Math 76 **Solutions to practice problems for test 1**

1. For each function, find its inverse, and sketch the graphs of the original function and its inverse.

   (a) $f(x) = e^x - 2$
   Write $y = e^x - 2$ and solve for x:
   $y + 2 = e^x$
   $\ln(y+2) = x$
   Therefore $f^{-1}(y) = \ln(y+2)$ or $f^{-1}(x) = \ln(x+2)$

   (b) $f(x) = 2^{-x}$
   $y = 2^{-x}$
   $\log_2 y = -x$
   $-\log_2 y = x$
   $f^{-1}(y) = -\log_2 y$ or $f^{-1}(x) = -\log_2 x$

   (c) $f(x) = \ln(x+3) + 2$
   $y = \ln(x+3) + 2$
   $y - 2 = \ln(x+3)$
   $e^{y-2} = x + 3$
   $e^{y-2} - 3 = x$
   $f^{-1}(y) = e^{y-2} - 3$ or $f^{-1}(x) = e^{x-2} - 3$

2. Find the exact value of

(a) $\log_5\left(\frac{1}{125}\right) = -3$ because $5^{-3} = \frac{1}{125}$

(b) $\log_6 2 + \log_6 3 = \log_6(2 \cdot 3) = \log_6 6 = 1$ because $6^1 = 6$

(c) $3\log_8 4 = \log_8(4^3) = \log_8 64 = 2$ because $8^2 = 64$

(d) $\arcsin(1) = \frac{\pi}{2}$ because $\sin\left(\frac{\pi}{2}\right) = 1$ and $-\frac{\pi}{2} \leq \frac{\pi}{2} \leq \frac{\pi}{2}$ (See the definition of arcsin)

(e) $\arccos\left(\frac{1}{2}\right) = \frac{\pi}{3}$ because $\cos\left(\frac{\pi}{3}\right) = \frac{1}{2}$ and $0 \leq \frac{\pi}{3} \leq \pi$

(f) $\sin\left(\arctan\left(\frac{3}{4}\right)\right) = \frac{3}{5}$

$\arctan\left(\frac{3}{4}\right)$ is an angle, say t, between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$, whose tan is $\frac{3}{4}$. Draw (I do mean draw, don't just read this!) a right triangle, and denote one of the acute angles by t. Since its tan is $\frac{3}{4}$, let the opposite side be 3 and the adjacent side be 4. Then the hypothenuse is 5. Therefore $\sin(t) = \frac{3}{5}$.

3. Evaluate the limits.

(a) $\lim_{x\to 0} \frac{\sin(5x)}{2\sin(3x)}$ This limit is of type $\frac{0}{0}$, so use L'Hospital's rule:

$= \lim_{x\to 0} \frac{5\cos(5x)}{6\cos(3x)} = \frac{5\cos(0)}{6\cos(0)} = \frac{5}{6}$

(b) $\lim_{x\to 0} \frac{e^x(\cos x - 1)}{\tan(3x)}$ This limit is also of type $\frac{0}{0}$, so use L'Hospital's rule:

$= \lim_{x\to 0} \frac{e^x(\cos x - 1) + e^x(-\sin x)}{3\sec^2(3x)} = \frac{0}{3} = 0$

(c) $\lim_{x\to 0} \frac{e^x - 1 - x}{x^2}$ Same here:

$= \lim_{x\to 0} \frac{e^x - 1}{2x}$ Still $\frac{0}{0}$, so use L'Hospital's rule again:

$= \lim_{x\to 0} \frac{e^x}{2} = \frac{1}{2}$

(d) $\lim_{x\to\infty} x^3 e^{-3x}$ This one is of type $\infty \cdot 0$, so rewrite the function as a quotient before using L'Hospital's rule (you will have to use L'Hospital's rule three times):

$= \lim_{x\to\infty} \frac{x^3}{e^{3x}} = \lim_{x\to\infty} \frac{3x^2}{3e^{3x}} = \lim_{x\to\infty} \frac{6x}{9e^{3x}} = \lim_{x\to\infty} \frac{6}{27e^{3x}} = 0$

