Practice test 3 - Solutions

Multiple choice questions: circle the correct answer

1. $\int \cos(2x)dx =$

   A. $\sin(2x) + C$ **B.** $-\sin(2x) + C$ **(C.)** $\frac{1}{2}\sin(2x) + C$ **D.** $-2\sin(2x)$ **E.** $x^2\sin(2x)$

2. $\int_{-3}^{1} |x|dx =$

   A. -5 **B.** -4 **C.** 0 **D.** 4 **(E.)** 5

3. 600 square cm of material is available to make a box with an open top. The length and the height of the box have to be equal. Find the largest possible volume of such a box.

   A. 600 cm^3 **B.** $\frac{200^{3/2}}{3}$ cm^3 **C.** 1 L **(D.)** $\frac{4000}{3}$ cm^3 **E.** 2000 cm^3

4. Find the derivative of $f(x) = \int_2^x \sin(t^2)dt$

   A. $2t\cos(t^2)$ **B.** $\sin(2x)$ **C.** $\cos(x^2)$ **D.** $\sin(x^2) - \sin(4)$ **(E.)** $\sin(x^2)$

   Note: the minus sign is not supposed to be there... sorry!

5. The area of the region enclosed by $y = \cos\left(\frac{\pi}{2}x\right)$ and $y = x^2 - 1$ is

   (A.) $\frac{12+4\pi}{3\pi}$ **B.** $\frac{4}{\pi+3}$ **C.** $\frac{10}{3}$ **D.** $\frac{14}{3}$ **E.** $\frac{4}{3} - \frac{4}{\pi}$

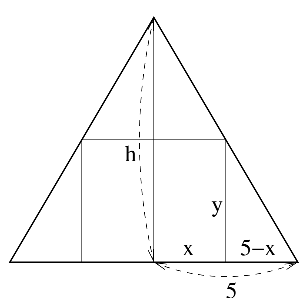
6. The region enclosed by $y = |x| - 1$ and the x-axis is rotated about the x-axis. The volume of the obtained solid is

   A. $\pi\int_{-1}^{0}(-x+1)^2dx + \pi\int_0^1 (x-1)^2dx$ **(B.)** $\pi\int_{-1}^{0}(x+1)^2dx + \pi\int_0^1(x-1)^2dx$

   C. $\pi\int_{-1}^{1}(x-1)^2dx$ **D.** $2\pi\int_{1}^{-1}(|x|-1)dx$ **E.** $2\pi\int_{-1}^{1}x(x-1)dx$

Regular problems: show all your work

7. Find the dimensions of the rectangle of largest area that can be inscribed in an equilateral triangle of side 10 if one side of the rectangle lies on the base of the triangle.



We want to find x and y such that the area of the rectangle, i.e. $2xy$, is a maximum.
Use similar triangles to find a relationship between x and y, e.g.

$\frac{5-x}{5} = \frac{y}{h}$ where $h = 5\tan 60^0 = 5\sqrt{3}$, so we have

$$\frac{5-x}{5} = \frac{y}{5\sqrt{3}}$$

Multiplying both sides by $5\sqrt{3}$ gives $\sqrt{3}(5-x) = y$.
Now we can express the area as a function of one variable x:
$A(x) = 2x\sqrt{3}(5-x) = 10\sqrt{3}x - 2\sqrt{3}x^2$. To find a maximum, we have to differentiate $A(x)$ and set the derivative equal to 0:

$A'(x) = 10\sqrt{3} - 4\sqrt{3}x = 0$
$10\sqrt{3} = 4\sqrt{3}x$
$x = 2.5$
It is easy to see that $A'(x)$ changes from positive to negative at 2.5, so this is a local maximum.
$y = \sqrt{3}(5 - x) = 2.5\sqrt{3}$, thus the width of the rectangle is $2x = 2 \cdot 2.5 = 5$, and the height is $y = 2.5\sqrt{3}$.

8. Use Newton's method to approximate the root of the equation $x^2 - 23 = 0$. Choose a reasonable initial approximation x_1, and use it to find the second approximation x_2.
   A reasonable approximation is $x_1 = 5$ because $5^2 = 25$ which is close to 23.
   Let $f(x) = x^2 - 23$, then $f'(x) = 2x$, and by Newton's method
   $x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 5 - \frac{2}{10} = 4.8.$

9. Find $f(x)$ if
   (a) $f'(x) = 1 - 8x^3 + 2\sin x - \cos x,\ f(0) = 5.$
   $f(x) = x - 2x^4 - 2\cos x - \sin x + c$
   $f(0) = -2 + c = 5 \quad \Rightarrow \quad c = 7$
   Therefore $f(x) = x - 2x^4 - 2\cos x - \sin x + 7$

