Math 75A **Maria Nogin**

Practice test 2 - Solutions

Multiple choice questions: circle the correct answer

1. Solve for x: $\log_{\frac{1}{2}} x = 3$.

   A. 6 **B.** $\frac{1}{6}$ **C.** 8 (**D.**) $\frac{1}{8}$
   E. None of the above

   (because $\left(\frac{1}{2}\right)^3 = \frac{1}{8}$)

2. How many vertical asymptotes does the curve $y = \dfrac{x+1}{x(x+2)(x+3)}$ have?

   A. 0 **B.** 1 **C.** 2 (**D.**) 3 **E.** 4

   ($x = 0$, $x = -2$, and $x = -3$)

3. $\displaystyle\lim_{x\to 2} \frac{5}{x-2} =$

   A. 0 **B.** 5 **C.** ∞ **D.** $-\infty$ (**E.**) Does not exist

   (because the limits from the right and from the left are not equal)

4. $\displaystyle\lim_{x\to -\infty} \frac{x+2}{3x+4} =$

   A. 1 **B.** $\frac{1}{2}$ (**C.**) $\frac{1}{3}$ **D.** 0 **E.** Does not exist

   (divide the numerator and denominator by x)

5. The function $f(x) = \begin{cases} -x-1 & \text{if } x < -1 \\ 0 & \text{if } -1 \le x \le 1 \\ x & \text{if } x > 1 \end{cases}$ is

   A. continuous everywhere
   B. continuous at 1 but discontinuous at -1
   (**C.**) continuous at -1 but discontinuous at 1
   D. continuous at all points except for 1 and -1
   E. discontinuous everywhere

   (because there is a break in the graph at 1 but not at -1)

6. Find the rate of change of $y = 3x + 5$ at $x = 4$.

   (**A.**) 3 **B.** 4 **C.** 5 **D.** 17
   E. None of the above

   (the rate of change is $y'(4)$, the slope of the graph, which is 3)

7. Find the equation of the line tangent to the curve $y = x^2$ at $(2, 4)$.

   A. $y = 4x$ (**B.**) $y = 4x - 4$ **C.** $y = 4x + 4$ **D.** $y = -4x$ **E.** $y = -4x - 4$

   (first find the slope of the tangent line: $y'(2)$; then use $y - 4 = y'(2)(x - 2)$)

Regular problems: show all your work

8. Solve the following equations:

   (a) $\ln(5x-2)=3$
   $e^{\ln(5x-2)}=e^3$
   $5x-2=e^3$
   $5x=e^3+2$
   $x=\dfrac{e^3+2}{5}$

   (b) $e^{3t+1}=100$
   $\ln(e^{3t+1})=\ln 100$
   $3t+1=\ln 100$
   $3t=\ln 100-1$
   $t=\dfrac{\ln 100-1}{3}$

   (c) $\log_2 t+\log_2(t+1)=1$
   $\log_2(t(t+1))=1$
   $2^{\log_2(t(t+1))}=2^1$
   $t(t+1)=2$
   $t^2+t-2=0$
   $(t+2)(t-1)=0$
   $t=-2,\ t=1$
   Howerver, $\log_2 t$ is undefined for negative t, therefore we disregard the root $t=-2$.
   Answer: $t=1$

   (d) $10^{4x+1}=300$
   $\log_{10}(10^{4x+1})=\log_{10}300$
   $4x+1=\log_{10}300$
   $4x=\log_{10}300-1$
   $x=\dfrac{\log_{10}300-1}{4}$

9. Evaluate the limits:

   (a) $\lim_{x\to 5}(7x-25)=7\cdot 5-25=10$

   (b) $\lim_{x\to -1}\dfrac{x^3+x^2}{x^2+3x+2}=\lim_{x\to -1}\dfrac{x^2(x+1)}{(x+1)(x+2)}=\lim_{x\to -1}\dfrac{x^2}{x+2}=1$

   (c) $\lim_{x\to 0}\dfrac{3-\sqrt{9+x}}{x}=\lim_{x\to 0}\dfrac{(3-\sqrt{9+x})(3+\sqrt{9+x})}{x(3+\sqrt{9+x})}=\lim_{x\to 0}\dfrac{3^2-(\sqrt{9+x})^2}{x(3+\sqrt{9+x})}=$
   $\lim_{x\to 0}\dfrac{9-(9+x)}{x(3+\sqrt{9+x})}=\lim_{x\to 0}\dfrac{-x}{x(3+\sqrt{9+x})}=\lim_{x\to 0}\dfrac{-1}{3+\sqrt{9+x}}=-\dfrac{1}{6}$

   (d) $\lim_{x\to 2^+}\dfrac{x^3-2}{x^2-x-2}=\lim_{x\to 2^+}\dfrac{x^3-2}{(x-2)(x+1)}\left[\dfrac{\text{pos.}}{(\text{small pos.})(\text{pos.})}\right]=+\infty$

   (e) $\lim_{x\to 2^-}\dfrac{x^3-2}{x^2-x-2}=\lim_{x\to 2^+}\dfrac{x^3-2}{(x-2)(x+1)}\left[\dfrac{\text{pos.}}{(\text{small neg.})(\text{pos.})}\right]=-\infty$

   (f) $\lim_{x\to 2}\dfrac{x^3-2}{x^2-x-2}$ DNE because the limits in (d) and (e) are not equal

(g) $\lim_{x\to 0} x^4 \cos\left(\frac{1}{x}\right) = 0$ by the squeeze theorem since $-x^4 \le x^4 \cos\left(\frac{1}{x}\right) \le x^4$ and $\lim_{x\to 0}(-x^4) = \lim_{x\to 0}(x^4) = 0$.

