MATH 171
Test 2 - Solutions

April 8, 2005

1. Give the definition of $\lim\limits_{x\to\infty} f(x) = L$.

   Let f be defined on some interval $(c, +\infty)$. We say that $\lim\limits_{x\to\infty} f(x) = L$ if for any $\varepsilon > 0$ there exists an $M \in \mathbb{R}$ such that $x > M$ implies $|f(x) - L| < \varepsilon$.

2. State Rolle's theorem.

   Let $a < b$. If $f(x)$ is continuous on $[a, b]$ and differentiable on (a, b), and $f(a) = f(b)$, then there exists a point $c \in (a, b)$ such that $f'(c) = 0$.

3. State and prove the sign-preserving property.

   Let I be an open (nondegenerate) interval. If $f(x)$ is continuous at a point $a \in I$ and $f(a) > 0$, then there exist positive numbers ε and δ such that for $x \in I$, $|x - a| < \delta$ implies $f(x) > \varepsilon$.

   Proof. *Let $\varepsilon = \dfrac{f(a)}{2}$. Since $f(x)$ is continuous at a, there exists $\delta > 0$ such that $|x - a| < \delta$ implies $|f(x) - f(a)| < \varepsilon$. Then $|f(x) - f(a)| < \dfrac{f(a)}{2}$, so $-\dfrac{f(a)}{2} < f(x) - f(a) < \dfrac{f(a)}{2}$. Adding $f(a)$ gives $\dfrac{f(a)}{2} < f(x) < \dfrac{3f(a)}{2}$, so we have $f(x) > \dfrac{f(a)}{2} = \varepsilon$.*

4. Find all $a \in \mathbb{R}$ such that $f(x) = \dfrac{ax + 2}{x + 1}$ is strictly increasing on $(1, 2)$.

   $f'(x) = \dfrac{a(x+1) - (ax+2)}{(x+1)^2} = \dfrac{a-2}{(x+1)^2}$.

   If $f(x)$ is strictly increasing then $f'(x) \geq 0$, so we have $a - 2 \geq 0$, or $a \geq 2$.

   If $a > 2$ then we have $f'(x) > 0$, so $f(x)$ is strictly increasing in its domain (which contains the interval $(1, 2)$).

   If $a = 2$, $f(x) = \dfrac{2x + 2}{x + 1} = 2$ is not strictly increasing.

   Answer: $a > 2$.

5. Let $f(x)$ and $g(x)$ be uniformly continuous on $\mathbb{R}$. Prove that $(f+g)(x)$ is uniformly continuous on $\mathbb{R}$.

   Let $\varepsilon > 0$.

   Since $f(x)$ is uniformly continuous on $\mathbb{R}$, there exists $\delta_1 > 0$ such that for any $x_1, x_2 \in \mathbb{R}$, $|x_1 - x_2| < \delta_1$ implies $|f(x_1) - f(x_2)| < \dfrac{\varepsilon}{2}$.

   Since $g(x)$ is uniformly continuous on $\mathbb{R}$, there exists $\delta_2 > 0$ such that for any $x_1, x_2 \in \mathbb{R}$, $|x_1 - x_2| < \delta_2$ implies $|g(x_1) - g(x_2)| < \dfrac{\varepsilon}{2}$.

   Let $\delta = \min(\delta_1, \delta_2)$. Then for any $x_1, x_2 \in \mathbb{R}$ such that $|x_1 - x_2| < \delta$ we have

   $|(f+g)(x_1) - (f+g)(x_2)| = |f(x_1) + g(x_1) - f(x_2) - g(x_2)| = |f(x_1) - f(x_2) + g(x_1) - g(x_2)|$

   $\leq |f(x_1) - f(x_2)| + |g(x_1) - g(x_2)| < \dfrac{\varepsilon}{2} + \dfrac{\varepsilon}{2} = \varepsilon.$

   Thus $(f+g)(x)$ is uniformly continuous on $\mathbb{R}$.

6. Prove or disprove each of the following statements:

   (a) If a function is continuously differentiable on $\mathbb{R}$ then it is twice differentiable on $\mathbb{R}$.

   The statement is false. Let $f(x) = \begin{cases} 0 & \text{if } x < 0 \\ x^2 & \text{if } x \geq 0 \end{cases}$. Then $f'(x) = \begin{cases} 0 & \text{if } x < 0 \\ 2x & \text{if } x \geq 0 \end{cases}$ is continuous but not differentiable.

   (b) If a function is continuously differentiable 100 times on $\mathbb{R}$ then it is differentiable 101 times on $\mathbb{R}$.

   The statement is false. Let $f(x) = \begin{cases} 0 & \text{if } x < 0 \\ x^{101} & \text{if } x \geq 0 \end{cases}$. Then $f^{(100)}(x) = \begin{cases} 0 & \text{if } x < 0 \\ 101!x & \text{if } x \geq 0 \end{cases}$ is continuous but not differentiable.