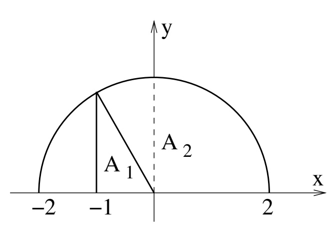
   (b) $f''(x) = 6 - 24x^2,\ f'(1) = -3,\ f(2) = -32.$
   $f'(x) = 6x - 8x^3 + c$
   $f'(1) = 6 - 8 + c = -3 \quad \Rightarrow \quad c = -1$
   $f'(x) = 6x - 8x^3 - 1$
   $f(x) = 3x^2 - 2x^4 - x + d$
   $f(2) = 12 - 32 - 2 + d = -32 \quad \Rightarrow \quad d = -10$
   $f(x) = 3x^2 - 2x^4 - x - 10$

10. Evaluate the following integrals

   (a) $\displaystyle\int_1^3 (3x^2 - 6x + 5)dx = (x^3 - 3x^2 + 5x)\Big|_1^3 = (27 - 27 + 15) - (1 - 3 + 5) =$
   $15 - 3 = 12$

   (b) $\displaystyle\int_\pi^{3\pi} \cos x dx = \sin x\Big|_\pi^{3\pi} = \sin(3\pi) - \sin(\pi) = 0 - 0 = 0$

   (c) $\displaystyle\int_{-1}^{2} \sqrt{4 - s^2}ds = A_1 + A_2 = \frac{1}{2} \cdot 1 \cdot \sqrt{3} + \frac{1}{3} \cdot \pi \cdot 2^2 = \frac{\sqrt{3}}{2} + \frac{4\pi}{3}$



   (d) $\displaystyle\int x\sin(x^2)dx$
   $u = x^2,\ du = 2xdx,\ \dfrac{du}{2} = xdx$, so
   $\displaystyle = \int \frac{1}{2}\sin(u)du = -\frac{1}{2}\cos(u) + c = -\frac{1}{2}\cos(x^2) + c$

   (e) $\displaystyle\int \frac{1}{(2 - 3s)^5}ds$
   $u = 2 - 3s,\ du = -3ds,\ -\dfrac{du}{3} = ds$, so
   $\displaystyle = -\frac{1}{3}\int \frac{1}{u^5}du = -\frac{1}{3}\int u^{-5}du = -\frac{1}{3} \cdot \frac{u^{-4}}{-4} + c = \frac{u^{-4}}{12} + c = \frac{(2 - 3s)^{-4}}{12} + c$

(f) $\int \sin x\sqrt{\cos x}dx$

$u = \cos x$, $du = -\sin x dx$, $-du = \sin x dx$, so

$\int \sin x\sqrt{\cos x}dx = -\int \sqrt{u}du = -\int u^{1/2}du = -\frac{u^{3/2}}{3/2}+c = -\frac{2}{3}u^{3/2}+c = -\frac{2}{3}(\cos x)^{3/2}+$
c

11. Find the derivative of $g(x) = \int_{3x}^{5x^2} \sqrt{t}\tan(3t)dt$

$g'(x) = \sqrt{5x^2}\tan(3\cdot 5x^2)\cdot 10x - \sqrt{3x}\tan(3\cdot 3x)\cdot 3 = 10x\sqrt{5x^2}\tan(15x^2) - 3\sqrt{3x}\tan(9x)$

12. Estimate the value of $\int_0^{10}(x^2+6)dx$ using 5 approximating rectangles and

We divide $[0,10]$ into 5 subintervals of length $\Delta x = \frac{10-0}{5} = 2$, i.e. $[0,2]$, $[2,4]$, $[4,6]$, $[6,8]$, $[8,10]$.

(a) left endpoints,

Left endpoints are 0, 2, 4, 6, and 8. So $L_5 = (f(0)+f(2)+f(4)+f(6)+f(8))\Delta x = (6+10+22+42+70)2 = 300$.

(b) right endpoints.

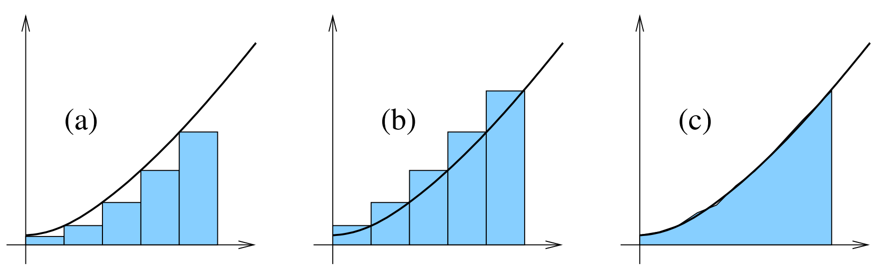
Right endpoints are 2, 4, 6, 8, and 10. So $R_5 = (f(2)+f(4)+f(6)+f(8)+f(10))\Delta x = (10+22+42+70+106)2 = 500$.

(c) Evaluate $\int_0^{10}(x^2+6)dx$ using the Fundamental Theorem of Calculus.

