Math 75 Maria Voloshina

Review - 2

THEORY

Differentiation rules

1. Sum and difference rules $(f+g)' = f' + g', \quad (f-g)' = f' - g'$
2. Constant multiple rule $(cf)' = cf'$ for any constant c
3. Product rule $(fg)' = f'g + fg'$
4. Quotient rule $\left(\dfrac{f}{g}\right)' = \dfrac{f'g - fg'}{g^2}$
5. Chain rule $(f \circ g)'(x) = (f(g(x)))' = f'(g(x))g'(x)$

Derivatives of some elementary functions

$$(x^n)' = nx^{n-1}, \qquad (c)' = 0,$$
$$(\sin x)' = \cos x, \qquad (\cos x)' = -\sin x, \qquad (\tan x)' = (\sec x)^2,$$
$$(\csc x)' = -\csc x \cot x, \qquad (\sec x)' = \sec x \tan x, \qquad (\cot x)' = -(\csc x)^2,$$

Higher derivatives

Definition. Let $y = f(x)$. The derivative of the derivative of $f(x)$ is called the second derivative of $f(x)$:

$$y'' = f''(x) = (y')' = (f'(x))' =$$
$$= \frac{d}{dx}\left(\frac{df(x)}{dx}\right) = \frac{d^2}{(dx)^2}(f(x)) = \frac{d^2 f(x)}{(dx)^2} = \frac{d^2 y}{dx^2}$$

The derivative of the second derivative is called the third derivative:

$$f'''(x) = (f''(x))'$$

In general, the n-th derivative of $f(x)$ is the derivative of the $(n-1)$-st derivative of $f(x)$:

$$f^{(n)}(x) = (f^{(n-1)}(x))'$$

Implicit differentiation

is differentiation without solving for y. Be sure to use the chain rule when you differentiate a function of $y = y(x)$ with respect to x:

$$(f(y))' = (f(y(x)))' = f'(y(x)) \cdot y'(x)$$

Example.
If $\sqrt[3]{xy} = x^2y - 7x$ and $y(2) = 4$, find $y'(2)$.
Solution:
First rewrite using $y(x)$ instead of just y:
$\sqrt[3]{xy(x)} = x^2y(x) - 7x$
Now differentiate both sides of the equation with respect to x:

$$\frac{1}{3}(xy(x))^{-2/3}(1 \cdot y(x) + xy'(x)) = 2xy(x) + x^2y'(x) - 7$$

$$\frac{y(x)}{3(xy(x))^{2/3}} + \frac{xy'(x)}{3(xy(x))^{2/3}} = 2xy(x) + x^2y'(x) - 7$$

$$\frac{xy'(x)}{3(xy(x))^{2/3}} - x^2y'(x) = 2xy(x) - 7 - \frac{y(x)}{3(xy(x))^{2/3}}$$

$$y'(x)\left(\frac{x}{3(xy(x))^{2/3}} - x^2\right) = 2xy(x) - 7 - \frac{y(x)}{3(xy(x))^{2/3}}$$

$$y'(x) = \frac{2xy(x) - 7 - \frac{y(x)}{3(xy(x))^{2/3}}}{\left(\frac{x}{3(xy(x))^{2/3}} - x^2\right)}$$

if $x = 2$ and $y(2) = 4$, $\quad y'(2) = \dfrac{2 \cdot 2 \cdot 4 - 7 - \frac{4}{3(2 \cdot 4)^{2/3}}}{\left(\frac{2}{3(2 \cdot 4)^{2/3}} - 2^2\right)} = -\dfrac{52}{23}$

Related rates

Strategy:

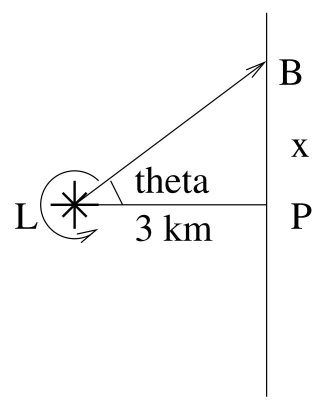
1. Read the problem carefully. Draw a diagram if possible.
2. Introduce notation. Assign symbols to all quantities that are functions of time.
3. Express the given information and the requited rate in terms of derivatives.
4. Write an equation that relates the various quantities of the problem.
5. Differentiate both sides of the equation with respect to t.
(Usually you will need the chain rule. Say if x is a function of t, then $\frac{d}{dt}(f(x)) = f'(x) \cdot \frac{dx}{dt}$.)
6. Substitute the given information into the resulting equation and solve for the unknown rate.

Example.

