Section 5.1. Sequences.

Examples.

1. If $a_n = \dfrac{n}{n+1}$, then $\lim\limits_{n\to\infty} a_n = 1$ (we say that this sequence converges).

2. If $a_n = n^2$, then $\lim\limits_{n\to\infty} a_n = \infty$ (we say that this sequence diverges).

3. If $a_n = \begin{cases} 0 & \text{if } n \text{ is odd} \\ n & \text{if } n \text{ is even} \end{cases}$, $\lim\limits_{n\to\infty} a_n$ does not exist (the sequence diverges).

4. If $a_n = \dfrac{\cos(n)}{n^2}$, then $\lim\limits_{n\to\infty} a_n = 0$ by the Squeeze Theorem because $-\dfrac{1}{n^2} \leq \dfrac{\cos(n)}{n^2} \leq \dfrac{1}{n^2}$ and $\lim\limits_{n\to\infty}\left(-\dfrac{1}{n^2}\right) = \lim\limits_{n\to\infty}\dfrac{1}{n^2} = 0.$

Section 5.2. Infinite Series.

Examples.

1. $\displaystyle\sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^n = 1$, so the series is convergent.

2. $\displaystyle\sum_{n=1}^{\infty} 2^n = \infty$, so the series is divergent.

3. $\displaystyle\sum_{n=1}^{\infty}(-1)^n$ does not exist, so the series is divergent.

4. Geometric series: $\displaystyle\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + ar^3 + \cdots = \frac{a}{1-r}$ if $|r| < 1$

   and $\displaystyle\sum_{n=1}^{\infty} ar^{n-1}$ diverges if $|r| \geq 1$.

5. Telescoping series: $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n(n+1)} = \sum_{n=1}^{\infty}\left(\frac{1}{n} - \frac{1}{n+1}\right) = 1.$

Exercises.

Determine whether the series converges or diverges.

1. $1 + \dfrac{e}{\pi} + \dfrac{e^2}{\pi^2} + \dfrac{e^3}{\pi^3} + \ldots$

   Solution. This is a geometric series with $r = \dfrac{e}{\pi}$. Since $|r| < 1$, it converges.

2. $1 - 2 + 4 - 8 + 16 - 32 + \ldots$

   Solution. This is a geometric series with $r = -2$. Since $|r| > 1$, it diverges.

Section 5.3. The Divergence and Integral Tests.

Examples.

1. $\displaystyle\sum_{n=1}^{\infty} \frac{n}{2n+1}$ diverges by the divergence test since $\displaystyle\lim_{n\to\infty} \frac{n}{2n+1} = \frac{1}{2} \neq 0$.

2. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ diverges by the integral test since $\displaystyle\int_1^{\infty} \frac{1}{x}\,dx = \infty$.

Exercises.

Determine whether the series converges or diverges.

1. $\displaystyle\sum_{n=1}^{\infty} (1 - (-1)^n)$

   Solution. The terms of the sequence are $a_n = 2$ when n is odd and $a_n = 0$ when n is even. Since they do not approach 0, the series diverges.

2. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$

   Solution. The series converges by the integral test since

   $$\int_1^{\infty} \frac{1}{x^2}\,dx = \lim_{t\to\infty} \int_1^{t} \frac{1}{x^2}\,dx = \lim_{t\to\infty} \left(-\frac{1}{x}\right)\Bigg|_1^t = \lim_{t\to\infty} \left(1 - \frac{1}{t}\right) = 1.$$

Section 5.4. Comparison Tests.

Examples.

1. The series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ converges by the comparison test since $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges (this is a p-series with $p = 2$) and $0 \le \dfrac{1}{n^2+1} \le \dfrac{1}{n^2}$.

2. The series $\displaystyle\sum_{n=2}^{\infty} \frac{1}{\ln n}$ diverges by the comparison test since $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n}$ diverges (this is a p-series with $p = 1$, also known as the harmonic series) and $0 \le \dfrac{1}{n} \le \dfrac{1}{\ln n}$.

3. The series $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n^2-1}$ converges by the limit comparison test since $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n^2}$ converges and $\displaystyle\lim_{n\to\infty} \frac{\frac{1}{n^2-1}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{n^2}{n^2-1} = 1 \neq 0$.

Exercises. Determine whether each of the following series converges or diverges.

1. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}+1}$

   Solution. The series diverges by the limit comparison test since $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ diverges (it's a p-series with $p = \frac{1}{2}$) and $\displaystyle\lim_{n\to\infty} \frac{\frac{1}{\sqrt{n}+1}}{\frac{1}{\sqrt{n}}} = \lim_{n\to\infty} \frac{\sqrt{n}}{\sqrt{n}+1} = \lim_{n\to\infty} \frac{1}{1+\frac{1}{\sqrt{n}}} = 1 \neq 0$.

2. $\displaystyle\sum_{n=1}^{\infty} \frac{2^n+1}{3^n-2}$

   Solution. The series converges by the limit comparison test since $\displaystyle\sum_{n=1}^{\infty} \frac{2^n}{3^n}$

converges (it's a geometric series with $r = \frac{2}{3}$) and $\lim_{n\to\infty} \frac{\frac{2^n+1}{3^n-2}}{\frac{2^n}{3^n}} = \lim_{n\to\infty} \frac{3^n(2^n+1)}{2^n(3^n-2)} = \lim_{n\to\infty} \frac{1+\frac{1}{2^n}}{1-\frac{2}{3^n}} = 1 \neq 0$.