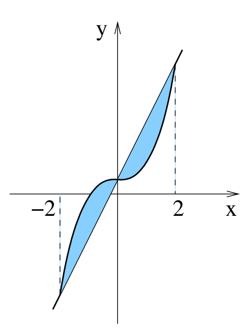
$= \left(\frac{1}{3}x^3 + 6x\right)\Big|_0^{10} = \left(\frac{1000}{3}+60\right) - 0 = 393.333333$

(d) Sketch the graph of $f(x) = x^2+6$ and explain the meaning of your answers in (a)-(c).

The Riemann sums in (a) and (b) and the intergral in (c) calculate the areas shown below.



13. Sketch the region enclosed by $y = x^3+1$ and $y = 4x+1$ and find its area.



To find the intersection points,
set $x^3+1 = 4x+1$
$x^3 - 4x = 0$
$x(x-2)(x+2) = 0$
$x = 0, 2$, and -2.

The region consists of 2 parts.

$A_1 = \int_{-2}^{0} (x^3 + 1 - (4x + 1))dx = \int_{-2}^{0} (x^3 - 4x)dx = \left(\frac{x^4}{4} - 2x^2\right)\Big|_{-2}^{0} = 0 - (4 - 8) = 4$

$A_2 = \int_{0}^{2} (4x + 1 - (x^3 + 1))dx = \int_{0}^{2} (4x - x^3)dx = \left(2x^2 - \frac{x^4}{4}\right)\Big|_{0}^{2} = (8 - 4) - 0 = 4$

Thus the area of the whole region is 8.

14. Find the area of the region between the curves $x = y^2 - 3y$ and $x = 5y$ from $y = 0$ to $y = 3$.

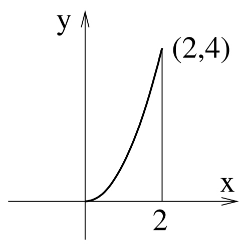
$A = \int_{0}^{3} (5y - (y^2 - 3y))dy =$

$= \int_{0}^{3} (8y - y^2)dy = \left(4y^2 - \frac{y^3}{3}\right)\Big|_{0}^{3} =$

$(36 - 9) - 0 = 27$

15. Find the volume of the solid obtained by rotating the region under the curve $y = x^2$ from $x = 0$ to $x = 2$ about

First of all, sketch the region:



(a) the x-axis,

Each cross-section is a disk. The area of a crossection is $A(x) = \pi(x^2)^2$, thus the volume is

$V = \int_0^2 \pi x^4 dx = \pi \frac{x^5}{5}\Big|_0^2 = \frac{32\pi}{5}.$

(b) the y-axis,

Cross-sections are too complicated, so we use cylindrical shells. The area of the base of a cylindrical shell is $2\pi x\Delta x$, and the height is x^2, so the volume is

$V = \int_0^2 2\pi x x^2 dx = 2\pi \int_0^2 x^3 dx = 2\pi \frac{x^4}{4}\Big|_0^2 = 8\pi.$

(c) the line $y = 4$,

Now each cross-section is a ring with outer radius 4 and inner radius $4 - x^2$. The area of the cross-section is

$A(x) = \pi 4^2 - \pi(4 - x^2)^2 = \pi 16 - \pi(16 - 8x^2 + x^4) = \pi(8x^2 - x^4).$

The volume is $V = \int_0^2 \pi(8x^2 - x^4)dx = \pi \left(\frac{8x^3}{3} - \frac{x^5}{5}\right)\Big|_0^2 = \pi\left(\frac{64}{3} - \frac{32}{5}\right) = \frac{224}{15}\pi$

(d) the line $x = -1$.

Using cylindrical shells again. The area of the base is $2\pi(x+1)\Delta x$, and the height is x^2, so the volume is

$V = \int_0^2 2\pi(x + 1)x^2 dx = 2\pi \int_0^2 (x^3 + x^2)dx = 2\pi \left(\frac{x^4}{4} + \frac{x^3}{3}\right)\Big|_0^2 = 2\pi\left(4 + \frac{8}{3}\right) = \frac{40}{3}\pi$

16. Find the average value of $f(x) = \cos(\pi x/2)$ on the interval $[-1, 1]$.

$f_{\text{ave}} = \frac{1}{1 - (-1)} \int_{-1}^{1} \cos\left(\frac{\pi x}{2}\right) dx =$ (Substitution: $u = \frac{\pi x}{2}$, $du = \frac{\pi}{2}dx$, $\frac{2}{\pi}du = dx$.

Change the limits of integration: when $x = -1$, $u = -\frac{\pi}{2}$, and when $x = 1$, $u = \frac{\pi}{2}$)

$= \frac{1}{2}\frac{2}{\pi} \int_{-\pi/2}^{\pi/2} \cos u du = \frac{1}{\pi}(\sin u)\Big|_{-\pi/2}^{\pi/2} = \frac{1}{\pi}(1 - (-1)) = \frac{2}{\pi}$