(e) $\lim_{x\to\infty}\left(\frac{x}{x+1}\right)^{3x} = \lim_{x\to\infty} e^{\ln\left(\left(\frac{x}{x+1}\right)^{3x}\right)} = \lim_{x\to\infty} e^{3x\ln\left(\frac{x}{x+1}\right)} = e^{\lim_{x\to\infty} 3x\ln\left(\frac{x}{x+1}\right)}$

The limit in the exponent is of type $\infty\cdot 1$, so use the same method as in (d):

$$\lim_{x\to\infty} 3x\ln\left(\frac{x}{x+1}\right) = \lim_{x\to\infty}\frac{3\ln\left(\frac{x}{x+1}\right)}{\frac{1}{x}} = \lim_{x\to\infty}\frac{3(\ln(x)-\ln(x+1))}{\frac{1}{x}} = \lim_{x\to\infty}\frac{3\left(\frac{1}{x}-\frac{1}{x+1}\right)}{-\frac{1}{x^2}} =$$

$$\lim_{x\to\infty}\frac{3\frac{1}{x(x+1)}}{-\frac{1}{x^2}} = \lim_{x\to\infty}\frac{-3x^2}{x(x+1)} = \lim_{x\to\infty}\frac{-3x}{x+1} = \lim_{x\to\infty}\frac{-3}{1} = -3$$

Then $\lim_{x\to\infty}\left(\frac{x}{x+1}\right)^{3x} = e^{\lim_{x\to\infty} 3x\ln\left(\frac{x}{x+1}\right)} = e^{-3}$

4. Evaluate the following integrals.

(a) $\int x\sin(x^2)dx =$ (Substitution: $u = x^2$, $du = 2xdx$, $\frac{1}{2}du = xdx$)

$= \int \frac{1}{2}\sin u du = -\frac{1}{2}\cos u + c = -\frac{1}{2}\cos(x^2) + c$

(b) $\int_{\pi/6}^{\pi/2}\cos^2 xdx =$ (Trig. integral) $= \int_{\pi/6}^{\pi/2}\frac{1}{2}(\cos(2x)+1)dx =$

(Substitution: $u = 2x$, $du = 2dx$, $\frac{1}{2}du = dx$. Change the limits of integration: when $x = \pi/6$, $u = \pi/3$, and when $x = \pi/2$, $u = \pi$)

$= \int_{\pi/3}^{\pi}\frac{1}{4}(\cos u + 1)du = \frac{1}{4}(\sin u + u)\Big|_{\pi/3}^{\pi} = \frac{1}{4}\left[(0+\pi) - \left(\frac{\sqrt{3}}{2}+\frac{\pi}{3}\right)\right] = \frac{\pi}{6} - \frac{\sqrt{3}}{8}$

(c) $\int \frac{4x+3}{x+1}dx$ (This is a rational function.)

Step I. Long division gives $\frac{4x+3}{x+1} = 4 - \frac{1}{x+1}$, so

$\int \frac{4x+3}{x+1}dx = \int 4dx - \int \frac{1}{x+1}dx = 4x - \ln|x+1| + c$

The other steps are not needed here.

(d) $\int_1^2 xe^x dx =$ (By parts: $u = x$, $dv = e^x dx$, $du = dx$, $v = e^x$)

$= xe^x\Big|_1^2 - \int_1^2 e^x dx = (2e^2 - 1e^1) - e^x\Big|_1^2 = 2e^2 - e - (e^2 - e) = e^2$

(e) $\int_0^{\pi}\sin^5 tdt =$ (Trig. integral) $= \int_0^{\pi}(\sin^2 t)^2\sin tdt = \int_0^{\pi}(1-\cos^2 t)^2\sin tdt =$

(Substitution: $u = \cos t$, $du = -\sin tdt$, $-du = \sin tdt$. Change the limits of integration: when $x = 0$, $u = \cos 0 = 1$, and when $x = \pi$, $u = \cos\pi = -1$)

$= -\int_1^{-1}(1-u^2)^2du = -\int_1^{-1}(1-2u^2+u^4)du = -\left(u - \frac{2}{3}u^3 + \frac{1}{5}u^5\right)\Big|_1^{-1} =$

