Section 5.5. Alternating Series.

Examples.

1. $\displaystyle\sum_{n=1}^{\infty}\left(-\frac{1}{2}\right)^{n} = -\frac{1}{2}+\frac{1}{4}-\frac{1}{8}+\frac{1}{16}-\dots$

2. $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n} = \frac{1}{1}-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\dots$

Exercises. Determine whether each of the following series converges absolutely, converges conditionally, or diverges.

1. $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{2n+1}$

2. $\displaystyle\sum_{n=1}^{\infty}\frac{\cos n}{n^2}$

3. $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n}n}{n+3}$

Section 5.6. Ratio and Root Tests.

Examples.

1. The series $\displaystyle\sum_{n=1}^{\infty} \frac{2^n}{n!}$ converges by the ratio test since

$$\rho = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \frac{\frac{2^{n+1}}{(n+1)!}}{\frac{2^n}{n!}} = \lim_{n\to\infty} \frac{2}{n+1} = 0.$$

2. The series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^n}$ converges by the root test since

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \sqrt[n]{\frac{1}{n^n}} = \lim_{n\to\infty} \frac{1}{n} = 0.$$

Exercises. Can the ratio and/or root test be used to determine convergence/divergence of the following series?

1. $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{3n+1}$

   Solution. The ratio test can be used. The limit is infinite, so the series diverges.

2. $\displaystyle\sum_{n=1}^{\infty} \frac{n^2+2}{n^3+n^2+1}$

   Solution. If the ratio test is used, the limit is 1, so the test is inconclusive. The limit does not seem to be simple enough for the root test. (Note: the limit comparison test can be used with $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ to establish that this series diverges.)

3. $\displaystyle\sum_{n=1}^{\infty} \frac{3^n}{(n+1)^n}$

   Solution. The root test can be used. The limit is 0, so the series converges.

Section 6.1. Power Series and Functions.

Examples. For 1-3, ind the radius and interval of convergence for each series.

1. $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!}$

2. $\displaystyle\sum_{n=0}^{\infty} n!x^n$

3. $\displaystyle\sum_{n=0}^{\infty} \frac{(x-2)^n}{(n+1)3^n}$

4. Use a power series to represent $\dfrac{1}{1+x^3}$ and find its interval of convergence.

Exercises. (Note: we did not actually have time in class to do these.)

1. Find the radius and interval of convergence for the series $\displaystyle\sum_{n=1}^{\infty} \frac{x^n}{\sqrt{n}}$.

   Solution. Using the ratio test, we have $\rho = \displaystyle\lim_{n\to\infty} \frac{x^{n+1}\sqrt{n+1}}{x^n\sqrt{n}} = x$. The series converges when $|x| < 1$ and diverges when $|x| > 1$, so the radius of convergence is 1. If $x = 1$, the series is $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$, and it diverges by the p-series test (as $p < 1$). If $x = -1$, the series is $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$, and it converges by the alternating series test. Thus the interval of convergence is $[-1, 1)$.

2. Use a power series to represent $\dfrac{x^2}{4-x^2}$ and find its interval of convergence.

   Solution. See example 6.3 (b) in the book.

Section 6.3. Taylor and Maclaurin Series.

Examples.

1. Find the Taylor series of $f(x) = \ln x$ at $a = 1$. Graph the first seven Taylor polynomials.

   Solution. $f(x) = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \frac{(x-1)^5}{5} - \frac{(x-1)^6}{6} + \dots$.

   $p_0(x) = 0,$

   $p_1(x) = x - 1,$

   $p_2(x) = (x-1) - \frac{(x-1)^2}{2},$

   $p_3(x) = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3},$

   $p_4(x) = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4},$

   $p_5(x) = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \frac{(x-1)^5}{5},$

   $p_6(x) = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \frac{(x-1)^5}{5} - \frac{(x-1)^6}{6}.$

2. Find the Maclaurin series of $f(x) = e^x$.

   Solution. $e^x = 1 + x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \dfrac{x^4}{4!} + \cdots = \displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!}.$

Exercises. (Note: we did not actually have time in class to do these.)
Find and graph the first four Maclaurin polynomials of

1. $f(x) = \sin x$.

2. $f(x) = \cos x$.

Solutions. See example 6.12 in the book.

Section 6.4. Working with Taylor and Series.

Examples.

1. Binomial series: $(1+x)^r = \sum_{n=0}^{\infty} \binom{r}{n} x^n = 1 + rx + \frac{r(r-1)}{2!}x^2 + \frac{r(r-1)(r-2)}{3!}x^3 + \cdots + \frac{r(r-1)\ldots(r-n+1)}{n!}x^n + \ldots$ for $|x| < 1$.

2. Maclaurin polynomial for $\ln(1+x)$: $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{x^n}{n}$.

Exercises.

1. Write the third Maclaurin polynomial for $\sqrt{1+x}$.

   Solution. See example 6.17 in the book.

2. Find the Maclaurin series for $\sin(x^2)$.

   Solution. Since the Maclaurin series for $\sin(x)$ is $\sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{(2n+1)!}$, the Maclaurin series for $\sin(x^2)$ is $\sum_{n=0}^{\infty}(-1)^n \frac{(x^2)^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty}(-1)^n \frac{x^{4n+2}}{(2n+1)!}$.

3. Use $(\tan^{-1} x)' = \frac{1}{1+x^2}$ to find the Maclaurin series for $\tan^{-1} x$.

   Solution. Integrating $\frac{1}{1+x^2} = \frac{1}{1-(-x^2)} = \sum_{n=0}^{\infty}(-x^2)^n = \sum_{n=0}^{\infty}(-1)^n x^{2n}$, we have $\tan^{-1} x = c_0 + \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{2n+1}$. Since $\arctan(0) = 0$, we have $\tan^{-1}(x) = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{2n+1}$.