MATH 171
Test 3 - Solutions

May 9, 2005

1. Give the definition of an integrable function.

   A function $f : [a, b] \to \mathbb{R}$ *is called integrable on* $[a, b]$ *if* f *is bounded on* $[a, b]$ *and for any* $\varepsilon > 0$ *there is a partition* P *of* $[a, b]$ *such that* $U(f, P) - L(f, P) < \varepsilon$.

2. State the Fundamental Theorem of Calculus.

   Let $f : [a, b] \to \mathbb{R}$.

   (i) If f *continuous on* $[a, b]$ *and* $F(x) = \int_a^x f(t)dt$, *then* F *is continuously differentiable on* $[a, b]$ *and* $F'(x) = f(x)$.

   (ii) If f *is differentiable on* $[a, b]$ *and* f' *is integrable on* $[a, b]$, *then* $\int_a^x f'(t)dt = f(x) - f(a)$

3. Prove that the harmonic series diverges.

   Since $\frac{1}{k} \geq \frac{1}{x}$ *for* $x \in [k, k+1]$, $s_n = \sum_{k=1}^{n} \frac{1}{k} \geq \sum_{k=1}^{n} \int_k^{k+1} \frac{1}{x}dx = \int_1^{n+1} \frac{1}{x}dx = \ln(n+1) - \ln(1) = \ln(n+1) \to +\infty$ *as* $n \to +\infty$.

   Therefore the series $\sum_{k=1}^{\infty} \frac{1}{k}$ *diverges.*

4. Evaluate the integral: $\int_1^{\infty} \frac{x}{(x^2+1)^3}dx$.

   Let $u = x^2 + 1$, *then* $du = 2xdx$, $\frac{1}{2}du = xdx$, *so*

   $$\int_1^{\infty} \frac{x}{(x^2+1)^3}dx = \lim_{t\to+\infty} \int_1^t \frac{x}{(x^2+1)^3}dx = \lim_{t\to+\infty} \left(\frac{1}{2} \int_2^{t^2+1} \frac{1}{u^3}du \right) = \frac{1}{2} \lim_{t\to+\infty} \left(\frac{1}{-2u^2} \Big|_2^{t^2+1} \right) = \frac{1}{2} \lim_{t\to+\infty} \left(\frac{1}{-2(t^2+1)^2} - \frac{1}{-8} \right) = \frac{1}{16}$$

5. (a) Prove that if $\sum_{k=1}^{\infty} a_k$ converges, then its partial sums s_n are bounded.

   If $\sum_{k=1}^{\infty} a_k$ *converges then the sequence of its partial sums* $\{s_n\}$ *converges. Since every convergent sequence is bounded (Theorem 2.8),* $\{s_n\}$ *is bounded.*

   (b) Show that the converse of part (a) is false. Namely, she that a series $\sum_{k=1}^{\infty} a_k$ may have bounded partial sums and still diverge.

   Let $a_k = (-1)^k$. *Then the sequence of partial sums of* $\sum_{k=1}^{\infty} a_k$ *is* $\{-1, 0, -1, 0, \ldots\}$. *It is bounded but divergent, so the series diverges.*