$= -\left[\left(-1+\frac{2}{3}-\frac{1}{5}\right) - \left(1-\frac{2}{3}+\frac{1}{5}\right)\right] = -\left[\frac{-8}{15} - \frac{8}{15}\right] = \frac{16}{15}$

(f) $\int_0^{\pi/4}\sec^2 xe^{\tan x}dx =$ (Substitution: $u = \tan x$, $du = \sec^2 xdx$. Change the limits of integration: when $x = 0$, $u = \tan(0) = 0$, and when $x = \pi/4$, $u = \tan(\pi/4) = 1$)

$= \int_0^1 e^u du = e^1 - e^0 = e - 1$

(g) $\int x^2 \sin(2x)dx =$ (Substitution: $y = 2x$, $dy = 2dx$, $x = \frac{y}{2}$, $dx = \frac{dy}{2}$)

$= \int \left(\frac{y}{2}\right)^2 \sin y \frac{dy}{2} = \frac{1}{8}\int y^2 \sin y dy =$

(By parts: $u = y^2$, $dv = \sin y dy$, $du = 2ydy$, $v = -\cos y$)

$= -\frac{1}{8}\left(y^2 \cos y + \int 2y \cos y dy\right) = -\frac{1}{8}y^2 \cos y + \frac{1}{4}\int y \cos y dy =$

(By parts again: $u = y$, $dv = \cos y dy$, $du = dy$, $v = \sin y$)

$= -\frac{1}{8}y^2 \cos y + \frac{1}{4}\left(y \sin y - \int \sin y dy\right) = -\frac{1}{8}y^2 \cos y + \frac{1}{4}y \sin y + \frac{1}{4}\cos y + c$

$= -\frac{1}{2}x^2 \cos(2x) + \frac{1}{2}x \sin(2x) + \frac{1}{4}\cos(2x) + c$

(Note: the substitution is not necessary here, you could start with $u = x^2$, $dv = \sin(2x)$.)

(h) $\int \frac{1}{x^2\sqrt{4-x^2}}dx$

(Trig substitution: $x = 2\sin t$, $dx = 2\cos t dt$, $\sqrt{4-x^2} = \sqrt{4 - 4\sin^2 t} = 2\cos t$)

$= \int \frac{1}{4\sin^2 t \cdot 2\cos t} 2\cos t dt = \frac{1}{4}\int \frac{1}{\sin^2 t}dt = \frac{1}{4}\int \csc^2 t dt = -\frac{1}{4}\cot t + c$

Change back to x: $x = 2\sin t$ implies $\sin t = \frac{x}{2}$. Draw a right triangle with an acute angle t, the opposite side x and hypothenus 2, so that $\sin t = \frac{x}{2}$. Then the adjacent side is $\sqrt{4-x^2}$, and $\cot t = \frac{\sqrt{4-x^2}}{x}$.

Thus $\int \frac{1}{x^2\sqrt{4-x^2}}dx = -\frac{1}{4}\frac{\sqrt{4-x^2}}{x} + c = -\frac{\sqrt{4-x^2}}{4x} + c$

(i) $\int \frac{7x+12}{x^3+4x^2}dx$

We skip **Step I** because the integrand is a proper rational function (the degree of the numberator is less than the degree of the denominator).

Step II. Factor the denominator: $x^3 + 4x^2 = x^2(x+4)$

Step III. Partial fraction decomposition: $\frac{7x+12}{x^3+4x^2} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+4}$

$7x + 12 = Ax(x+4) + B(x+4) + Cx^2$

$7x + 12 = (A+C)x^2 + (4A+B)x + 4B$

$12 = 4B \;\Rightarrow\; B = 3$

$7 = 4A + B \;\Rightarrow\; A = 1$

$0 = A + C \;\Rightarrow\; C = -1$

Thus $\frac{7x+12}{x^3+4x^2} = \frac{1}{x} + \frac{3}{x^2} - \frac{1}{x+4}$

Step IV. $\int \frac{7x+12}{x^3+4x^2}dx = \int \frac{1}{x}dx + \int \frac{3}{x^2}dx - \int \frac{1}{x+4} = \ln|x| - \frac{3}{x} - \ln|x+4| + c$