A lighthouse is located on a small island 3 km away from the nearest point P on a straight shoreline and its light makes four revolutions per minute. How fast is the beam of light moving along the shoreline when it is 3 km from P?

Solution:

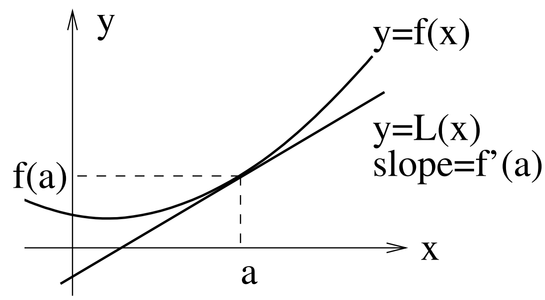
1.



2. Let θ denote the angle between the beam of light LB and LP, and $x = |BP|$.
3. Since the light makes four revolutions ($=4 \cdot 2\pi$) per minute, the rate of change of θ is $\frac{d\theta}{dt} = 8\pi/\text{min}$.
We need to find $\frac{dx}{dt}$ at when $x = 3$.
4. $\tan\theta = \frac{x}{3} \quad \Rightarrow \quad x = 3\tan\theta$
5. Differentiate both sides with respect to t:
$\frac{dx}{dt} = 3\sec^2\theta \cdot \frac{d\theta}{dt}$
6. When $x = 3$, $\theta = \frac{\pi}{4}$, and
$\frac{dx}{dt} = 3\sec^2\frac{\pi}{4} \cdot 8\pi = 3 \cdot 2 \cdot 8\pi = 48\pi$ (km/min)

Linear approximations and differentials

Let $y = f(x)$. The curve lies very close to its tangent line near the point of tangency. An equation of the tangent line at the point $(a, f(a))$ is $y - f(a) = f'(a)(x - a)$, or $y = f(a) + f'(a)(x - a)$, so $f(x) \approx f(a) + f'(a)(x - a)$ is called the linear approximation of f at a.

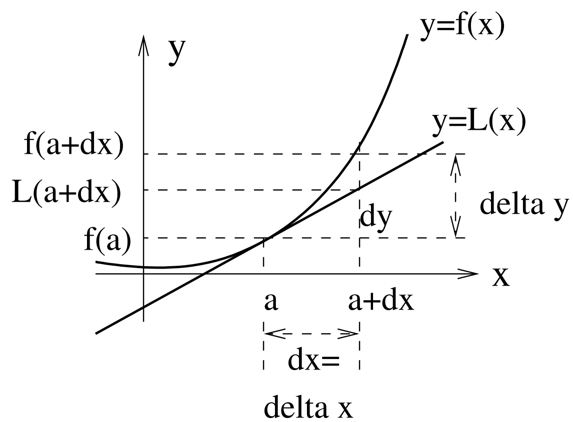


Def. The linear function whose graph is this tangent line, that is,

$$L(x) = f(a) + f'(a)(x - a)$$

is called the linearization of f at a.

Def. $dx = \Delta x$ is the change in x.
$\Delta y = f(a + \Delta x) - f(a)$ represents the amount that the curve $y = f(x)$ rises or falls (the change in $f(x)$) when x changes by an amount dx.
$dy = f'(a)dx$ represents the amount that the tangent line rises or falls (the change in the linearization) when x changes by an amount dx.



Example.

(a) Find the linearization $L(x)$ of $y = x^2 + 2x$ at $a = 1$ and use it to approximate the value of the function at 1.05.

(b) Compute Δy and dy for $x = 1$ and $dx = 0.05$.

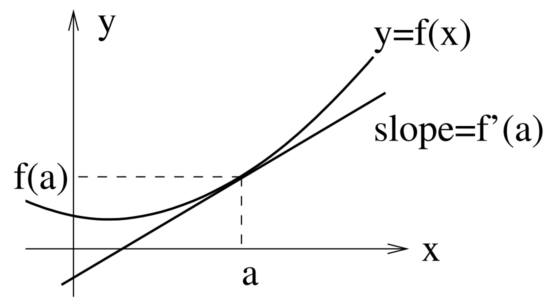
Solution:

(a) $f'(x) = (x^2 + 2x)' = 2x + 2$
$f(1) = 3, \quad f'(1) = 4$
$L(x) = f(1) + f'(1)(x - 1) = 3 + 4(x - 1) = 4x - 1$
Linearization: $L(x) = 4x - 1$.
$L(1.05) = 4 \cdot 1.05 - 1 = 3.2$
Therefore, $f(1.05) \approx 3.2$