(h) $\lim_{x\to\infty} \frac{5x^3 - x - 3}{4x^3 + 3x^2 - 3} = \lim_{x\to\infty} \frac{5 - \frac{1}{x^2} - \frac{3}{x^3}}{4 + \frac{3}{x} - \frac{3}{x^3}} = \frac{5}{4}$

(i) $\lim_{x\to -\infty} \frac{5x^2 - x - 3}{4x^3 + 3x^2 - 3} = \lim_{x\to\infty} \frac{\frac{5}{x} - \frac{1}{x^2} - \frac{3}{x^3}}{4 + \frac{3}{x} - \frac{3}{x^3}} = \frac{0}{4} = 0$

(j) $\lim_{x\to\infty} \frac{5x^3 - x - 3}{4x^2 + 3x - 3} = \lim_{x\to\infty} \frac{5x - \frac{1}{x} - \frac{3}{x^2}}{4 + \frac{3}{x} - \frac{3}{x^2}} = \infty$

(k) $\lim_{x\to\infty} \frac{\sqrt{4x^2+5}}{3x-3} = \lim_{x\to\infty} \frac{\frac{\sqrt{4x^2+5}}{x}}{3 - \frac{3}{x}} = \lim_{x\to\infty} \frac{\frac{\sqrt{4x^2+5}}{\sqrt{x^2}}}{3 - \frac{3}{x}} = \lim_{x\to\infty} \frac{\sqrt{\frac{4x^2+5}{x^2}}}{3 - \frac{3}{x}} = \lim_{x\to\infty} \frac{\sqrt{4 + \frac{5}{x^2}}}{3 - \frac{3}{x}} = \frac{2}{3}$

(l) $\lim_{x\to -\infty} \frac{\sqrt{4x^2+5}}{3x-3} = \lim_{x\to -\infty} \frac{\frac{\sqrt{4x^2+5}}{x}}{3 - \frac{3}{x}} = \lim_{x\to -\infty} \frac{\frac{\sqrt{4x^2+5}}{-\sqrt{x^2}}}{3 - \frac{3}{x}} = \lim_{x\to -\infty} \frac{-\sqrt{\frac{4x^2+5}{x^2}}}{3 - \frac{3}{x}} = \lim_{x\to -\infty} \frac{-\sqrt{4 + \frac{5}{x^2}}}{3 - \frac{3}{x}} = -\frac{2}{3}$

(m) $\lim_{x\to\infty} (3 - x + 2x^2 - 5x^3) = \lim_{x\to\infty} x^3 \left(\frac{3}{x^3} - \frac{1}{x^2} + \frac{2}{x} - 5\right) = -\infty$

(n) $\lim_{x\to -\infty} (3 - x + 2x^2 - 5x^3) = \lim_{x\to -\infty} x^3 \left(\frac{3}{x^3} - \frac{1}{x^2} + \frac{2}{x} - 5\right) = \infty$

10. Show that the equation $x^5 - 4x + 2 = 0$ has at least one solution in the interval $(1, 2)$.

    Let $f(x) = x^5 - 4x + 2$. Then $f(x)$ is a continuous function with $f(1) = -1 < 0$ and $f(2) = 26 > 0$. By the intermediate value theorem, there is a point c between 1 and 2 such that $f(c) = 0$.

11. Find all values of c such that the function $f(x)$ is continuous everywhere.

    (a) $f(x) = \begin{cases} cx & \text{if} \quad x \ge 2 \\ 5 - x & \text{if} \quad x < 2 \end{cases}$

    Since linear functions are continuous everywhere, $f(x)$ is continuous at all poits except possibly at 2. It is continuous at 2 if and only if the functions cx and $5 - x$ agree at 2 (that is, they have the same value at 2. The graph of $f(x)$ then has no jump at 2.) So we set the values of cx and $5 - x$ at 2 equal:

    $c \cdot 2 = 5 - 2$

    $2c = 3$

    $c = \frac{3}{2}$

    (b) $f(x) = \begin{cases} x^2 & \text{if} \quad x \le c \\ x^3 & \text{if} \quad x > c \end{cases}$

    Since polynomial functions are continuous everywhere, $f(x)$ is continuous at all poits except possibly at c. It is continuous at c if and only if the values of x^2 and x^3 agree at c, i.e.

