasing/decreasing test,* $f(x)$ *is decreasing on that interval. Therefore if* $f(x) = ax^2 + 3x + 5$ *is strictly increasing then* $f'(x) \ge 0$ *on* $(1,2)$*. So* $2ax + 3 \ge 0$ *on* $(1,2)$*. Since* $f'(x) = 2ax + 3$ *is continuous,* $f'(2) \ge 0$*. So we have* $4a + 3 \ge 0$*, so* $a \ge -\frac{3}{4}$*.*

Conversely, if $a \ge -\frac{3}{4}$*, then* $2ax + 3 > 0$ *on* $(1,2)$*, and then* $f(x)$ *is strictly increasing on* $(1,2)$*.*

Another solution (not using calculus): if $a = 0$, then $f(x) = 3x + 5$ is strictly increasing everywhere.

If $a \neq 0$, then the graph of this function is a parabola. If $a > 0$ then the parabola opens upward, and if $a < 0$ then the parabola opens downward. Recall that the vertex of the parabola $y = ax^2 + bx + c$ has x-coordinate $-\dfrac{b}{2a}$, so for our function it is $-\dfrac{3}{2a}$. Now, $f(x) = ax^2 + 3x + 5$ is strictly increasing on $(1, 2)$ if either $a > 0$ and $-\dfrac{3}{2a} \leq 1$ (i.e. the vertex of the parabola is to the left of the interval), or $a < 0$ and $-\dfrac{3}{2a} \geq 2$ (i.e. the vertex of the parabola is to the right of the interval). Solving these inequalities, and adding the solution $a = 0$, gives $\left[-\dfrac{3}{4}, +\infty\right)$.

2 Let f and g be 1-1 and continuous on $\mathbb{R}$. If $f(0) = 2$, $g(1) = 2$, $f'(0) = \pi$, and $g'(1) = e$, compute the following derivatives.
(a) $(f^{-1})'(2)$**.**

$(f^{-1})'(2) = \dfrac{1}{f'(f^{-1}(2))} = \dfrac{1}{f'(0)} = \dfrac{1}{\pi}$.

(b) $(g^{-1})'(2)$**.**

$(g^{-1})'(2) = \dfrac{1}{g'(g^{-1}(2))} = \dfrac{1}{g'(1)} = \dfrac{1}{e}$.

(c) $(f^{-1} \cdot g^{-1})'(2)$**.**

$(f^{-1} \cdot g^{-1})'(2) = f^{-1}(2)(g^{-1})'(2) + (f^{-1})'(2)g^{-1}(2) = 0 \cdot \dfrac{1}{e} + \dfrac{1}{\pi} \cdot 1 = \dfrac{1}{\pi}$.

4 Using the Inverse Function Theorem, prove that $(\arcsin x)' = \dfrac{1}{\sqrt{1 - x^2}}$ **for** $x \in (-1, 1)$ **and** $(\arctan x)' = \dfrac{1}{1 + x^2}$ **for** $x \in (-\infty, \infty)$.

Since $\arcsin x = \sin^{-1} x$, $\arcsin^{-1} x = \sin x$, *so by the Inverse Function Theorem* $(\arcsin x)' = \dfrac{1}{\sin'(\arcsin x)} = \dfrac{1}{\cos(\arcsin x)}$.

Let $y = \arcsin x$, *then* $\sin y = x$, $\sin^2 y = x^2$, $\cos^2 y = 1 - \sin^2 y = 1 - x^2$, *so* $\cos y = \pm\sqrt{1 - x^2}$. *Since* $-\dfrac{\pi}{2} \leq \arcsin x = y \leq \dfrac{\pi}{2}$, $\cos y \geq 0$, *so* $\cos y = \sqrt{1 - x^2}$.
Thus $\cos(\arcsin x) = \cos y = \sqrt{1 - x^2}$, *and* $(\arcsin x)' = \dfrac{1}{\sqrt{1 - x^2}}$.

Similarly for $\arctan x$*:* $(\arctan x)' = \dfrac{1}{\tan'(\arctan x)} = \dfrac{1}{\sec^2(\arctan x)}$.

Let $y = \arctan x$, *then* $\tan y = x$, $\sec^2 y = 1 + \tan^2 y = 1 + x^2$. *Then* $(\arctan x)' = \dfrac{1}{1 + x^2}$.