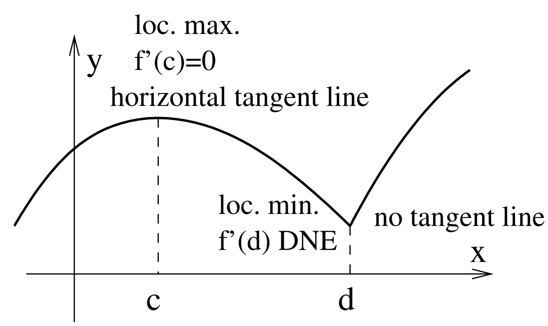
(b) $\Delta y = f(1.05) - f(1) = 3.2025 - 3 = 0.2025$
$dy = f'(a)dx = f'(1) \cdot 0.05 = 4 \cdot 0.05 = 0.2$

Applications of derivatives to graphs

The slope of the tangent line to a curve $y = f(x)$ at a point $x = a$ is equal to the derivative of $f(x)$ at a, i.e. $f'(a)$.



Fermat's theorem.
If $f(x)$ is continuous near b, and b is a local minimum or maximum, then b is a critical number of f, i.e. either $f'(b) = 0$ or $f'(b)$ does not exist.
On the picture: $f'(c) = 0$ and $f'(d)$ DNE.



The Closed Interval Method. To find the absolute maximum and minimum values of a continous function f on a closed interval $[a, b]$:

1. Find the critical numbers of f on (a, b), i.e. the points where $f'(x) = 0$ or $f'(x)$ doesn't exist.
2. Find the values of f at the critical numbers.
3. Find the values of f at the endpoints of the interval, i.e. $f(a)$ and $f(b)$.
4. The largest of the values from Steps 2 and 3 is the absolute maximum value; the smallest of those values is the absolute minimum value.

Example.
Find the absolute maximim and minimum values of $f(x) = x^4 - 4x^3 + 15$ on $[-1, 4]$.
Solution:

1. $f'(x) = 4x^3 - 12x^2 = 4x^2(x - 3) \Rightarrow$ 0 and 3 are critical numbers of f.
2. $f(0) = 15, \quad f(3) = -12.$
3. $f(-1) = 20, \quad f(4) = 15.$
4. Therefore, 20 is the absolute maximum value of f,
   and -12 is the absolute minimum value of f.

Summary of curve sketching

Def. The domain of a function $f(x)$ is the set of all values of x for which the function is defined.

To find the x-intercepts, solve $f(x) = 0$ for x. The y-intercept is $f(0)$.

Def. $f(x)$ is called even if $f(x) = f(-x)$ for all x. $f(x)$ is called odd if $f(x) = -f(-x)$ for all x. E.g. $\cos x$ is even, and $\sin x$ is odd. $f(x)$ is called periodic if there exists a number p such that $f(x) = f(x+p)$ for all x. The smallest positive number p such that $f(x) = f(x+p)$ for all x is called the period of $f(x)$. E.g. $\cos x$ and $\sin x$ are periodic with period 2π.

Def. If either $\lim_{x\to+\infty} f(x) = L$ or $\lim_{x\to-\infty} f(x) = L$ (or both), then $y = L$ is called a horizontal asymtote. If either $\lim_{x\to a^+} = \pm\infty$ or $\lim_{x\to a^-} = \pm\infty$ (or both), then $x = a$ is a vertical asymptote.

Def. A number c in the domain of $f(x)$ is called a critical number of $f(x)$ if either $f'(c)$ doesn't exist or $f'(c) = 0$.

Increading/Decreasing test. $f(x)$ is increasing when $f'(x) > 0$, and decreasing when $f'(x) < 0$.

First derivative test. If $f'(x)$ changes from positive to negative at c, then c is a local maximum for $f(x)$. If $f'(x)$ changes from negative to positive at c, then c is a local minimum for $f(x)$. (In both cases c must be in the domain of $f(x)$, and $f(x)$ must be continuous at c.)

Concavity test. $f(x)$ is CU (concave upward) when $f''(x) > 0$, and CD (concave down) when $f''(x) < 0$.

Def. If $f(x)$ changes the direction of concavity at c (and c is in the domain of $f(x)$), then c is an inflection point.

Curve sketching. Draw asymptotes as dashed lines. Plot all the intercepts, local maximum and minimum points, and inflection points. Then draw the graph of the function, making it pass throught all the points, approach the asymptotes, fall and rise according to the increasing/decreasing test, and with concavity according to the concavity test. If the function is even, odd, or periodic, use this fact (the graph of an even function is symmetric about the y-axis, the graph of an odd function is symmetric about the origin, and the graph of a periodic function is periodic).

Example. There is a separate handout with an example, $f(x) = \dfrac{2x^2}{x^2-1}$. Take a copy from the tray outside of my office if you were not in class when this example was done.