(j) $\int (\ln(5x))^3 dx =$ (Substitution: $y = 5x$, $dy = 5dx$, $\frac{1}{5}dy = dx$)

$= \frac{1}{5}\int (\ln y)^3 dy =$ (By parts: $u = (\ln y)^3$, $dv = dy$, $du = 3(\ln y)^2 \frac{1}{y}dy$, $v = y$)

$= \frac{1}{5}\left((\ln y)^3 y - 3\int (\ln y)^2 dy\right) = \frac{1}{5}(\ln y)^3 y - \frac{3}{5}\int (\ln y)^2 dy =$

(By parts again: $u = (\ln y)^2$, $dv = dy$, $du = 2(\ln y)\frac{1}{y}dy$, $v = y$)

$= \frac{1}{5}(\ln y)^3 y - \frac{3}{5}\left((\ln y)^2 y - \int 2(\ln y)dy\right) = \frac{1}{5}(\ln y)^3 y - \frac{3}{5}(\ln y)^2 y + \frac{6}{5}\int (\ln y) y dy =$

(One more time by parts: $u = \ln y$, $dv = dy$, $du = \frac{1}{y}dy$, $v = y$)

$= \frac{1}{5}(\ln y)^3 y - \frac{3}{5}(\ln y)^2 y + \frac{6}{5}\left((\ln y)y - \int 1 dy\right) = \frac{1}{5}(\ln y)^3 y - \frac{3}{5}(\ln y)^2 y + \frac{6}{5}(\ln y)y - \frac{6}{5}y + c$

$= (\ln(5x))^3 x - 3(\ln(5x))^2 x + 6\ln(5x)x - 6x + c$

(Note: as in (g), the substitution is not necessary here. You could start with integration by parts.)

(k) $\int \frac{4x+3}{x^2+1}dx = \int \frac{4x}{x^2+1} + \int \frac{3}{x^2+1} =$

(Substitution for the first integral: $u = x^2 + 1$, $du = 2xdx$)

$= \int \frac{2}{u}du + 3\arctan x = 2\ln|u| + 3\arctan x + c = 2\ln(x^2+1) + 3\arctan x + c$

(l) $\int (\cos x)^3(\sin x)^2 dx =$ (Trig. integral) $= \int (\cos x)^2(\sin x)^2 \cos x dx =$

$= \int (1 - (\sin x)^2)(\sin x)^2 \cos x dx =$ (Substitution: $u = \sin x$, $du = \cos x dx$)

$= \int (1 - u^2)u^2 du = \int (u^2 - u^4)du = \frac{1}{3}u^3 - \frac{1}{5}u^5 + c = \frac{1}{3}\sin^3 x - \frac{1}{5}\sin^5 x + c$

(m) $\int \frac{1}{(2-3s)^5}ds =$ (Substitution: $u = 2 - 3s$, $du = -3ds$, $-\frac{1}{3}du = ds$)

$= -\frac{1}{3}\int \frac{1}{u^5}du = -\frac{1}{3}\int u^{-5}du = -\frac{1}{3}\cdot\frac{u^{-4}}{-4} + c = \frac{(2-3s)^{-4}}{12} + c$

(n) $\int \frac{1}{\sqrt{x^2+9}}dx =$

(Trig substitution: $x = 3\tan t$, $dx = 3\sec^2 t dt$, $\sqrt{x^2+9} = \sqrt{9\tan^2 t + 9} = 3\sec t$)

$= \int \frac{1}{3\sec t}3\sec^2 t dt = \int \sec t dt = \ln|\sec t + \tan t| + c = \ln\left|\frac{\sqrt{x^2+9}}{3} + \frac{x}{3}\right| + c$

(o) $\int \frac{e^x+5}{e^x+3}dx$ (Substitution: $u = e^x$, $du = e^x dx$, $\frac{1}{e^x}du = dx$, $\frac{1}{u}du = dx$)

$= \int \frac{u+5}{(u+3)u}du$ (This is a rational function) $\frac{u+5}{(u+3)u} = \frac{A}{u+3} + \frac{B}{u}$

