MATH 171
Test 1 - Solutions
February 28, 2005

1. Give the definition of a Cauchy sequence. *A sequence $\{x_n\}$ is called Cauchy if for any $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that for any $n, m \geq N$, $|x_n - x_m| < \epsilon$.*

2. State the Well-ordering Principle. *Every nonempty subset of $\mathbb{N}$ has a least element.*

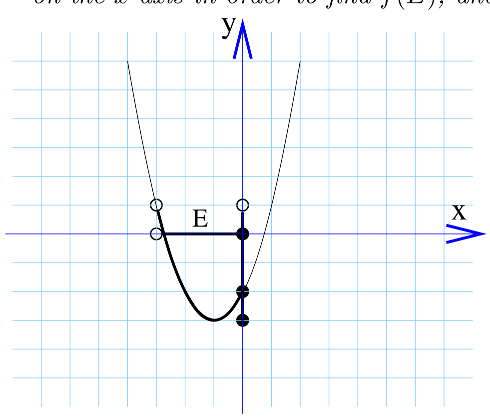
3. State and prove the Approximation Property for Suprema.

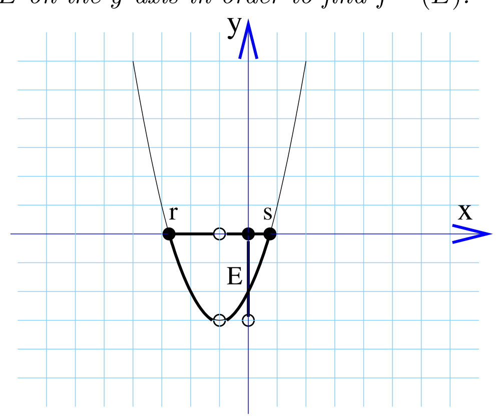
   Let E be a subset of $\mathbb{R}$ that has a supremum. Then for any $\epsilon > 0$ there exists $a \in E$ such that $\sup E - \epsilon < a \leq \sup E$.

   Proof. Suppose that the statement is false, i.e. there exists an $\epsilon > 0$ such that no point $a \in E$ satisfies $\sup E - \epsilon < a \leq \sup E$. Then for all $a \in E$, $a \leq \sup E - \epsilon$. Then $\sup E - \epsilon$ is an upper bound of E. Since any upper bound of E is greater than or equal to $\sup E$, $\sup E - \epsilon \geq \sup E$, so $0 \geq \epsilon$. This contradicts to the statement that $\epsilon > 0$.

4. Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = (x+1)^2 - 3$ and let $E = (-3, 0]$. Find $f(E)$ and $f^{-1}(E)$. (Explain how you find these!)

   Sketch the graph of $f(x)$. Actually, it is convenient to have two separate graphs, and show E on the x-axis in order to find $f(E)$, and show E on the y-axis in order to find $f^{-1}(E)$:





   From the above graphs, we see that

   $f(E) = \{y \in \mathbb{R} | y = f(x)$ for some $x \in E\} = [-3, 1)$, and

   $f^{-1}(E) = \{x \in \mathbb{R} | f(x) = y$ for some $y \in E\} = [r_1, -1) \cup (-1, r_2]$ where r and s are the roots of the equation $(x+1)^2 - 3 = 0$. Solving this equation gives: $(x+1)^2 = 3$, $x + 1 = \pm\sqrt{3}$, so $r = -\sqrt{3} - 1$ and $s = \sqrt{3} - 1$. Therefore we have $f^{-1}(E) = [-\sqrt{3} - 1, -1) \cup (-1, \sqrt{3} - 1]$.

5. Prove that for all $n \in \mathbb{N}$, $1+2+3+\ldots+(n-2)+(n-1)+n+(n-1)+(n-2)+\ldots+3+2+1 = n^2$.

   Proof by induction. Basis step: if $n = 1$, the formula becomes $1 = 1^2$ which is true.

   Inductive step. Assume the formula holds for $n = k$, i.e.

$$1 + 2 + 3 + \ldots + (k-2) + (k-1) + k + (k-1) + (k-2) + \ldots + 3 + 2 + 1 = k^2.$$

   We want to show that the formula holds for $n = k+1$, i.e.

$$1 + 2 + 3 + \ldots + (k-1) + k + (k+1) + k + (k-1) + \ldots + 3 + 2 + 1 = (k+1)^2.$$

   Adding $(k+1) + k$ to both sides of

$$1 + 2 + 3 + \ldots + (k-2) + (k-1) + k + (k-1) + (k-2) + \ldots + 3 + 2 + 1 = k^2,$$

   we have:

   $1 + 2 + 3 + \ldots + (k-2) + (k-1) + k + (k+1) + k + (k-1) + (k-2) + \ldots + 3 + 2 + 1 = k^2 + (k+1) + k = k^2 + 2k + 1 = (k+1)^2$.

6. (**For extra credit**, 10 points) Prove or disprove each of the following statements:

(a) If $\lim_{x \to a} f(x) = L$ then $\lim_{x \to a} |f(x)| = |L|$.

This statement is true.

If $L = 0$, *then for any* $\epsilon > 0$ *there exists* $\delta > 0$ *such that* $0 < |x - a| < \delta$ *implies* $|f(x) - 0| < \epsilon$, *which implies* $||f(x)| - 0| < \epsilon$, *so* $\lim_{x \to a} |f(x)| = 0$.

Now consider $L \neq 0$. *Given* $\epsilon > 0$, *let* $\epsilon_1 = min\left(\epsilon, \frac{|L|}{2}\right)$. *Since* $\lim_{x \to a} f(x) = L$, *there exists* $\delta > 0$ *such that* $0 < |x - a| < \delta$ *implies* $|f(x) - L| < \epsilon_1$, *i.e.* $L - \epsilon_1 < f(x) < L + \epsilon_1$. *Since* $\epsilon_1 \leq \frac{|L|}{2}$, *the numbers* $L - \epsilon_1$, L, *and* $L + \epsilon_1$ *are either all positive or all negative.*

Case I. The numbers $L - \epsilon_1$, L, *and* $L + \epsilon_1$ *are all positive. Then* $f(x)$ *is also positive for* $0 < |x - a| < \delta$, *and we have* $|L| - \epsilon_1 < |f(x)| < |L| + \epsilon_1$. *Since* $\epsilon_1 \leq \epsilon$, *we have* $|L| - \epsilon < |f(x)| < |L| + \epsilon$. *Therefore* $\lim_{x \to a} |f(x)| = |L|$.

Case II. The numbers $L - \epsilon_1$, L, *and* $L + \epsilon_1$ *are all negative. Then* $f(x)$ *is also negative for* $0 < |x - a| < \delta$, *and we have* $-|L| - \epsilon_1 < -|f(x)| < -|L| + \epsilon_1$ *which implies* $|L| - \epsilon_1 < |f(x)| < |L| + \epsilon_1$. *Again, since* $\epsilon_1 \leq \epsilon$, *we have* $|L| - \epsilon < |f(x)| < |L| + \epsilon$. *Therefore* $\lim_{x \to a} |f(x)| = |L|$.

(b) If $\lim_{x \to a} |f(x)| = |L|$ then $\lim_{x \to a} f(x) = L$ or $\lim_{x \to a} f(x) = -L$.

This statement is false. Counterexample: let $f(x) = \begin{cases} 1, & \text{if } x \text{ is rational} \\ -1, & \text{if } x \text{ is irrational} \end{cases}$

Then $\lim_{x \to 0} |f(x)| = \lim_{x \to 0} 1 = 1$, *but* $\lim_{x \to 0} f(x)$ *does not exist.*