    $c^2 = c^3$

    $c^2 - c^3 = 0$

    $c^2(1 - c) = 0$

    $c = 0$ or $c = 1$.

12. Find the vertical and horizontal asymptotes of $f(x) = \dfrac{(x+2)(3x-4)}{(x-5)(x+7)}$.

Since rational functions are continuous in their domains, $f(x)$ can have vertical asymptotes only at 5 and -7 (where it is undefined). Check the limits of $f(x)$ as x approaches 5 and -7:

$$\lim_{x\to 5^+} \frac{(x+2)(3x-4)}{(x-5)(x+7)} \left[\frac{(\text{pos.})(\text{pos.})}{(\text{small pos.})(\text{pos.})}\right] = +\infty$$

$$\lim_{x\to -7^+} \frac{(x+2)(3x-4)}{(x-5)(x+7)} \left[\frac{(\text{neg.})(\text{neg.})}{(\text{neg.})(\text{small pos.})}\right] = -\infty$$

Since the limits are infinite, $f(x)$ has vertical asymptotes $x = 5$ and $x = -7$.

To find the horizontal asymptotes, we find the limits at infinity and negative infinity:

$$\lim_{x\to\infty} \frac{(x+2)(3x-4)}{(x-5)(x+7)} = \lim_{x\to\infty} \frac{\frac{(x+2)}{x}\frac{(3x-4)}{x}}{\frac{(x-5)}{x}\frac{(x+7)}{x}} = \lim_{x\to\infty} \frac{(1+\frac{2}{x})(3-\frac{4}{x})}{(1-\frac{5}{x})(1+\frac{7}{x})} = \frac{3}{1} = 3$$

$$\lim_{x\to-\infty} \frac{(x+2)(3x-4)}{(x-5)(x+7)} = \lim_{x\to-\infty} \frac{\frac{(x+2)}{x}\frac{(3x-4)}{x}}{\frac{(x-5)}{x}\frac{(x+7)}{x}} = \lim_{x\to-\infty} \frac{(1+\frac{2}{x})(3-\frac{4}{x})}{(1-\frac{5}{x})(1+\frac{7}{x})} = \frac{3}{1} = 3$$

Thus $y = 3$ is the only horizontal asymptotes.

13. Find the following derivatives:

(a) $f'(1)$ if $f(x) = 5$

$f'(1) = 0$ because the graph of $f(x)$ is a horizontal line, and its slope at 5 (as well as at any other point) is 0.

(b) $f'(2)$ if $f(x) = 7x - 3$

$f'(2) = 7$ because the graph of $f(x)$ is a line with slope 7. In particular, its slope at 2 is equal to 7.

(c) $f'(3)$ if $f(s) = s^2 + 5s - 6$

$$f'(3) = \lim_{h\to 0} \frac{f(3+h) - f(3)}{h} = \lim_{h\to 0} \frac{(3+h)^2 + 5(3+h) - 6 - 18}{h} =$$
$$\lim_{h\to 0} \frac{9 + 6h + h^2 + 15 + 5h - 24}{h} = \lim_{h\to 0} \frac{11h + h^2}{h} = \lim_{h\to 0}(11 + h) = 11$$

(Equivalently, could use $f'(3) = \lim_{s\to 3} \dfrac{f(s) - f(3)}{s - 3}$.)

(d) $f'(4)$ if $f(t) = \sqrt{t}$

$$f'(4) = \lim_{h\to 0} \frac{f(4+h) - f(4)}{h} = \lim_{h\to 0} \frac{\sqrt{4+h} - 2}{h} = \lim_{h\to 0} \frac{(\sqrt{4+h} - 2)(\sqrt{4+h} + 2)}{h(\sqrt{4+h} + 2)} =$$
$$\lim_{h\to 0} \frac{(4+h) - 4}{h(\sqrt{4+h} + 2)} = \lim_{h\to 0} \frac{h}{h(\sqrt{4+h} + 2)} = \lim_{h\to 0} \frac{1}{\sqrt{4+h} + 2} = \frac{1}{4}$$

(Could use $f'(4) = \lim_{t\to 4} \dfrac{f(t) - f(4)}{t - 4}$.)

(e) $f'(5)$ if $f(x) = \dfrac{2}{x}$

$$f'(5) = \lim_{x\to 5} \frac{f(x) - f(5)}{x - 5} = \lim_{x\to 5} \frac{\frac{2}{x} - \frac{2}{5}}{x - 5} = \lim_{x\to 5} \frac{\frac{10-2x}{5x}}{x - 5} = \lim_{x\to 5} \frac{\frac{10-2x}{5x}}{\frac{x-5}{1}} = \lim_{x\to 5} \frac{10 - 2x}{5x(x-5)} =$$
$$\lim_{x\to 5} \frac{2(5-x)}{5x(x-5)} = \lim_{x\to 5} \frac{-2}{5x} = -\frac{2}{25}$$

(Could use $f'(5) = \lim_{h\to 0} \dfrac{f(5+h) - f(5)}{h}$.)