$u + 5 = Au + B(u+3)$

$u + 5 = (A+B)u + 3B$

$1 = A + B$

$5 = 3B \Rightarrow B = \frac{5}{3} \Rightarrow A = -\frac{2}{3}$

Thus $\int \frac{u+5}{(u+3)u}du = -\frac{2}{3}\int \frac{1}{u+3}du + \frac{5}{3}\int \frac{1}{u}du = -\frac{2}{3}\ln|u+3| + \frac{5}{3}\ln|u| + c =$

$= -\frac{2}{3}\ln(e^x+3) + \frac{5}{3}\ln e^x + c = -\frac{2}{3}\ln(e^x+3) + \frac{5}{3}x + c$

(p) $\displaystyle\int \frac{x^2}{x^2+5x+6}dx$

Step I. Long division

$$\begin{array}{r|l} & 1 \\ x^2+5x+6 \ ) & \ x^2 \\ - & \underline{x^2+5x+6} \\ & \ -5x-6 \end{array}$$

$$\frac{x^2}{x^2+5x+6} = 1 + \frac{-5x-6}{x^2+5x+6}$$

Step II. Factor the denominator: $x^2+5x+6 = (x+2)(x+3)$.

Step III. Partial fraction decomposition: $\dfrac{-5x-6}{x^2+5x+6} = \dfrac{A}{x+2} + \dfrac{B}{x+3}$

$-5x-6 = A(x+3) + B(x+2)$

$-5x-6 = (A+B)x + (3A+2B)$

$-5 = A+B \quad \Rightarrow \quad B = -5-A$

$-6 = 3A+2B \quad \Rightarrow \quad -6 = 3A-10-2A \quad \Rightarrow \quad A = 4 \quad \Rightarrow \quad B = -9$

Thus $\dfrac{x^2}{x^2+5x+6} = 1 + \dfrac{4}{x+2} - \dfrac{9}{x+3}$

Step IV. $\displaystyle\int \frac{x^2}{x^2+5x+6}dx = \int 1dx + \int \frac{4}{x+2}dx - \int \frac{9}{x+3}dx$

$= x + 4\ln|x+2| - 9\ln|x+3| + c$

(q) $\displaystyle\int x\sqrt{4x^2-1}dx$

We could use a trig substitution here, but the easiest way to do this problem is to make the substitution $u = 4x^2-1$. Then $du = 8xdx$, $\frac{1}{8}du = xdx$, and we have

$$\int x\sqrt{4x^2-1}dx = \frac{1}{8}\int \sqrt{u}du = \frac{1}{8}\frac{u^{3/2}}{3/2} + c = \frac{u^{3/2}}{12} + c = \frac{(4x^2-1)^{3/2}}{12} + c$$

For the sake of practice, let's try a trig substitution too. First we have to factor out 4 under the square root to get $\sqrt{x^2-a^2}$:

$$\int x\sqrt{4x^2-1}dx = \int x\sqrt{4\left(x^2-\tfrac{1}{4}\right)}dx = 2\int x\sqrt{x^2-\tfrac{1}{4}}dx$$

(Substitution: $x = \frac{1}{2}\sec t$, $dx = \frac{1}{2}\sec t\tan t dt$, $\sqrt{x^2-\frac{1}{4}} = \sqrt{\frac{1}{4}\sec^2 t - \frac{1}{4}} = \frac{1}{2}\tan t$)

$$2\int x\sqrt{x^2-\frac{1}{4}}dx = 2\int \frac{1}{2}\sec t \cdot \frac{1}{2}\tan t \cdot \frac{1}{2}\sec t\tan t dt = \frac{1}{4}\int \tan^2 t\sec^2 t dt$$

(Substitution: $u = \tan t$, $du = \sec^2 t dt$)

$$= \frac{1}{4}\int u^2 du = \frac{1}{4}\frac{u^3}{3} + c = \frac{u^3}{12} + c = \frac{(\tan t)^3}{12} + c$$

to change back to x, recall $\sqrt{x^2-\dfrac{1}{4}} = \dfrac{1}{2}\tan t$, so $\tan t = 2\sqrt{x^2-\dfrac{1}{4}} = \sqrt{4x^2-1}$

Thus $\displaystyle\int x\sqrt{4x^2-1}dx = \frac{(\sqrt{4x^2-1})^3}{12} + c$